A study on homogeneous Reinhardt domains

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博士論文

A study on homogeneous Reinhardt domains
（等質ラインハルト領域に関する研究）

木村　光一

平成２５年

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Abstract

In this paper, we give the following two different partial answers to the same conjecture:
(i) As is well known, a homogeneous Reinhardt domain in $\mathbf{C}^*$ coincides with $\mathbf{C}^*$. Generalizing this fact, we show that a pseudoconvex homogeneous Reinhardt domain in $(\mathbf{C}^*)^n$ coincides with $(\mathbf{C}^*)^n$ itself.
(ii) We classify Liouville foliations which are defined on pseudoconvex Reinhardt domains containing the origin in $\mathbf{C}^3$. From this, when the preceding domains are homogeneous, we classify these domains by means of algebraic equivalence and determine their canonical forms.

Contents

1 Introduction

1.1 Holomorphic equivalence problem

Poincaré showed that there is no holomorphic isomorphism of the polydisc $B_1 \times B_1$ to the unit ball B_2 in $\mathbf{C}^2$ (cf. [6]). This is in sharp contrast to the Riemann mapping theorem in one complex variable. Sunada and Shimizu generalized the Poincaré example. In order to state their results, we need two notions.

First, a holomorphic automorphism $(z_i) \mapsto (w_i)$ of $(\mathbf{C}^*)^n$, where $\mathbf{C}^*$ is the set of non-zero complex numbers, is called an algebraic automorphism if its components are given by Laurent monomials, that is, they are of the form

$$w_i = \alpha_i \, {z_i}^{a_{i1}} \cdots {z_n}^{a_{in}}, \qquad i = 1, \cdots, n\,; \tag{1.1}$$

where $(a_{ij}) \in GL(n, \mathbf{Z})$ and $(\alpha_i) \in (\mathbf{C}^*)^n$. The group of all algebraic automorphisms of $(\mathbf{C}^*)^n$ is denoted by $\mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$. This is a 2n-dimensional Lie group with respect to the compact-open topology, and its identity component is the multiplicative group $(\mathbf{C}^*)^n$ which acts on $\mathbf{C}^n$ by the rule that

$$\alpha \cdot z := (\alpha_1 z_1, \cdots, \alpha_n z_n)$$

for every $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbf{C}^*)^n$ and every $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$. Suppose that D and D' are domains in $\mathbf{C}^n$ and a holomorphic isomorphism $\varphi : D \to D'$ is induced by an algebraic automorphism Φ of $(\mathbf{C}^*)^n$, that is, $\varphi = \Phi$ in $D^* = D \cap (\mathbf{C}^*)^n$. Then φ is said to be an algebraic isomorphism and the two domains D, D' are called algebraically equivalent.

Secondly, suppose that a domain D in $\mathbf{C}^n$ is stable under rotations about the coordinate axes, that is, $\alpha \cdot z \in D$ for every $\alpha \in U(1)^n \subset (\mathbf{C}^*)^n$ and every $z \in D$, where $U(1)$ is the one-dimensional unitary group. Then D is called a Reinhardt domain. For example, the polydisc and the unit ball are Reinhardt domains.

Theorem 1.1. (Sunada [12]). *If two bounded Reinhardt domains in $\mathbf{C}^n$ containing the origin are holomorphically equivalent, then there exists an algebraic isomorphism between them induced by an algebraic automorphism $(z_i) \mapsto (w_i)$ of $(\mathbf{C}^*)^n$ of the form*

$$w_i = \alpha_i z_{\sigma(i)}, \qquad i = 1, \cdots, n\,; \tag{1.2}$$

where σ is a permutation of the indices and $(\alpha_1, \cdots, \alpha_n) \in (\mathbf{C}^*)^n$.

There is no algebraic isomorphism of the form (1.2) between the polydisc and the unit ball in $\mathbf{C}^2$. Hence, by the above theorem, they are not holomorphically equivalent. In other words, Theorem 1.1 includes the Poincaré example. Moreover, Shimizu [8] removed the assumption, that is, that the domains contain the origin, from Theorem 1.1:

Theorem 1.2. *If two bounded Reinhardt domains in* $\mathbf{C}^n$ *are holomorphically equivalent, then they are algebraically equivalent.*

If two Reinhardt domains contain the origin, then every algebraic isomorphism between them must be of the form (1.2). Hence, Theorem 1.2 implies Theorem 1.1 as a corollary and gives a partial answer in the bounded case of the following problem:

Holomorphic equivalence problem. *If two Reinhardt domains* D *and* D' *in* $\mathbf{C}^n$ *are holomorphically equivalent, then are they also algebraically equivalent?*

Unfortunately, in the unbounded case, or when $\log D^*$ and $\log(D')^*$ contain a complete straight line (see Section 2.1), we know little. This is because, in such a case, we cannot use the following theorem:

Cartan's theorem (cf. [1]) *Let* D *be a bounded domain in* $\mathbf{C}^n$*. Then* Aut(D)*, which is the holomorphic automorphism group of* D*, has the structure of a Lie group with respect to the compact-open topology and acts as a Lie transformation group on* D*.*

For this reason, Shimizu [9], [10] introduced the notion of Liouville foliation (cf. Definition 2.4) and obtained the following result: For a pair (a, b) of non-negative constants with $(a, b) \neq (0, 0)$ and positive constant r, we define an unbounded Reinhardt domain $D_{a,b}(r)$ in $\mathbf{C}^2$ by

$$D_{a,b}(r) := \{(z, w) \in \mathbf{C}^2 \mid |z|^a |w|^b < r\}.$$

For example, in the case of $b = 0$, the domain $D_{a,b}(r)$ is considered to be $\{(z, w) \in \mathbf{C}^2 \mid |z|^a < r\}$.

Theorem 1.3. *If* $D_{a,b}(r)$ *and* $D_{u,v}(s)$ *are holomorphically equivalent, then they are algebraically equivalent under an automorphism of* $\mathbf{C}^2$ *given by*

$$(z, w) \mapsto (\alpha z, \beta w) \qquad or \qquad (z, w) \mapsto (\gamma w, \delta z),$$

where $\alpha, \beta, \gamma, \delta$ *are non-zero complex constants.*

Note that the proof of Theorem 1.3 yields the classification of the Liouville foliations which are defined on unbounded pseudoconvex Reinhardt domains containing the origin in $\mathbf{C}^2$ (cf. Theorem 4.1).

1.2 Conjecture

Related to the holomorphic equivalence problem, we have the problem of determining canonical forms of homogeneous Reinhardt domains. Here, a complex manifold M is said to be homogeneous if Aut(M) (that is, the holomorphic automorphism group of M) acts on M transitively. If a Reinhardt domain is bounded, then the following result has been shown [8].

Theorem 1.4. *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$. If D is homogeneous, then D is algebraically equivalent to the direct product $B_{n_1} \times \cdots \times B_{n_k}$ of balls, where B_{n_i} denotes the unit ball in $\mathbf{C}^{n_i}$.*

For the general case, canonical forms of homogeneous Reinhardt domains are conjectured to be as follows:

Conjecture. *For every homogeneous Reinhardt domain D in $\mathbf{C}^n$, there exist k positive integers $n_1, \cdots, n_k$ (k may be 0) and non-negative integers l, m such that D is algebraically equivalent to the direct product $B_{n_1} \times \cdots \times B_{n_k} \times \mathbf{C}^l \times (\mathbf{C}^*)^m$.*

Theorem 1.4 means that this conjecture is true when D is bounded. Note also that this conjecture includes the remarkable assertion that every homogeneous Reinhardt domain is pseudoconvex. However, for when D is unbounded, no proof has yet been reported. For that reason, under the additional condition that D be pseudoconvex, Shimizu and Kimura gave a partial answer for the unbounded case [11]:

Theorem 1.5. *Let D be a pseudoconvex Reinhardt domain in $(\mathbf{C}^*)^n$. If D is homogeneous, then D coincides with $(\mathbf{C}^*)^n$.*

On the other hand, in the case of $n = 2$, it follows from Corollary 4.2 herein that the conjecture is true if D is pseudoconvex and contains the origin. Note that the condition that D contain the origin conflicts with the condition of Theorem 1.5 that D contain no coordinate hyperplanes.

The purposes of this paper consist of giving the proof of Theorem 1.5, and generalizing Theorem 4.1 herein to the three-dimensional case which implies the same dimensional generalization of Corollary 4.2.

2 Preliminaries

2.1 Covering tube domain

Suppose that D is a Reinhardt domain and D^* is defined by $D \cap (\mathbf{C}^*)^n$. If D is pseudoconvex, then we can explicitly construct a universal covering of D^* which is a useful tool for investigating D.

A tube domain T_Ω is a domain in $\mathbf{C}^n$ defined by $T_\Omega = \Omega + \sqrt{-1}\mathbf{R}^n$, where Ω is a domain in $\mathbf{R}^n$ and is called the base of T_Ω. Now, consider a mapping $\log : (\mathbf{C}^*)^n \to \mathbf{R}^n$ defined by

$$\log(z_1, \cdots, z_n) = (-(2\pi)^{-1} \log |z_1|, \cdots, -(2\pi)^{-1} \log |z_n|).$$

We make each Reinhardt domain D in $\mathbf{C}^n$ correspond to a tube domain T_Ω in $\mathbf{C}^n$ such that $\Omega = \log D^*$. Then, T_Ω is a covering manifold of D^*. Its covering map $\varpi : T_\Omega \to D^*$ is the restriction of a natural covering map $\varpi : \mathbf{C}^n \to (\mathbf{C}^*)^n$ defined by

$$\varpi(\zeta_1, \cdots, \zeta_n) = (\exp(-2\pi\zeta_1), \cdots, \exp(-2\pi\zeta_n)).$$

Its transformation group $G(T_\Omega/D^*)$ is given by $\{\sigma_\eta | \eta \in \mathbf{Z}\}$, where σ_η is a translation of $\mathbf{C}^n$: $\sigma_\eta(\zeta) = \zeta + \eta$ for $\zeta \in \mathbf{C}^n$.

It is well known that the following relationships between pseudoconvexity of D and convexity of $\log D^*$ hold (cf. [4]):

Proposition 2.1. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Then D is pseudoconvex if and only if the following hold:*
(1) *$\log D^*$ is convex,*
(2) *D is relatively complete, i.e., for every $j = 1, \cdots, n$, if $D \cap H_j \neq \phi$, then $\hat{D}^j \subset D$, where $H_j = \{(z_1, \cdots, z_n) \in \mathbf{C}^n | z_j = 0\}$ and $\hat{D}^j = \{(z_1, \cdots, \alpha z_j, \cdots, z_n) | (z_1, \cdots, z_n) \in D, \alpha \in B_1\}$.*

In particular, if D contains the origin, then D must be complete.

Therefore, if D is pseudoconvex, then $\varpi : T_\Omega \to D^*$ is the universal covering of D^*. Indeed, since $\Omega = \log D^*$ is convex, Ω is simply connected, and so is T_Ω.

2.2 Lifting of an algebraic isomorphism.

Suppose that D and D' are Reinhardt domains in $\mathbf{C}^n$, $\varphi : D \to D'$ is an algebraic isomorphism, and $\Phi : T_\Omega \to T_{\Omega'}$ is a lifting of φ, where T_Ω and $T_{\Omega'}$

denote the covering tube domains of D and D', respectively. We will discuss the relationship between φ and its lifting Φ.

Let $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ be the group of all complex affine transformations of $\mathbf{C}^n$ whose linear parts belong to $GL(n,\mathbf{Z})$. Let Φ be any element of $GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ of the form $\Phi(\zeta) = A\zeta + \alpha$, where $A = (a_{ij}) \in GL(n,\mathbf{Z})$ and $\alpha = (\alpha_i) \in \mathbf{C}^n$. We associate Φ with an element $\varphi : (z_i) \mapsto (w_i)$ of $\mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$ defined by

$$w_i = e^{-2\pi\alpha_i} z_1^{a_{i1}} \cdots z_n^{a_{in}}, \quad i = 1, 2, \cdots, n.$$

The mapping $\rho : GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n \to \mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^n)$ assigning φ to Φ is a surjective group homomorphism, whose kernel is given by $\{\sigma_\eta | \eta \in \mathbf{Z}\}$. Then, the following holds:

$$\varpi \circ \Phi = \rho(\Phi) \circ \varpi \tag{2.1}$$

for every $\Phi \in GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$, where $\varpi : \mathbf{C}^n \to (\mathbf{C}^*)^n$ is the covering map.

From (2.1), we see that $\rho(\Phi)(D) = D'$ if and only if $\Phi(T_\Omega) = T_{\Omega'}$. Consequently, we have the following result:

Proposition 2.2. *Let D and D' be Reinhardt domains in $\mathbf{C}^n$. Then, for every algebraic isomorphism $\varphi : D \to D'$, its lifting $\Phi : T_\Omega \to T_{\Omega'} \in GL(n,\mathbf{Z}) \ltimes \mathbf{C}^n$ is uniquely determined up to an element of $\{\sigma_\eta | \eta \in \mathbf{Z}\}$, and vice versa.*

In particular, if D contains the origin, then every algebraic isomorphism $\varphi : D \to D'$ must be of the form (1.2). Therefore, its lifting $\Phi : T_\Omega \to T_{\Omega'}$ is the composition of a permutation of the coordinates and a translation of $\mathbf{C}^n$.

2.3 $\ell(D)$

Suppose that D is a pseudoconvex Reinhardt domain in $\mathbf{C}^n$, and set $\Omega = \log(D^*)$. By Proposition 2.1, Ω is convex in $\mathbf{R}^n$. Hence, we have the maximal affine subspace $A(D)$ of Ω, which is uniquely determined, and have the linear subspace $V(D)$ in $\mathbf{R}^n$ which consists of all translations of $A(D)$. Then, we set

$$\ell(D) := \dim_{\mathbf{R}} V(D).$$

The integer $\ell(D)$ between 0 and n associated with D represents the essential size of D. In fact, there exists an affine transformation f of $\mathbf{R}^n$ such that

$$f(\Omega) = \Xi^{(1)} \times \mathbf{R}^\ell,$$

where $\ell = \ell(D)$, and $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^{n-\ell}$ containing no complete straight lines (cf. [4]). Now, we write $f(\xi) = A\xi + a$ for $\xi \in \mathbf{R}^n$, where $A \in GL(n, \mathbf{R}), a \in \mathbf{R}^n$, then f naturally gives rise to an affine transformation of $\mathbf{C}^n$, that is, $f(\eta) = A\eta + a$ for $\eta \in \mathbf{C}^n$. As a consequence, we have

$$f(T_\Omega) = T_{f(\Omega)} = T_{\Xi^{(1)} \times \mathbf{R}^\ell} = T_{\Xi^{(1)}} \times T_{\mathbf{R}^\ell} = T_{\Xi^{(1)}} \times \mathbf{C}^\ell,$$

and hence T_Ω is holomorphically equivalent to $T_{\Xi^{(1)}} \times \mathbf{C}^\ell$. In particular, when $\ell(D) = 0$, we may assume that f is an element of $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$, and we have

$$f(\Omega) = \Xi^{(1)} \subset \{(\xi_1, \cdots, \xi_n) \in \mathbf{R}^n |\, \xi_i > 0, i = 1, \cdots, n\}.$$

Hence, by Proposition 2.2, the domain D^* is algebraically equivalent to a bounded Reinhart domain in $(\mathbf{C}^*)^n$ (cf. [4]).

Proposition 2.3. *Suppose that D is a Reinhart domain in $\mathbf{C}^n$. If D is pseudoconvex, then the tube domain T_Ω of D^* is holomorphically equivalent to $T_{\Xi^{(1)}} \times \mathbf{C}^{\ell(D)}$, where $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^{n-\ell}$ containing no complete straight lines. In particular:*
(1) *If $\ell(D) = 0$, then D^* is algebraically equivalent to a bounded Reinhardt domain.*
(2) *If $\ell(D) = n$, then D^* coincides with $(\mathbf{C}^*)^n$.*

2.4 Liouville foliation

The notion of Liouville foliation was introduced by Shimizu (see [9], [10]) in order to analyze unbounded Reinhardt domains and look for their holomorphic automorphism groups.

For a complex manifold M, let $f_1, \cdots, f_m$ be bounded holomorphic functions on M and $g_1, \cdots, g_n$ be bounded plurisubharmonic (briefly, psh) functions on M. Then, a map $\varphi_{m,n} := (f_1, \cdots, f_m, g_1, \cdots, g_n)$ on M is called a Liouville map.

Definition 2.4. *Let M be a complex manifold. A collection $\mathcal{F} = \{\Sigma_\alpha\}_{\alpha \in A}$ of subsets of M is called a Liouville foliation on M if the following conditions are satisfied:*
(L1) *If $\alpha_1 \neq \alpha_2$, then* $\Sigma_{\alpha_1} \cap \Sigma_{\alpha_2} = \phi$;
(L2) $\bigcup_{\alpha \in A} \Sigma_\alpha = M$;
(L3) *Any bounded psh function on M takes a constant value on each Σ_α;*

(L4) *For every $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \neq \alpha_2$, there exist a pair (m, n) of non-negative integers and a Liouville map $\varphi_{m,n}$ on M such that the constant values of $\varphi_{m,n}$ on Σ_{α_1} and Σ_{α_2} are different. Then, we say that $\varphi_{m,n}$ separates Σ_{α_1} and Σ_{α_2}.*

By condition (L3), any Liouville map on M takes a constant value on each Σ_α. In addition, if we can choose all integers n to be 0 in (L4), then we may replace psh function by holomorphic function in (L3).

Definition 2.5. *Let E be a subset of a complex manifold M. Then a collection $\mathcal{F}_E = \{\Sigma'_\beta\}_{\beta \in B}$ of subsets of E is called a sub-Liouville foliation on E if all conditions in Definition 2.3 are satisfied when we replace Σ_α by Σ'_β.*

Suppose $M = E \sqcup E'$, i.e., the disjoint union of E and E', and suppose that E' has a sub-Liouville foliation $\mathcal{F}_{E'} = \{\Sigma''_\gamma\}_{\gamma \in C}$. If any $\Sigma'_\beta \in \mathcal{F}_E$ and any $\Sigma''_\gamma \in \mathcal{F}_{E'}$ are separated by some Liouville map on M, then we see that $\mathcal{F} = \mathcal{F}_E \sqcup \mathcal{F}_{E'}$ is a Liouville foliation on M.

Note that each subset of M has at most one structure of sub-Liouville foliation (cf. [9], [10]):

Proposition 2.6. *Let M be a complex manifold and E be a subset of M. If $\mathcal{F}_1 = \{\Sigma_\alpha\}_{\alpha \in A}$ and $\mathcal{F}_2 = \{\Sigma'_\beta\}_{\beta \in B}$ are two sub-Liouville foliations on E, then they coincide, that is, there exists a bijection $\tau : A \to B$ such that $\Sigma_\alpha = \Sigma'_{\tau(\alpha)}$ for every $\alpha \in A$.*

The following lemma is easily proved by using the notion of Liouville foliation, but it is important for deciding the canonical forms of Reinhardt domains.

Lemma 2.7. *Let $E \times \mathbf{C}^\ell$ and $E' \times \mathbf{C}^{\ell'}$ be two domains in $\mathbf{C}^n$, where E and E' are domains in $\mathbf{C}^{n-\ell}$ and $\mathbf{C}^{n-\ell'}$, respectively, that are holomorphically equivalent to bounded domains. Suppose that there exists a biholomorphic mapping Φ of $E \times \mathbf{C}^\ell$ onto $E' \times \mathbf{C}^{\ell'}$. Then ℓ and ℓ' coincide. Moreover, if each point $w \in \mathbf{C}^n = \mathbf{C}^{n-\ell} \times \mathbf{C}^\ell$ is written as*

$$w = (w^{(1)}, w^{(2)}), \quad w^{(1)} \in \mathbf{C}^{n-\ell},\ w^{(2)} \in \mathbf{C}^\ell,$$

then $\Phi : E \times \mathbf{C}^\ell \to E' \times \mathbf{C}^{\ell'}$ has the form

$$w = (w^{(1)}, w^{(2)}) \mapsto (\Phi^{(1)}(w^{(1)}), \Phi^{(2)}(w)),$$

where $\Phi^{(1)} : E \ni w^{(1)} \mapsto \Phi^{(1)}(w^{(1)}) \in E'$ gives a biholomorphic mapping of E onto E'.

Next, we define the dimension of a Liouville foliation $\mathcal{F} = \{\Sigma\}$ on a Reinhardt domain D in $\mathbf{C}^n$.

If Σ is the real submanifold of D, then Σ is said to be non-singular. On the other hand, suppose that there exists C^∞-map $f = (f_1, \cdots, f_p) : D \to \mathbf{R}^p$ such that

$$\Sigma = \{z = (z_1, \cdots, z_n) \in D | f(z) = 0\}.$$

If $\{z \in \Sigma \,|\, df(z) = 0\} \neq \phi$ and $\Sigma \setminus \{z \in \Sigma \,|\, df(z) = 0\}$ is the finite disjoint union of real submanifolds Σ^k of $D, k = 1, \cdots, q$, then Σ is called quasi-singular. We call the number

$$\max \{ \dim \Sigma^k \,|\, k = 1, \cdots, q\}$$

the dimension of Σ.

Definition 2.8. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Suppose that D has a Liouville foliation $\mathcal{F} = \{\Sigma_\alpha\}_{\alpha \in A}$, and set*

$$\mathcal{F}' := \{\Sigma_\alpha \in \mathcal{F} \,|\, \Sigma_\alpha \cap D^* \neq \phi\}.$$

If all elements of $\mathcal{F}'$ are non-singular and have the same dimension m, then the number m is called the dimension of $\mathcal{F}$, and is denoted by $\dim \mathcal{F}$.

3 Some Reinhardt domains in $(\mathbf{C}^*)^n$

In this section, we prove Theorem 1.5 according to [11]. Let D be a pseudoconvex homogeneous Reinhardt domain in $(\mathbf{C}^*)^n$. Then, D coincides with $(\mathbf{C}^*)^n$ if and only if $\ell(D) = n$. Therefore, suppose $\ell(D) < n$, and we shall derive a contradiction.

If $\ell(D) = 0$, then D is algebraically equivalent to a bounded Reinhardt domain. By Theorem 1.4, every homogeneous bounded Reinhardt domain is algebraically equivalent to the direct product $D' := B_{n_1} \times \cdots \times B_{n_k}$ of balls, which contains the origin. Hence, there exists a holomorphic isomorphism of D onto D' induced by an algebraic automorphism $(z_i) \mapsto (w_i)$ of $(\mathbf{C}^*)^n$ of the form $w_i = \alpha_i z_{\tau(i)}$, $i = 1, \cdots, n$, where τ is a permutation of $\{1, 2, \cdots, n\}$ and $(\alpha_1, \cdots, \alpha_n) \in (\mathbf{C}^*)^n$. Consequently, D contains the origin, which contradicts the assumption $D \subset (\mathbf{C}^*)^n$.

Now we suppose $0 < \ell(D) < n$, and set $k = n - \ell(D)$. We divide the proof into four steps.

Step 1. Let T_Ω be the covering tube domain of D and let $\Gamma := \{\sigma_\eta | \eta \in \mathbf{Z}^n\}$ be its covering transformation group. We denote by $\tilde{\varphi} \in \mathrm{Aut}(T_\Omega)$ a lifting of $\varphi \in \mathrm{Aut}(D)$. The set G of all liftings $\tilde{\varphi}$ forms a subgroup of $\mathrm{Aut}(T_\Omega)$ and is given as the normalizer of Γ in $\mathrm{Aut}(T_\Omega)$. Since the covering transformation group Γ is isomorphic to the additive group $\mathbf{Z}^n$, for every $\tilde{\varphi} \in G$ there exists a unique $A \in GL(n, \mathbf{Z})$ such that

$$\tilde{\varphi} \circ \sigma_\eta \circ \tilde{\varphi}^{-1} = \sigma_{A\eta}. \tag{3.1}$$

From this, we have a linear representation ρ of G on $GL(n, \mathbf{Z})$ such that $G \ni \tilde{\varphi} \mapsto A \in GL(n, \mathbf{Z})$. By assumption, $\mathrm{Aut}(D)$ acts on D transitively. Therefore, G acts on T_Ω transitively.

Next, we shall see what influence a permutation of coordinates has on the linear representation ρ of G. We consider a linear transformation of $\mathbf{C}^n$ represented by a matrix $P_\tau := (\delta_{\tau(i)j}) \in GL(n, \mathbf{Z})$, where τ is a permutation of $\{1, 2, \cdots, n\}$. Then, the universal covering $\varpi : T_\Omega \to D$ is replaced by the covering $P_\tau \circ \varpi \circ P_\tau^{-1} : T_{P_\tau(\Omega)} \to P_\tau(D)$, and a lifting of $P_\tau \circ \varphi \circ P_\tau^{-1} \in \mathrm{Aut}(P_\tau(D))$ is given by $P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1} \in \mathrm{Aut}(T_{P_\tau(\Omega)})$. Equation (3.1) means that

$$\tilde{\varphi}(\zeta + \sqrt{-1}m) = \tilde{\varphi}(\zeta) + \sqrt{-1}Am \tag{3.2}$$

for every $\zeta \in T_\Omega$ and every $m \in \mathbf{Z}^n$. Hence, we have

$$\begin{aligned} P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1}(\zeta + \sqrt{-1}m) &= P_\tau \circ \tilde{\varphi}\,(P_\tau^{-1}(\zeta) + \sqrt{-1}P_\tau^{-1}(m)) \\ &= P_\tau\,(\tilde{\varphi}\,(P_\tau^{-1}(\zeta)) + \sqrt{-1}AP_\tau^{-1}(m)) \\ &= P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1}(\zeta) + \sqrt{-1}P_\tau A P_\tau^{-1} m \end{aligned}$$

for every $\zeta \in T_{P_\tau(\Omega)}$ and every $m \in \mathbf{Z}^n$. Consequently, the linear representation $\rho : G \to GL(n, \mathbf{Z})$ is just replaced by $\rho' : P_\tau G P_\tau^{-1} \ni P_\tau \circ \tilde{\varphi} \circ P_\tau^{-1} \mapsto P_\tau A P_\tau^{-1} \in GL(n, \mathbf{Z})$.

Step 2. We would like to represent the domain $\Omega = \log(D)$ in $\mathbf{R}^n$ as simply as possible. By means of a linear transformation L on $\mathbf{R}^n$ induced by a suitable permutation of coordinates, we make $V(D)$ parallel to some coordinate axes.

For simplicity, write $\ell := \ell(D)$. Since $\dim V(D) = \ell$, there exists a basis $\{v_1, \cdots, v_\ell\}$ of $V(D)$ over $\mathbf{R}$. We write $V := (v_1, \cdots, v_\ell)$, which is an $n \times \ell$ matrix consisting of column vectors $v_1, \cdots, v_\ell$. As $\mathrm{rank}\, V = \ell$, doing a suitable permutation of coordinates, we have

$$V = \begin{pmatrix} V^{(1)} \\ V^{(2)} \end{pmatrix},$$

where $V^{(1)}$ is a $k \times \ell$ matrix and $V^{(2)}$ is a non-singular $\ell \times \ell$ matrix. Moreover, by means of elementary transformations to column vectors of V if necessary, we have $V^{(2)} = I_\ell$ and

$$V = \begin{pmatrix} V^{(1)} \\ I_\ell \end{pmatrix}.$$

We define a matrix L by

$$L := \begin{pmatrix} I_k & -V^{(1)} \\ O & I_\ell \end{pmatrix} \in GL(n, \mathbf{R}). \tag{3.3}$$

Then we see

$$LV = \begin{pmatrix} I_k & -V^{(1)} \\ O & I_\ell \end{pmatrix} \begin{pmatrix} V^{(1)} \\ I_\ell \end{pmatrix} = \begin{pmatrix} O \\ I_\ell \end{pmatrix}.$$

Writing $\Xi := L\Omega$, we have

$$\Xi = \Xi^{(1)} \times \mathbf{R}^\ell, \tag{3.4}$$

where $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^k$ containing no complete straight lines. If we consider L as a linear transformation of $\mathbf{C}^n$, then

$$L(T_\Omega) = T_{L(\Omega)} = T_\Xi = T_{\Xi^{(1)}} \times \mathbf{C}^l,$$

and $\Phi := L \circ \tilde{\varphi} \circ L^{-1} \in \mathrm{Aut}(T_\Xi)$ satisfies

$$\Phi(w + \sqrt{-1}Lm) = \Phi(w) + \sqrt{-1}LAm \tag{3.5}$$

for every $w \in T_\Xi$ and every $m \in \mathbf{Z}^n$. Indeed, by (3.2),

$$\begin{aligned} \Phi(w + \sqrt{-1}Lm) &= L \circ \tilde{\varphi} \circ L^{-1}(w + \sqrt{-1}Lm) \\ &= L \circ \tilde{\varphi}\,(L^{-1}w + \sqrt{-1}m) \\ &= L(\tilde{\varphi}\,(L^{-1}w) + \sqrt{-1}Am) \\ &= \Phi(w) + \sqrt{-1}LAm. \end{aligned}$$

Note that LGL^{-1} acts on T_Ξ transitively.

Step 3. We consider a holomorphic automorphism:

$$\Phi : T_{\Xi^{(1)}} \times \mathbf{C}^\ell \ni w = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix} \longmapsto \begin{pmatrix} \Phi^{(1)}(w) \\ \Phi^{(2)}(w) \end{pmatrix} \in T_{\Xi^{(1)}} \times \mathbf{C}^\ell,$$

where $w^{(1)}, \Phi^{(1)}(w) \in T_{\Xi^{(1)}}$ and $w^{(2)}, \Phi^{(2)}(w) \in \mathbf{C}^\ell$. Since $\Xi^{(1)}$ is a convex domain in $\mathbf{R}^k$ containing no complete straight lines, $T_{\Xi^{(1)}}$ is holomorphically equivalent to a bounded domain in $\mathbf{C}^k$. Hence, Lemma 2.7 implies that the first component $\Phi^{(1)}$ depends only on $w^{(1)}$ and $\Phi^{(1)} : T_{\Xi^{(1)}} \ni w^{(1)} \mapsto \Phi^{(1)}(w^{(1)}) \in T_{\Xi^{(1)}}$ is a holomorphic automorphism.

We shall next see a more precise form of $\Phi^{(1)}$. For every $A = \rho(\tilde{\varphi})$ with $\tilde{\varphi} \in G$, set

$$LA =: \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix}, \tag{3.6}$$

where $A^{(1)}$ is a $k \times k$ matrix, and $A^{(2)}, A^{(3)}, A^{(4)}$ are $k \times \ell, \ell \times k, \ell \times \ell$ matrices, respectively. Note that the matrix L defined by (3.3) is independent of A. We will later see that $A^{(1)}$ is an element of $GL(k, \mathbf{R})$. Set

$$L^{(1)} := (I_k, -V^{(1)}), \quad (LA)^{(1)} := (A^{(1)}, A^{(2)}). \tag{3.7}$$

Then, by (3.5), $\Phi^{(1)}$ satisfies

$$\Phi^{(1)}(w^{(1)} + \sqrt{-1}L^{(1)}m) = \Phi^{(1)}(w^{(1)}) + \sqrt{-1}(LA)^{(1)}m \tag{3.8}$$

for every $w^{(1)} \in T_{\Xi^{(1)}}$ and every $m \in \mathbf{Z}^n$. In particular, setting

$$m = {}^t(m_1, \cdots, m_k, 0, \cdots, 0),$$

and writing $m^{(1)} := {}^t(m_1, \cdots, m_k)$, by (3.7) we have

$$L^{(1)}m = m^{(1)}, \quad (LA)^{(1)}m = A^{(1)}m^{(1)}.$$

Hence, (3.8) implies

$$\Phi^{(1)}(w^{(1)} + \sqrt{-1}m^{(1)}) = \Phi^{(1)}(w^{(1)}) + \sqrt{-1}A^{(1)}m^{(1)} \tag{3.9}$$

for every $w^{(1)} \in T_{\Xi^{(1)}}$ and every $m^{(1)} \in \mathbf{Z}^k$, and by [11, Lemma 2.2] the matrix $A^{(1)}$ is non-singular. Therefore, by [11, Proposition 2.2], we see $\Phi^{(1)} \in GL(k, \mathbf{R}) \ltimes \mathbf{C}^k$, that is, there exist $B^{(1)} \in GL(k, \mathbf{R})$ and $\beta^{(1)} \in \mathbf{C}^k$ such that

$$\Phi^{(1)}(w^{(1)}) = B^{(1)}w^{(1)} + \beta^{(1)}. \tag{3.10}$$

Substituting (3.10) into (3.9) yields $B^{(1)}m^{(1)} = A^{(1)}m^{(1)}$ for every $m^{(1)} \in \mathbf{Z}^k$. Consequently, we see that $B^{(1)} = A^{(1)}$, or

$$\Phi^{(1)}(w^{(1)}) = A^{(1)}w^{(1)} + \beta^{(1)} \quad \text{for every } w^{(1)} \in T_{\Xi^{(1)}}. \tag{3.11}$$

Note that $G^{(1)} := \left\{\Phi^{(1)} | \, \Phi \in LGL^{-1}\right\}$ acts on $T_{\Xi^{(1)}}$ transitively.

In (3.11), we can decompose $\Phi^{(1)}$ into real and imaginary components, since $A^{(1)} \in GL(k, \mathbf{R})$. Namely, we can write

$$w^{(1)} =: \xi^{(1)} + \sqrt{-1}\,\eta^{(1)}, \quad \beta^{(1)} =: a^{(1)} + \sqrt{-1}\,b^{(1)},$$

where $\xi^{(1)}, \eta^{(1)}, a^{(1)}, b^{(1)}$ are elements of $\mathbf{R}^k$. Then, we have

$$\Phi^{(1)}\,(w^{\,(1)}) = A^{\,(1)}\xi^{\,(1)} + a^{\,(1)} + \sqrt{-1}\,(A^{\,(1)}\eta^{\,(1)} + b^{\,(1)})$$

and the real component

$$\operatorname{Re}\Phi^{(1)}\,(\xi^{\,(1)}) := A^{\,(1)}\xi^{\,(1)} + a^{\,(1)} \tag{3.12}$$

gives an affine automorphism of the domain $\Xi^{(1)}$. Then, $H^{(1)} := \left\{\operatorname{Re}\Phi^{(1)} | \Phi^{(1)} \in G^{(1)}\right\}$ acts on $\Xi^{(1)}$ transitively.

Step 4. We see that in (3.12) the translational part $a^{(1)}$ is uniquely determined by the linear part $A^{(1)}$:

Lemma 3.1 If $A^{\,(1)}\xi^{\,(1)} + a^{\,(1)}$ and $A^{\,(1)}\xi^{\,(1)} + b^{\,(1)}$ are elements of $H^{(1)}$, then $a^{\,(1)} = b^{\,(1)}$.

Proof. By the assumptions, we have

$$A^{\,(1)}(\Xi^{\,(1)}) + a^{\,(1)} = \Xi^{\,(1)}, \quad A^{\,(1)}(\Xi^{\,(1)}) + b^{\,(1)} = \Xi^{\,(1)}.$$

Since $\Xi^{\,(1)} - a^{\,(1)} = A^{\,(1)}(\Xi^{\,(1)})$, it follows that

$$\Xi^{\,(1)} + (b^{(1)} - a^{(1)}) = (\Xi^{\,(1)} - a^{(1)}) + b^{(1)} = A^{\,(1)}(\Xi^{\,(1)}) + b^{(1)} = \Xi^{\,(1)}.$$

Since $\Xi^{\,(1)}$ is a convex domain containing no complete straight lines, this can only happen when $b^{(1)} - a^{(1)} = 0$. □

Note that in (3.6), $A^{\,(1)}$ is the $k \times k$ principal matrix of LA, where $L \in GL(n, \mathbf{R})$ is the fixed matrix determined by the domain D and $A \in \rho(G) \subset GL(n, \mathbf{Z}^n)$. By Lemma 3.1, we have a surjection of $\rho(G)$ onto $H^{(1)}$. Since $\rho(G)$ is at most countable, so is $H^{(1)}$. This contradicts the fact that $H^{(1)}$ acts on $\Xi^{(1)}$ transitively, and the proof of Theorem 1.5 is completed.

4 Some Reinhardt domains in $\mathbf{C}^2$

Let D be a unbounded, proper, pseudoconvex Reinhardt domain containing the origin in $\mathbf{C}^2$. By Proposition 2.1, D must be complete. Furthermore, by Proposition 2.3, we have $\ell(D) = 1$. Hence, there exists non-negative real constants a, b with $(a, b) \neq (0, 0)$ such that D is algebraically equivalent to

$$D_{a,b} := \{(z, w) \in \mathbf{C}^2 |\, |z|^a |w|^b < 1\}.$$

Therefore, D may be identified with $D_{a,b}$. We define the number associated with D as

$$\delta(D) := \dim\{a, b\}_{\mathbf{Q}},$$

where $\{a, b\}_{\mathbf{Q}}$ is the linear subspace generated by a and b over $\mathbf{Q}$. The number $\delta(D)$, either 1 or 2, plays a specific role in the classification of the Liouville foliation which is defined on a Reinhardt domain D. Note that, by the last part of Section 2.2, both $\ell(D)$ and $\delta(D)$ are algebraic invariants. The following classification is substantially due to Shimizu [9], [10]:

Theorem 4.1. *Suppose that D is an unbounded, proper, pseudoconvex Reinhardt domain containing the origin in $\mathbf{C}^2$. Then the Liouville foliation $\mathcal{F}$ on D is as follows.*
(1) *If $\delta(D) = 1$, then we may assume that (a, b) is a primitive element of the free module $\mathbf{Z}^2$.*

(i) *if $ab = 0$, then we may assume without loss of generality $(a, b) = (1, 0)$, and D may be identified with $B_1 \times \mathbf{C}$. The foliation $\mathcal{F}$ on D consists of*

$$\Sigma_\zeta := \mathbf{C} \times \{\zeta\}, \quad \zeta \in B_1.$$

Since $\mathcal{F}' = \mathcal{F} \setminus \Sigma_0$, we have $\dim \mathcal{F} = 2$.

(ii) *if $ab \neq 0$, then the foliation $\mathcal{F}$ on D consists of*

$$\Sigma_\zeta := \{(z, w) |\, z^a w^b = \zeta\}, \quad \zeta \in B_1.$$

Since $\mathcal{F}' = \mathcal{F} \setminus \Sigma_0$, we have $\dim \mathcal{F} = 2$, and $\mathbf{\Sigma}_0 = \{(z, w) |\, zw = 0\}$ is quasi-singular.
(2) *If $\delta(D) = 2$ and $b/a = c \notin \mathbf{Q}$, then D may be identified with $D_{1,c}$. The foliation $\mathcal{F}$ on D consists of*

$$\Sigma_t := \{(z, w) |\, |z||w|^c = t\}, \quad t \in I := \{t \in \mathbf{R} |\, 0 \leq t < 1\}.$$

Since $\mathcal{F}' = \mathcal{F} \setminus \Sigma_0$*, we see* $\dim \mathcal{F} = 3$*, and* $\Sigma_0 = \{(z, w) |\, zw = 0\,\}$ *is quasi-singular.*

Let φ be a holomorphic automorphism of D, and $\mathcal{F}$ be a Liouville foliation on D. Since $\varphi(\mathcal{F})$ is also a Liouville foliation on D, they must coincide by Proposition 2.6. Therefore, if $\mathcal{F}$ has both singular and non-singular elements, then D is not homogeneous. From this and Theorem 4.1, the following holds.

Corollary 4.2. *Let* D *be a Reinhardt domain as in Theorem 4.1. If* D *is homogeneous, then* D *is algebraically equivalent to the product* $B_1 \times \mathbf{C}$*.*

5 Some Reinhardt domains in $\mathbf{C}^3$

From now on, all the Reinhardt domains are unbounded, proper, pseudoconvex, and containing the origin in $\mathbf{C}^3$, unless otherwise noted.

5.1 $\delta(D)$

First, we define $\delta(D)$ in the case of three-dimensional complex space. By Proposition 2.3, we have

$$\ell(D) = 1, 2.$$

If $\ell(D) = 2$, then, since D is complete, there exists a triplet (a_1, a_2, a_3) of non-negative real constants such that D is algebraically equivalent to a Reinhardt domain

$$D_{a_1, a_2, a_3} = \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, |z_1|^{a_1} |z_2|^{a_2} |z_3|^{a_3} < 1\}, \tag{5.1}$$

where we may assume $a_1 \neq 0$ without loss of generality. Set $\vec{\pi} := (a_1, a_2, a_3)$, and define $\delta(D)$ by

$$\delta(D) := \dim\{a_1, a_2, a_3\}_{\mathbf{Q}}.$$

If $\ell(D) = 1$, then, without loss of generality, the group of translations $V(D)$ may be generated by a vector

$$\vec{l} = (-l_1, -l_2, 1),$$

and we define $\delta(D)$ by

$$\delta(D) := \dim\{-l_1, -l_2, 1\}_{\mathbf{Q}}\,.$$

Note that a pair $(\ell(D), \delta(D))$ associated with D is an algebraic invariant, as in the case of $\mathbf{C}^2$.

5.2 Main Theorem

As mentioned at the end of Section 1.2, one purpose of this paper is to prove the following theorem, which is a generalization of Theorem 4.1 to the case of $\mathbf{C}^3$.

Theorem 5.1. *The dimension of a Liouville foliation $\mathcal{F}$ which is defined on Reinhardt domain D in $\mathbf{C}^3$ is classified according to the following table.*

$\ell(D)$	$\delta(D)$	$\dim \mathcal{F}$
1	1	2
	2	
	3	4
2	1	4
	2	5
	3	5

Except for the cases of $\ell(D) = 1$, $\vec{l} = (0, 0, 1)$ and $\ell(D) = 2$, $\vec{\pi} = (1, 0, 0)$, which are the exceptional types of $(\ell(D), \delta(D)) = (1, 1)$ and $(2, 1)$, respectively, the Liouville foliation $\mathcal{F}$ has both singular and non-singular elements.

Theorem 4.1 implies that the notion of non-singular is equivalent to the notion of quasi-singular in the case of $\mathbf{C}^2$. In the following, we shall see that the two notions also coincide in the case of $\mathbf{C}^3$.

If $(\ell(D), \delta(D)) = (1, 2)$, then we do not know whether the Liouville foliation $\mathcal{F}$ has a dimension in the sense of Definition 2.8. However, investigating the cases of $(\ell(D), \delta(D)) = (1, 1)$ and $(2, 1)$ in detail, we have the following corollary as a partial solution to the conjecture.

Corollary 5.2. *If a Reinhardt domain D is homogeneous, then D is algebraically equivalent to either $B_1 \times B_1 \times \mathbf{C}$ or $B_2 \times \mathbf{C}$ or $B_1 \times \mathbf{C}^2$.*

6 Domains of Section 5 with $\ell(\mathbf{D}) = 2$

Since $\ell(D) = 2$, a Reinhardt domain D is of the form (5.1):

$$D_{a_1, a_2, a_3} = \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, |z_1|^{a_1} |z_2|^{a_2} |z_3|^{a_3} < 1\},$$

where (a_1, a_2, a_3) is a triplet of non-negative real constants with $a_1 \neq 0$.

6.1 Case of $\delta(\mathbf{D}) = \mathbf{1}$

Since $\delta(D) = 1$, we may assume that (a_1, a_2, a_3) is a primitive element of $\mathbf{Z}^3$.
Case 1. If $a_1 a_2 a_3 \neq 0$, then for every $\zeta \in B_1$, we define Σ_ζ by

$$\Sigma_\zeta := \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, z_1^{a_1} z_2^{a_2} z_3^{a_3} = \zeta\},$$

and then $\mathcal{F} = \{\Sigma_\zeta\}_{\zeta \in B_1}$ satisfies conditions (L1) and (L2).

(1) If $\zeta \in B_1^*$, then, since (a_1, a_2, a_3) is primitive, there exists an algebraic automorphism $\varphi : (z_i) \mapsto (w_i)$ of $(\mathbf{C}^*)^3$ such that

$$w_1 = z_1^{a_1} z_2^{a_2} z_3^{a_3},\ w_2 = z_1^{b_1} z_2^{b_2} z_3^{b_3},\ w_3 = z_1^{c_1} z_2^{c_2} z_3^{c_3}.$$

Hence, $\varphi(\Sigma_\zeta) = \{\zeta\} \times (\mathbf{C}^*)^2$, that is, Σ_ζ is algebraically equivalent to $(\mathbf{C}^*)^2$. Therefore, Σ_ζ is non-singular and has dimension 4.

(2) If $\zeta = 0$, we have

$$\Sigma_0 = \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, z_1 z_2 z_3 = 0\},$$

which is quasi-singular and has dimension 4.

Hence, by Riemann's removable singularities theorem, any bounded holomorphic function on D takes a constant value on each Σ, that is, $\mathcal{F}$ satisfies (L3). Additionally, a bounded holomorphic function $h(z_1, z_2, z_3) := z_1^{a_1} z_2^{a_2} z_3^{a_3}$ separates any pair of elements of $\mathcal{F}$. Therefore, $\mathcal{F}$ also satisfies (L4) and so is a Liouville foliation on D.

Case 2. If only one element of $\{a_1, a_2, a_3\}$ equals 0, then, without loss of generality, we may assume $a_1 a_2 \neq 0, a_3 = 0$. Therefore, D is of the form

$$D_{a_1, a_2, 0} = \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, |z_1|^{a_1} |z_2|^{a_2} < 1\}.$$

In little change from case 1, we have the Liouville foliation $\mathcal{F} = \{\Sigma_\zeta\}_{\zeta \in B_1}$ defined by

$$\Sigma_\zeta := \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, z_1^{a_1} z_2^{a_2} = \zeta\} \quad \text{for } \zeta \in B_1.$$

(1) If $\zeta \in B_1^*$, then Σ_ζ is algebraically equivalent to $\mathbf{C}^* \times \mathbf{C}$, which is non-singular and has dimension 4.

(2) If $\zeta = 0$, then $\Sigma_0 = \{(z_1, z_2, z_3) \in \mathbf{C}^3 |\, z_1 z_2 = 0\}$, which is quasi-singular and has dimension 4.

Any pair of elements of $\mathcal{F}$ are separated by a bounded holomorphic function $h(z_1, z_2, z_3) := z_1^{a_1} z_2^{a_2}$ on D.

Case 3. If $a_2 = a_3 = 0$, then D is of the form

$$D_{a_1,0,0} = \{(z_1, z_2, z_3) \in \mathbf{C}^3 | \, |z_1| < 1\} = B_1 \times \mathbf{C}^2.$$

Hence,

$$\Sigma_\zeta = \{\zeta\} \times \mathbf{C}^2, \quad \zeta \in B_1$$

can be pieced together to give a Liouville foliation $\mathcal{F}$ on D. Any element $\Sigma \in \mathcal{F}$ is non-singular and has dimension 4.

6.2 Case of $\delta(\mathbf{D}) = 2$

Since $a_1 \neq 0$, we may replace (a_1, a_2, a_3) with $(1, b, c)$, where b and c are non-negative real constants. In addition, since $\delta(D) = 2$, without loss of generality, we may assume $c \notin \mathbf{Q}$ and either

(i) $b \in \mathbf{Q}$ or

(ii) $b \notin \mathbf{Q}$ and $b/c \in \mathbf{Q}$.

In either case, D is algebraically equivalent to

$$D_{1,b,c} = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \,|\, |z_1||z_2|^b|z_3|^c < 1\}.$$

Case 1. Suppose $b \in \mathbf{Q}$ and $c \notin \mathbf{Q}$. In order to decide a Liouville foliation on D, we need the following lemma, which is a straightforward generalization of a lemma in [10].

Lemma 6.1. *Suppose that b, c are given as above and α is a complex constant $\xi + \eta\sqrt{-1}$ $(\xi, \eta \in \mathbf{R})$. Let Π_α be the complex hyperplane defined by*

$$\Pi_\alpha := \{(w_1, w_2, w_3) \in \mathbf{C}^3 \,|\, w_1 + bw_2 + cw_3 = \alpha\}.$$

Then, the image of Π_α under the covering map $\varpi : \mathbf{C}^n \to (\mathbf{C}^*)^n$ is dense in the set

$$\Sigma_\xi := \{(z_1, z_2, z_3) \in (\mathbf{C}^*)^3 \,|\, |z_1||z_2|^b|z_3|^c = e^{-2\pi\xi}\}.$$

Proof. Write $w_2 = u + v\sqrt{-1}$ and $w_3 = x + y\sqrt{-1}$. Then we have $\varpi(\Pi_\alpha) = \{(\zeta_1, \zeta_2, \zeta_3) \,|\, x, y, u, v \in \mathbf{R}\}$, where

$$\begin{cases} \zeta_1 & = \exp\left(-2\pi(\alpha - bw_2 - cw_3)\right) \\ & = \exp\left(-2\pi\{\xi + \eta\sqrt{-1} - b(u + v\sqrt{-1}) - c(x + y\sqrt{-1})\}\right), \\ \zeta_2 & = \exp\left(-2\pi(u + v\sqrt{-1})\right), \\ \zeta_3 & = \exp\left(-2\pi(x + y\sqrt{-1})\right). \end{cases}$$

By setting

$$\rho_{u,\,x} = \exp\left(-2\pi(\xi - bu - cx)\right), \quad e_v = \exp\left(-2\pi(\eta - bv)\sqrt{-1}\,\right),$$

the following hold:

$$\begin{cases} \zeta_1 &= \rho_{u,\,x} \cdot e_v \exp(2\pi cy\sqrt{-1}\,), \\ \zeta_2 &= \exp(-2\pi u) \exp(-2\pi v\sqrt{-1}\,), \\ \zeta_3 &= \exp(-2\pi x) \exp(-2\pi y\sqrt{-1}\,). \end{cases} \tag{6.1}$$

Fix real parameters u, v, x of Π_α, and consider the real line $L_{u,\,v,\,x}$ which is a subset of a complex hyperplane Π_α and the two-dimensional torus $T_{u,v,x} = \{(\tau_1, \tau_2, \tau_3) \,|\, (e_1, e_2) \in T = U(1)^2\}$, where

$$\begin{cases} \tau_1 &= \rho_{u,v} \cdot e_1, \\ \tau_2 &= \exp(-2\pi u) \exp(-2\pi v\sqrt{-1}), \\ \tau_3 &= \exp(-2\pi x) \cdot e_2. \end{cases} \tag{6.2}$$

Then we see

$$\Pi_\alpha = \bigsqcup_{u,\,v,\,x \in \mathbf{R}} L_{u,v,x}, \quad \Sigma_\xi = \bigsqcup_{u,\,v,\,x \in \mathbf{R}} T_{u,v,x}. \tag{6.3}$$

It follows from (6.1) and (6.2) that $\varpi(L_{u,v,x}) \subset T_{u,v,x}$. Hence, by (6.3), the proof will be complete once we show

$$\varpi(L_{u,v,x})^- = T_{u,v,x}.$$

Define two maps $\iota : \mathbf{R} \to T$ and $\varphi : T \to T_{u,v,w}$ by

$$\iota(y) = \left(\exp(2\pi cy\sqrt{-1}),\, \exp(-2\pi y\sqrt{-1})\right),$$
$$\varphi(e_1, e_2) = (\tau_1 \cdot e_v,\, \tau_2,\, \tau_3).$$

Then we have $\varpi(L_{u,v,x}) = \varphi \circ \iota(\mathbf{R})$. Since φ is a homeomorphism of T onto $T_{u,v,w}$, the following holds:

$$\varpi(L_{u,v,x})^- = \varphi(\iota(\mathbf{R})^-).$$

On the other hand, with respect to the periods of ι,

$$2\pi c / -2\pi = -c \notin \mathbf{Q}.$$

Consequently, $\iota(\mathbf{R})^- = T$, that is, $\varpi(L_{u,v,x})^- = T_{u,v,x}$. $\square$

Now, we look for the Liouville foliation $\mathcal{F}$ of $D_{1,b,c}$ with $c \notin \mathbf{Q}$. For each $\rho \in I := \{t \in \mathbf{R} \,|\, 0 \leq t < 1\}$, we define Σ_ρ by

$$\Sigma_\rho := \{(z_1, z_2, z_3) \in \mathbf{C}^3 \,|\, |z_1||z_2|^b|z_3|^c = \rho\}.$$

Since a family $\mathcal{F} = \{\Sigma_\rho \,|\, \rho \in I\}$ satisfies conditions (L1) and (L2), we shall see $\mathcal{F}$ also satisfies condition (L3).

(1) If $\rho > 0$, then Σ_ρ is non-singular of dimension 5, and $\Sigma_\rho \subset D^*_{1,b,c} \subset D_{1,b,c}$. The logarithmic image of $D^*_{1,b,c}$ is

$$\Omega_{1,b,c} = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_1 + bx_2 + cx_3 > 0\},$$

and the universal covering of $D^*_{1,b,c}$ is $T_{\Omega_{1,b,c}}$.

It suffices that for every bounded psh function u on $D_{1,b,,c}$ and any two points p, q of Σ_ρ we show $u(p) = u(q)$. Choosing a point $\tilde{p}\,(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ of a fiber $\varpi^{-1}(p)$, set

$$\alpha := \tilde{w}_1 + b\tilde{w}_2 + c\tilde{w}_3,$$

and consider the complex hyperplane Π_α given by

$$\Pi_\alpha := \{(w_1, w_2, w_3) \in \mathbf{C}^3 \,|\, w_1 + bw_2 + cw_3 = \alpha\},$$

which is a subset of the covering space $T_{\Omega_{1,b,c}}$. Then, by Lemma 6.1, $\varpi(\Pi_\alpha)$ is a dense subset of Σ_ρ containing the point p.

On the other hand, $u \circ \varpi \,|\, \Pi_\alpha$ may be considered a bounded psh function on $\mathbf{C}^2$. Now, we need the following theorem (cf. [7], [4]):

Liouville's Theorem *Every plurisubharmonic function on $\mathbf{C}^n$ which is bounded above must be constant.*

From this, $u \circ \varpi \,|\, \Pi_\alpha$ takes a constant value, that is, u is constant on $\varpi(\Pi_\alpha)$. Hence, since a psh function u is upper semicontinuous,

$$u(p) = \limsup_{\varpi(\Pi_\alpha) \ni p_1 \to q} u(p_1) \leq u(q).$$

In the same way, we have $u(p) \geq u(q)$, and, consequently, $u(p) = u(q)$.

(2) If $\rho = 0$, then the following holds:

$$\Sigma_0 = \begin{cases} \{(z_1, z_2, z_3) \,|\, z_1 z_2 z_3 = 0\,\}, & \text{when } b \neq 0; \\ \{(z_1, z_2, z_3) \,|\, z_1 z_2 = 0\,\}, & \text{when } b = 0. \end{cases}$$

In either case, Σ_0 is quasi-singular of dimension 4, and any bounded psh function on $D_{1,b,c}$ takes a constant value on Σ_0.

Finally, any pair of elements of $\mathcal{F}$ are separated by the bounded psh function $\varphi(z_1, z_2, z_3) := |z_1||z_2|^b|z_3|^c$ on $D_{1,b,c}$. Therefore, $D_{1,b,c}$ has a Liouville foliation $\mathcal{F}$.

Case 2. Suppose $b \notin \mathbf{Q}, c \notin \mathbf{Q}$ and $b/c \in \mathbf{Q}$. This case is the same as case 1 except for the possibility that $b = 0$ because Lemma 6.1 also holds when $b \notin \mathbf{Q}$.

6.3 Case of $\delta(\mathbf{D}) = 3$

From Section 6.2,

$$b \notin \mathbf{Q},\ c \notin \mathbf{Q},\ \text{and}\ b/c \notin \mathbf{Q}.$$

Note that the condition $b/c \notin \mathbf{Q}$ has no influence on Lemma 6.1. Therefore, we have the same results in this case as seen in case 2 for $\delta(D) = 2$.

7 Domains of Section 5 with $\ell(\mathbf{D}) = 1$

We first recall a fact mentioned following Theorem 1.1: Let D and D' be Reinhardt domains. Since D contains the origin, every algebraic isomorphism $\varphi : D \to D'$ must be of the form (1.2). Therefore, its lifting $\Phi : T_\Omega \to T_{\Omega'}$ is the composition of a permutation of the coordinates and a translation of $\mathbf{C}^n$, where Ω and Ω' are logarithmic images of D^* and $(D')^*$, respectively.

7.1 Fundamental Reinhardt domains

For a Reinhardt domain D, we may assume that the group of translations $V(D)$ of the maximal affine subspace $A(D)$ of $\log(D^*)$ is generated by

$$\vec{l} = (-l_1, -l_2, 1)$$

with a suitable permutation of the coordinates, if necessary (cf. Section 5.1). Then, there exists a convex domain Ξ in the x_1x_2-plane containing no complete straight lines such that

$$\log D^* = \Xi \oplus \{\vec{l}\}_{\mathbf{R}}. \tag{7.1}$$

We will show in Section 7.3 that, if $(l_1, l_2) = (0, 0)$, then the case of either $B_1 \times B_1 \times \mathbf{C}$ or $B_2 \times \mathbf{C}$ occurs in Corollary 5.2. Hence, we exclude the case described in Section 7.3 and suppose

$$(l_1, l_2) \neq (0, 0). \tag{7.2}$$

Note that since D is complete, we have

$$l_1 \geq 0 \quad \text{or} \quad l_2 \geq 0. \tag{7.3}$$

If Ξ is a sector in the x_1x_2-plane, then a Reinhardt domain D is called fundamental. The vertex of a sector Ξ may be made the origin by a suitable translation of the x_1x_2-plane, if necessary. Since D is complete and Ξ contains no complete straight lines, Ξ may be spanned by two vectors

$$\vec{m} := (1, -m, 0) \quad \text{and} \quad \vec{n} := (-n, 1, 0),$$

where $m \geq 0$, $n \geq 0$, and ordered pairs $(\vec{m}, \vec{n})$ constitute a a right-handed system. From this, it follows that

$$\det \begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix} = 1 - mn > 0,$$

and we have

$$mn < 1. \tag{7.4}$$

By (7.1) and the two outer products

$$\vec{l} \times \vec{m} = (m, 1, l_1 m + l_2), \quad \vec{n} \times \vec{l} = (1, n, l_1 + l_2 n),$$

$\log D^*$ is represented by

$$\{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, mx_1 + x_2 + (l_1 m + l_2)x_3 > 0,\ x_1 + nx_2 + (l_1 + l_2 n)x_3 > 0\,\}.$$

For simplicity, instead of the above, we only write

$$\log D^* : mx_1 + x_2 + (l_1 m + l_2)x_3 > 0,\ x_1 + nx_2 + (l_1 + l_2 n)x_3 > 0.$$

Then, the image of $\log D^*$ by the canonical projection $\varpi : \mathbf{C}^3 \to (\mathbf{C}^*)^3$ is given by

$$D^* : 0 < |z_1|^m |z_2|\, |z_3|^{l_1 m + l_2} < 1,\ 0 < |z_1|\, |z_2|^n |z_3|^{l_1 + l_2 n} < 1.$$

Consequently, since D is complete, we have

$$D: |z_1|^m |z_2| \, |z_3|^{l_1 m + l_2} < 1, \ |z_1| \, |z_2|^n |z_3|^{l_1 + l_2 n} < 1,$$

where

$$l_1 m + l_2 \geq 0, \ l_1 + l_2 n \geq 0, \ (l_1 m + l_2, l_1 + l_2 n) \neq (0, 0). \tag{7.5}$$

To see this, suppose to the contrary that $l_1 m + l_2 = l_1 + l_2 n = 0$. Then (7.2) implies $mn - 1 = 0$, which contradicts (7.4). Note that by (7.5), for the projection $\vec{l'} = (-l_1, -l_2, 0)$ of $\vec{l}$, the two ordered pairs $(\vec{l'}, \vec{m})$ and $(\vec{n}, \vec{l'})$ form generalized right-handed systems. Conditions (7.2), (7.3), and (7.5) imply

$$l_1 > 0 \quad \text{or} \quad l_2 > 0.$$

Summarizing the preceding arguments, we have the following lemma:

Lemma 7.1. *If D is a fundamental Reinhardt domain, then there exist three vectors $\vec{l} = (-l_1, -l_2, 1), \vec{m} = (1, -m, 0)$, and $\vec{n} = (-n, 1, 0)$ with the following properties:*
(1) *$l_1 > 0$ or $l_2 > 0$;*
(2) *$m \geq 0$, $n \geq 0$, $mn < 1$;*
(3) *$l_1 m + l_2 \geq 0, \ l_1 + l_2 n \geq 0, \ (l_1 m + l_2, l_1 + l_2 n) \neq (0, 0)$.*
In addition, the following hold:
(4) *D is represented by*

$$|z_1|^m |z_2| \, |z_3|^{l_1 m + l_2} < 1, \ |z_1| \, |z_2|^n |z_3|^{l_1 + l_2 n} < 1; \tag{7.6}$$

(5) *$V(D)$ is generated by $\vec{l}$ on $\mathbf{R}$;*
(6) *the sector component Ξ of $\log(D^*)$ is spanned by $\vec{l}$ and $\vec{m}$.*

Conversely, for any real constants l_1, l_2, m, n with (1), (2), and (3), a Reinhardt domain D given by (7.6) is fundamental and satisfies (5) and (6).

The fundamental Reinhardt domain D mentioned above is denoted by $D_{m,n}(\vec{l})$, or by only $D_{m,n}$.

7.2 Classification of fundamental Reinhardt domains

In order to investigate a Liouville foliation on a fundamental Reinhardt domain D, we classify all of the fundamental Reinhardt domains given by (7.6).

For any non-negative integer k, the signature of k is defined by

$$\mathrm{sign}(k) := +, \text{ if } k > 0; \quad \mathrm{sign}(k) := 0, \text{ if } k = 0.$$

The signature of D, or of defining expression (7.6), is defined as

$$\mathrm{sign}(D) := (\mathrm{sign}(m), \mathrm{sign}(l_1 m + l_2); \mathrm{sign}(n), \mathrm{sign}(l_1 + l_2 n)).$$

We will construct a sub-Liouville foliation on $D^0 := D \setminus D^*$, which is denoted by $\mathcal{F}^0(D)$ (simply, $\mathcal{F}^0$), so that $\mathcal{F}^0$ depends on the signature $\mathrm{sign}(D)$. If two signatures coincide by a suitable permutation of the coordinates, then we choose either of them. However, note that we cannot treat l_1, l_2 and m, n equally, because whether l_1 and l_2 are rational plays a specific role in the forms of a sub-Liouville foliation $\mathcal{F}^0$ and a sub-Liouville foliation on D^*, which is denoted by $\mathcal{F}^*(D)$ (simply, $\mathcal{F}^*$). In addition, note that replacing z_1 with z_2 implies replacing l_1 with l_2.

From this, except for the case of $\delta(D) = 2$, we may classify fundamental Reinhardt domains as follows.

Type	Zero-signature	Defining expression (7.6)
O	none	$\lvert z_1\rvert^m \lvert z_2\rvert \, \lvert z_3\rvert^{l_1 m + l_2} < 1, \ \lvert z_1\rvert \, \lvert z_2\rvert^n \lvert z_3\rvert^{l_1 + l_2 n} < 1$
$\mathrm{I_a}$	$m = 0$	$\lvert z_2\rvert \, \lvert z_3\rvert^{l_2} < 1, \ \lvert z_1\rvert \, \lvert z_2\rvert^n \lvert z_3\rvert^{l_1 + l_2 n} < 1$
$\mathrm{I_b}$	$l_1 m + l_2 = 0$	$\lvert z_1\rvert^{-l_2/l_1} \lvert z_2\rvert < 1, \ \lvert z_1\rvert \, \lvert z_2\rvert^n \lvert z_3\rvert^{l_1 + l_2 n} < 1$
$\mathrm{II_a}$	$m = n = 0$	$\lvert z_2\rvert \, \lvert z_3\rvert^{l_2} < 1, \ \lvert z_1\rvert \, \lvert z_3\rvert^{l_1} < 1$
$\mathrm{II_b}$	$m = l_1 m + l_2 = 0$	$\lvert z_2\rvert < 1, \ \lvert z_1\rvert \, \lvert z_2\rvert^n \lvert z_3\rvert^{l_1} < 1$
$\mathrm{II_c}$	$l_1 m + l_2 = n = 0$	$\lvert z_1\rvert^{-l_2/l_1} \lvert z_2\rvert < 1, \ \lvert z_1\rvert \, \lvert z_3\rvert^{l_1} < 1$
III	$m = l_1 m + l_2 = n = 0$	$\lvert z_2\rvert < 1, \ \lvert z_1\rvert \, \lvert z_3\rvert^{l_1} < 1$

A sub-Liouville foliation $\mathcal{F}^*$ will be constructed so that the logarithmic image of each element of $\mathcal{F}^*$ is a subset of a line parallel to $\vec{l}$. Therefore, m and n have no influence on $\mathcal{F}^*$ except for the case of $\delta(D) = 2$, and so by increasing them a little if necessary, we may assume that they are rational numbers. Note that when we consider $\mathcal{F}^0$, we must increasing them to maintain its type.

If a Reinhardt domain D is not necessary fundamental, then set

$$(7.7) \qquad m := -\liminf_{(x_1,x_2)\in\Xi,\, x_1\to\infty} \frac{x_2}{x_1}, \quad n := -\liminf_{(x_1,x_2)\in\Xi,\, x_2\to\infty} \frac{x_1}{x_2},$$

where Ξ is the logarithmic image of D^*. We may assume that Ξ is contained in a sector with a vertex which is the origin by a suitable translation. Hence, m and n are finite and independent of the choice of translations. Then the sub-Liouville foliations $\mathcal{F}^*$ and $\mathcal{F}^0$ on D are induced by $\mathcal{F}^*$ and $\mathcal{F}^0$ on $D_{m,n}$ as follows:
(1) $\mathcal{F}^*(D) = \mathcal{F}^*(D_{m,n})|D^*$,
(2) $\mathcal{F}^0(D) = \mathcal{F}^0(D_{m,n})$.
(1) is trivial. Since $D^0 = D^0_{m,n}$ implies (2), for (2) to hold, it suffices that

$$S_{m,n} := \{(x_1, x_2) \,|\, x_2 > -mx_1, x_1 > -nx_2\}$$

be a minimal sector containing Ξ. For the family $\{S_i\}_{i\in I}$ of the sectors containing Ξ, set

$$S = (\bigcap_{i\in I} S_i)^o \quad \text{(open kernel)}.$$

Then it is easy to see that $S_{m,n} = S$, and S is minimal.

Therefore, we may investigate $D_{m,n}$ instead of D from the viewpoint of the structures of the Liouville foliation defined on D.

7.3 Case of $(l_1, l_2) = (0, 0)$

In this subsection, we assume that a Reinhardt domain D is homogeneous.

Since we have

$$\log D^* = \Xi \times \{\vec{l}\}_{\mathbf{R}}, \quad \vec{l} = (0, 0, 1),$$

there exists a complete Reinhardt domain E in $\mathbf{C}^2$ containing the origin such that

$$D = E \times \mathbf{C}, \quad \log(E^*) = \Xi.$$

Since Ξ contains no complete straight lines, by the fact mentioned in Section 2.3, for some $\tilde{\varphi} \in GL(2, \mathbf{Z}) \ltimes \mathbf{R}^2$, the following holds:

$$\tilde{\varphi}(\Xi) \subset \{(x_1, x_2) |\, x_1 > 0, x_2 > 0\}.$$

Hence, there exists a $\varphi \in \mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^2)$, and $\varphi(E^*)$ is a bounded domain in $(\mathbf{C}^*)^2$. Thus, by Riemann's removable singularities theorem, φ has a holomorphic extension to E. However, since E contains the origin, $\varphi : (z_1, z_2) \to (w_1, w_2)$ is of the form

$$w_1 = \alpha_1 z_{\sigma(1)}, \quad w_2 = \alpha_2 z_{\sigma(2)},$$

where $\alpha_1, \alpha_2 \in \mathbf{C}^*$ and σ is a permutation of $\{1, 2\}$. Therefore, E is also algebraically equivalent to a bounded domain in $\mathbf{C}^2$. From this and Lemma 2.7, since D is homogeneous, so is E. Hence, by Theorem 1.4, E is algebraically equivalent to either

$$B_1 \times B_1 \quad \text{or} \quad B_2.$$

Consequently, D is algebraically equivalent to either

$$B_1 \times B_1 \times \mathbf{C} \quad \text{or} \quad B_2 \times \mathbf{C}.$$

7.4 Case of $\delta(D) = 1$

First, we construct a sub-Liouville foliation $\mathcal{F}^*$ on D^*, which is independent of the type of D. In this case, all exponents of (7.6) are rational numbers, so we may assume

$$D : |z_1|^{a_1}|z_2|^{a_2}\,|z_3|^{a_3} < 1,\ |z_1|^{b_1}|z_2|^{b_2}|z_3|^{b_3} < 1,$$

where $a_i, b_i, i = 1, 2, 3$ are non-negative integers and (a_1, a_2, a_3), (b_1, b_2, b_3) are primitive elements of $\mathbf{Z}^3$. Then, we have

$$D^* : 0 < |z_1|^{a_1}|z_2|^{a_2}\,|z_3|^{a_3} < 1,\ 0 < |z_1|^{b_1}|z_2|^{b_2}|z_3|^{b_3} < 1.$$

For each pair (ζ_1, ζ_2) with $\zeta_1, \zeta_2 \in B_1^*$, we define a subset Σ_{ζ_1,ζ_2} of D^* as

$$\Sigma_{\zeta_1,\zeta_2} : z_1^{a_1} z_2^{a_2} z_3^{a_3} = \zeta_1,\ z_1^{b_1} z_2^{b_2} z_3^{b_3} = \zeta_2.$$

Since (a_1, a_2, a_3) is primitive, there exists an algebraic automorphism $\varphi : (z_i) \to (w_i)$ of $(\mathbf{C}^*)^3$ of the form

$$w_1 = z_1^{a_1} z_2^{a_2} z_3^{a_3},\ w_2 = z_1^{c_1} z_2^{c_2} z_3^{c_3},\ w_3 = z_1^{d_1} z_2^{d_2} z_3^{d_3},$$

and the image of $A : z_1^{a_1} z_2^{a_2} z_3^{a_3} = \zeta_1$ with φ is given by

$$w_1 = \zeta_1. \tag{7.8}$$

Therefore, if $\varphi^{-1} \in \mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^3)$ has the expression

$$z_1 = w_1^{p_1} w_2^{p_2} w_3^{p_3},\ z_2 = w_1^{q_1} w_2^{q_2} w_3^{q_3},\ z_3 = w_1^{r_1} w_2^{r_2} w_3^{r_3},$$

then

$$\varphi^{-1}(\zeta_1, \cdot, \cdot) : (\mathbf{C}^*)^2 \ni (w_2, w_3) \longmapsto (\zeta_1^{p_1} w_2^{p_2} w_3^{p_3}, \zeta_1^{q_1} w_2^{q_2} w_3^{q_3}, \zeta_1^{r_1} w_2^{r_2} w_3^{r_3}) \in A$$

gives a holomorphic isomorphism of $(\mathbf{C}^*)^2$ onto A.

By substituting the expression of $\varphi^{-1}(\zeta_1, \cdot, \cdot)$ for $B : z_1^{b_1} z_2^{b_2} z_3^{b_3} = \zeta_2$ and setting

$$n_1 = b_1 p_1 + b_2 q_1 + b_3 r_1,\ n_2 = b_1 p_2 + b_2 q_2 + b_3 r_2,\ n_3 = b_1 p_3 + b_2 q_3 + b_3 r_3,$$

we have

$$w_2^{n_2} w_3^{n_3} = \zeta_1^{-n_1} \zeta_2.$$

Since $(n_2, n_3) \neq (0, 0)$, the greatest common divisor $g(> 0)$ of the pair (n_2, n_3) such that

$$n_2 = m_2 g,\ n_3 = m_3 g$$

exists, and (m_2, m_3) is a primitive element of $\mathbf{Z}^2$. Then we have

$$(w_2^{m_2} w_3^{m_3})^g = \zeta_1^{-n_1} \zeta_2. \tag{7.9}$$

Therefore, by setting

$$(\, \zeta_1^{-n_1} \zeta_2 \,)^{1/g} = \eta_i, \quad i = 0, 1, \cdots, g-1, \tag{7.10}$$

we define subsets of $\Sigma_{\zeta_1, \zeta_2}$ as follows:

$$\Sigma^i_{\zeta_1, \zeta_2} : w_1 = \zeta_1,\ w_2^{m_2} w_3^{m_3} = \eta_i\,; \quad i = 0, 1, \cdots, g-1. \tag{7.11}$$

From (7.8)–(7.11), we have

$$\Sigma_{\zeta_1, \zeta_2} = \sqcup_{i=o, \cdots, g-1} \Sigma^i_{\zeta_1, \zeta_2}.$$

Now, since (m_2, m_3) is primitive, there exists an algebraic automorphism $\psi : (w_i) \to (\xi_i)$ of $(\mathbf{C}^*)^3$ of the form

$$\xi_1 = w_1,\ \xi_2 = w_2^{m_2} w_3^{m_3},\ \xi_3 = w_2^{k_2} w_3^{k_3}.$$

The image of $\Sigma^i_{\zeta_1, \zeta_2}$ with $\psi \circ \phi \in \mathrm{Aut}_{\mathrm{alg}}((\mathbf{C}^*)^3)$ is given by

$$\xi_1 = \zeta_1,\ \xi_2 = \eta_i.$$

Hence, $\Sigma^i_{\zeta_1, \zeta_2}$ is algebraically equivalent to $\mathbf{C}^*$, where $\zeta_1, \zeta_2 \in B_1^*$ and $i = 0, 1, \cdots, g-1$. Consequently, if we set

$$\mathcal{F}^* := \{\, \Sigma^i_{\zeta_1, \zeta_2} |\, \zeta_1, \zeta_2 \in B_1^*,\ i = 0, 1, \cdots, g-1\},$$

then $\mathcal{F}^*$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation on D^*.

We shall see that $\mathcal{F}^*$ satisfies condition (L4). For some positive integer ν with $\nu g - n_1 > 0$, we define Laurent monomials f_1 and f_2 as follows:

$$\begin{aligned} f_1(z_1, z_2, z_3) &= w_1 \\ &= z_1^{a_1} z_2^{a_2} z_3^{a_3}, \\ f_2(z_1, z_2, z_3) &= w_1^{\nu} \cdot w_2^{m_2} w_3^{m_3} \\ &= (z_1^{a_1} z_2^{a_2} z_3^{a_3})^{\nu} \cdot (z_1^{c_1} z_2^{c_2} z_3^{c_3})^{m_2} (z_1^{d_1} z_2^{d_2} z_3^{d_3})^{m_3}. \end{aligned}$$

Then, it follows from (7.11) that

$$f_1(z_1, z_2, z_3) = \zeta_1, \quad f_2(z_1, z_2, z_3) = (\zeta_1)^{\nu} \eta_i,$$

and so

$$|f_2(z_1, z_2, z_3)|^g = |\zeta_1|^{\nu g} |\zeta_1^{-n_1} \zeta_2| = |\zeta_1|^{\nu g - n_1} |\zeta_2|$$

on every $\Sigma^i_{\zeta_1, \zeta_2}$. Since $\nu g - n_1 > 0$, both f_1 and f_2 are bounded on D^*, and they have separate holomorphic extensions to D by means of Riemann's removable singularities theorem. Obviously, any pair of the elements of $\mathcal{F}^*$ are separated by a Liouville map (f_1, f_2).

If D has a Liouville foliation $\mathcal{F}$, then

$$\dim \mathcal{F} = 2,$$

because $\Sigma^i_{\zeta_1, \zeta_2}$ is algebraically equivalent to $\mathbf{C}^*$.

Next, we look for a sub-Liouville foliation $\mathcal{F}^0$ on $D^0 = D \setminus D^*$ type by type, and construct a Liouville foliation $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ on D.

Case 1. Suppose that D is O type. we may assume

$$D : |z_1|^{a_1} |z_2|^{a_2} |z_3|^{a_3} < 1, \ |z_1|^{b_1} |z_2|^{b_2} |z_3|^{b_3} < 1,$$

where all of the exponents are positive integers. As $D^0 = D \setminus D^*$ is the union of three coordinate hyperplanes, set

$$\Gamma^3 : z_1 z_2 z_3 = 0.$$

Then the collection consisting of a single element

$$\mathcal{F}^0 = \{\Gamma^3\}$$

is a sub-Liouville foliation on D^0. In addition, a bounded holomorphic function

$$z_1^{a_1} z_2^{a_2} z_3^{a_3}$$

on D separates every element $\Sigma^i_{\zeta_1,\zeta_2}$ of $\mathcal{F}^*$ and Γ^3. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 2. Suppose that D is $\mathrm{I_a}$ or $\mathrm{I_b}$ type. In the case of $\mathrm{I_b}$ type, permute z_1 and z_3. Then, both of them may be expressed by

$$D : |z_2|^{a_2}|z_3|^{a_3} < 1,\ |z_1|^{b_1}|z_2|^{b_2}|z_3|^{b_3} < 1,$$

where all the exponents are positive integers.

If $(z_1, z_2, z_3) \in D^0$ and $z_2 z_3 \neq 0$, then

$$z_1 = 0,\ 0 < |z_2|^{a_2}|z_3|^{a_3} < 1.$$

Therefore, set

$$\begin{aligned} &\Gamma^1_\zeta : z_1 = 0,\ z_2^{a_2} z_3^{a_3} = \zeta; \\ &\Gamma^2 : z_2 z_3 = 0, \end{aligned}$$

where $\zeta \in B_1^*$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \Gamma^2 |\, \zeta \in B_1^*\}.$$

In the same way as in Section 6.1, case 1, (1), we can show that Γ^1_ζ is algebraically equivalent to $\mathbf{C}^*$. Hence, we see that $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. Additionally, a pair

$$\left(z_2^{a_2} z_3^{a_3},\ z_1^{b_1} z_2^{b_2} z_3^{b_3}\right)$$

of bounded holomorphic functions on D separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 3. Suppose that D is $\mathrm{II_a}$ or $\mathrm{II_c}$ type. In the case of $\mathrm{II_c}$ type, permute z_1 and z_3. Then, both of them may be expressed by

$$D : |z_2|^{a_2}|z_3|^{a_3} < 1,\ |z_1|^{b_1}|z_3|^{b_3} < 1,$$

where all of the exponents are positive integers.

Suppose $(z_1, z_2, z_3) \in D^0$, $z_3 \neq 0$ and $(z_1, z_2) \neq (0,0)$. Then, depending on whether $z_1 = 0$ or $z_2 = 0$, we have either

$$0 < |z_2|^{a_2}|z_3|^{a_3} < 1 \quad \text{or} \quad 0 < |z_1|^{b_1}|z_3|^{b_3} < 1.$$

Therefore, set

$$\begin{aligned}
&\Gamma^1_\zeta : z_1 = 0,\ z_2^{a_2} z_3^{a_3} = \zeta;\\
&\overline{\Gamma^1_\zeta} : z_2 = 0,\ z_1^{b_1} z_3^{b_3} = \zeta;\\
&\Gamma^e : z_3 = 0 \quad \text{or} \quad (z_1, z_2) = (0,0),
\end{aligned}$$

where $\zeta \in B_1^*$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \overline{\Gamma^1_\zeta},\ \Gamma^e \,|\, \zeta \in B_1^*\}.$$

As in Section 6.1, case 1, (1), we see that $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. In addition, a pair

$$(z_2^{a_2} z_3^{a_3},\ z_1^{b_1} z_3^{b_3})$$

of bounded holomorphic functions on D takes a constant value $(\zeta, 0)$ on Γ^1_ζ, a constant value $(0, \zeta)$ on $\overline{\Gamma^1_\zeta}$, and a constant value $(0,0)$ on Γ^e. Hence, this Liouville map separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 4. Suppose that D is $\mathrm{II_b}$ type. We may assume

$$D : |z_2| < 1,\ |z_1|^{b_1}|z_2|^{b_2}|z_3|^{b_3} < 1,$$

where all of the exponents are positive integers.

If $(z_1, z_2, z_3) \in D^0$ and $z_2 \neq 0$, then

$$z_1 z_3 = 0,\ 0 < |z_2| < 1.$$

Therefore, set

$$\begin{aligned}
&\Gamma^1 : z_2 = 0;\\
&\Gamma^2_\zeta : z_1 z_3 = 0,\ z_2 = \zeta,
\end{aligned}$$

where $\zeta \in B_1^*$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\Gamma^1,\ \Gamma^2_\zeta \,|\, \zeta \in B_1^*\}.$$

Obviously, $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. In addition, a pair

$$(z_2,\ z_1^{b_1} z_3^{b_3})$$

of bounded holomorphic functions on D separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 5. Suppose that D is III type. We may assume

$$D : |z_2| < 1,\ |z_1|^{b_1} |z_3|^{b_3} < 1,$$

where all of the exponents are positive integers. Suppose $(z_1, z_2, z_3) \in D^0$. If $z_1 z_3 \neq 0$, then

$$z_2 = 0,\ 0 < |z_1|^{b_1} |z_3|^{b_3} < 1.$$

If $z_1 z_3 = 0$, then we have $|z_2| < 1$. Therefore, set

$$\begin{aligned} \overline{\Gamma^1_\zeta} &: z_2 = 0,\ z_1^{b_1} z_3^{b_3} = \zeta; \\ \Gamma^2_\eta &: z_1 z_3 = 0,\ z_2 = \eta, \end{aligned}$$

where $\zeta \in B_1^*$ and $\eta \in B_1$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\, \overline{\Gamma^1_\zeta},\ \Gamma^2_\eta \,|\, \zeta \in B_1^*, \eta \in B_1 \}.$$

As in Section 6.1, case 1, we see that $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. In addition, a pair

$$(z_2,\ z_1^{b_1} z_3^{b_3})$$

of bounded holomorphic functions on D separates any pair of the elements of$\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

The next theorem follows from the above.

Theorem 7.1. *Let D be a Reinhardt domain in $\mathbf{C}^3$ with $(\ell(D), \delta(D)) = (1, 1)$ and $\vec{l}$ be parallel to no coordinate axes. Then, D can be identified with fundamental Reinhardt domain*

$$D_{m,n}(\vec{l}) : |z_1|^{a_1} |z_2|^{a_2} |z_3|^{a_3} < 1,\ |z_1|^{b_1} |z_2|^{b_2} |z_3|^{b_3} < 1$$

from the viewpoint of the structures of the Liouville foliations defined on them, where all of the exponents are non-negative integers.

The sub-Liouville foliation $\mathcal{F}^*$ *on* D^* *is given by*

$$\mathcal{F}^* := \{\, \Sigma^i_{\zeta_1,\zeta_2} \,|\, \zeta_1, \zeta_2 \in B_1^*,\ i = 0, 1, \cdots, g-1\},$$

and the sub-Liouville foliation $\mathcal{F}^0$ *on* D^0 *is given by the following, according to its type:*

If D *is* O *type, then* $\mathcal{F}^0 = \{\Gamma^3\}$;

If D *is* $\mathrm{I_a}$ *or* $\mathrm{I_b}$ *type, then* $\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \Gamma^2 \,|\, \zeta \in B_1^*\}$;

If D *is* $\mathrm{II_a}$ *or* $\mathrm{II_c}$ *type, then* $\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \overline{\Gamma^1_\zeta},\ \Gamma^e \,|\, \zeta \in B_1^*\}$;

If D *is* $\mathrm{II_b}$ *type, then* $\mathcal{F}^0 = \{\Gamma^1,\ \Gamma^2_\zeta \,|\, \zeta \in B_1^*\}$;

If D *is* III *type, then* $\mathcal{F}^0 = \{\, \overline{\Gamma^1_\zeta},\ \Gamma^2_\eta \,|\, \zeta \in B_1^*, \eta \in B_1\}$.

Regardless of type, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ *is a Liouville foliation on* D*. Therefore, we have*

$$\dim \mathcal{F} = 2.$$

Moreover, the dimension and the singularity of an element of $\mathcal{F}^0$ *are classified as follows*:

	4-dimensional	*2-dimensional*
non-singular	$\Gamma^1 : z_2 = 0$	$\Gamma^1_\zeta : z_1 = 0,\ z_2^{a_2} z_3^{a_3} = \zeta$
quasi-singular	$\Gamma^2 : z_2 z_3 = 0$ $\Gamma^3 : z_1 z_2 z_3 = 0$ $\Gamma^e : z_2 = 0$ *or* $z_1 = z_3 = 0$	$\Gamma^2_\zeta : z_1 z_3 = 0,\ z_2 = \zeta$

7.5 Case of $\delta(D) = 3$

We may assume

$$D : \ |z_1|^m |z_2|\, |z_3|^\kappa < 1,\ |z_1|\, |z_2|^n |z_3|^\lambda < 1,$$

where $\kappa = l_1 m + l_2$, $\lambda = l_1 + l_2 n$, $mn < 1$ and $\dim\{-l_1, -l_2, 1\}_{\mathbf{Q}} = 3$.

First, we construct a sub-Liouville foliation $\mathcal{F}^*$ on D^*, which is independent of the type of D. Set $I = [\,0, 1)$. For every pair (ρ_1, ρ_2) with $\rho_i \in I^* = (0, 1), i = 1, 2$, we define a subset Σ_{ρ_1,ρ_2} of D^* by

$$\Sigma_{\rho_1,\rho_2} : |z_1|^m |z_2|\, |z_3|^\kappa = \rho_1,\ |z_1|\, |z_2|^n |z_3|^\lambda = \rho_2, \tag{7.12}$$

and set

$$\mathcal{F}^* = \{\, \Sigma_{\rho_1,\rho_2} \,|\, \rho_1, \rho_2 \in I^* \}.$$

Then, $\mathcal{F}^*$ satisfies conditions (L1) and (L2) of a sub-Liouville foliation on D^*. We shall see that $\mathcal{F}^*$ satisfies condition (L3). By setting $|z_3| = e^t$ and solving (7.12) with respect to $|z_1|$ and $|z_2|$, we have

$$|z_1| = (\rho_1^{-n}\rho_2)^{\frac{1}{1-mn}} e^{-l_1 t}, \quad |z_2| = (\rho_1\rho_2^{-m})^{\frac{1}{1-mn}} e^{-l_2 t}.$$

Then, by setting

$$R_1 := (\rho_1^{-n}\rho_2)^{\frac{1}{1-mn}}, \quad R_2 := (\rho_1\rho_2^{-m})^{\frac{1}{1-mn}},$$

the following hold:

$$|z_1| = R_1 e^{-l_1 t}, \quad |z_2| = R_2 e^{-l_2 t}, \quad |z_3| = e^t. \tag{7.13}$$

For every fixed t, (7.13) represents a three-dimensional torus $T^3(t)$ and Σ_{ρ_1,ρ_2} is the union of all such tori:

$$\Sigma_{\rho_1,\rho_2} = \bigsqcup_{t\in\mathbf{R}} T^3(t).$$

Note that since a continuous surjection

$$\begin{aligned}\mathbf{R}^4 \ni (t, \theta_1, \theta_2, \theta_3)& \\ &\longmapsto (R_1 e^{-l_1 t + \sqrt{-1}\theta_1},\, R_2 e^{-l_2 t+\sqrt{-1}\theta_2},\, e^{-t+\sqrt{-1}\theta_3}) \in \Sigma_{\rho_1,\rho_2}\end{aligned}$$

is a local diffeomorphism, Σ_{ρ_1,ρ_2} is a four-dimensional real manifold.

Now, we consider the complex affine line L in $\mathbf{C}^3$ given by

$$L : {}^t(w_1, w_2, w_3) = {}^t(-l_1, -l_2, 1)w + {}^t(\alpha_1, \alpha_2, \alpha_3),$$

where $\alpha_i = a_i + \sqrt{-1}\, b_i$, $i = 1, 2, 3$ are complex numbers such that

$$\begin{cases} a_1 &= -1/2\pi \log R_1 - l_1 t_0; \\ a_2 &= -1/2\pi \log R_2 - l_2 t_0; \\ a_3 &= t_0, \end{cases} \tag{7.14}$$

and t_0 is an arbitrary real constant.

Set $w = x + \sqrt{-1}y$. Looking for the image of L under the canonical projection $\varpi : \mathbf{C}^3 \to (\mathbf{C}^*)^3$, we have

$$\begin{cases} e^{-2\pi w_1} &= R_1\, e^{2\pi l_1(x+t_0)} \cdot e^{-2\pi(-l_1 y + b_1)\sqrt{-1}}, \\ e^{-2\pi w_2} &= R_2\, e^{2\pi l_2(x+t_0)} \cdot e^{-2\pi(-l_2 y + b_2)\sqrt{-1}}, \\ e^{-2\pi w_3} &= e^{-2\pi(x+t_0)} \cdot e^{-2\pi(y+b_3)\sqrt{-1}}. \end{cases}$$

Hence, for every fixed t, if we take x as $x_t := -(t_0 + t/2\pi)$, then by (7.13)

$$\varpi(L|x_t) \subset T^3(t).$$

Now, we need the following theorem (cf. [3]):

Kronecker's Theorem *Let $\theta_1, \theta_2, \theta_3$ be linearly independent real numbers on $\mathbf{Z}$ and $\alpha_1, \alpha_2, \alpha_3$ be arbitrary real numbers. Then, for every pair (T, ϵ) of positive numbers, there exist a real number t and integers n_1, n_2, n_3 such that*

$$t > T \quad \textit{and} \quad |t\theta_i - n_i - \alpha_i| < \epsilon,\ i = 1, 2, 3.$$

From this and the assumption $\dim\{-l_1, -l_2, 1\}_{\mathbf{Q}} = 3$, by moving the variable y on $\mathbf{R}$, we see $\varpi(L|x_t)^- = T^3(t)$, and, consequently,

$$\varpi(L)^- = \Sigma_{\rho_1,\rho_2}.$$

Suppose $p(z_1^0, z_2^0, z_3^0)$ and q to be two arbitrary points of Σ_{ρ_1,ρ_2}. Then, there exists a real number T such that

$$|z_1^0| = R_1 e^{-l_1 T},\ |z_2^0| = R_2 e^{-l_2 T},\ |z_3^0| = e^T. \tag{7.15}$$

Hence, there exists a complex affine line $L \subset T_\Omega$ with $p \in \varpi(L)$, where T_Ω is the covering tube domain of D. Indeed, choose a point $\tilde{p} = (w_1^0, w_2^0, w_3^0)$ of the fiber $\varpi^{-1}(p)$, whose coordinates satisfy

$$\mathrm{Re}\,(w_i^0) = -1/2\pi \log |z_i^0|, \quad i = 1, 2, 3. \tag{7.16}$$

If we take $b_i = \mathrm{Im}\,(\alpha_i)$ as $\mathrm{Im}\,(w_i^0)$, then by (7.15) and (7.16), the equation $(\alpha_1, \alpha_2, \alpha_3) = (w_1^0, w_2^0, w_3^0)$ is equivalent to the following:

$$\begin{cases} a_1 &= -1/2\pi \log R_1 + l_1 T/2\pi, \\ a_2 &= -1/2\pi \log R_2 + l_2 T/2\pi, \\ a_3 &= -T/2\pi. \end{cases} \tag{7.17}$$

Therefore, if we set $t_0 = -T/2\pi$ in (7.14), then (7.17) holds.

Consequently, in the same way as in Section 6.2, case 1, (1), we can show that $u(p) = u(q)$ for every bounded psh function u on D. Thus, $\mathcal{F}^*$ satisfies condition (L3), and a Liouville map

$$(|z_1|^m |z_2|\, |z_3|^\kappa,\ |z_1||z_2|^n |z_3|^\lambda)$$

on D separates any pair of the elements of $\mathcal{F}^*$, and therefore $\mathcal{F}^*$ is a sub-Liouville foliation on D^*.

Next, we look for a sub-Liouville foliation $\mathcal{F}^0$ on $D^0 = D \setminus D^*$ type by type, and construct a Liouville foliation $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ on D.

Case 1. Suppose that D is O type. We may assume

$$D:\ |z_1|^m |z_2|\, |z_3|^\kappa < 1,\ |z_1|\, |z_2|^n |z_3|^\lambda < 1,$$

where $\kappa = l_1 m + l_2$, $\lambda = l_1 + l_2 n$. This case is the same as case 1 in Section 7.4, that is, a sub-Liouville foliation on D^0 is given by

$$\mathcal{F}^0 = \{\Gamma^3\}, \quad \Gamma^3 : z_1 z_2 z_3 = 0,$$

and a bounded psh function

$$|z_1|^m |z_2|\, |z_3|^\kappa$$

on D separates every element of $\mathcal{F}^*$ and Γ^3. Then, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 2. Suppose that D is $\mathrm{I_a}$ or $\mathrm{I_b}$ type. Since $-l_2/l_1 \notin \mathbf{Q}$, being $\mathrm{I_b}$ type is the same as being $\mathrm{I_a}$ type. Hence, we may assume

$$D : |z_2|\, |z_3|^{l_2} < 1,\ |z_1|\, |z_2|^n |z_3|^\lambda < 1.$$

If $(z_1, z_2, z_3) \in D^0$ and $z_2 z_3 \neq 0$, then

$$z_1 = 0,\ 0 < |z_2|\, |z_3|^{l_2} < 1.$$

Therefore, set

$$\Gamma^1_\rho : z_1 = 0,\ |z_2|\, |z_3|^{l_2} = \rho;$$
$$\Gamma^2 : z_2 z_3 = 0,$$

where $\rho \in I^*$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\Gamma^1_\rho,\ \Gamma^2 \,|\, \rho \in I^*\}.$$

In the same way as in Section 6.2, case 1, (1), we can show that every Γ^1_ρ satisfies condition (L3). Thus, a Liouville map

$$(\,|z_2|\,|z_3|^{l_2},\ |z_1|\,|z_2|^n|z_3|^\lambda\,)$$

on D separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 3. Suppose that D is $\mathrm{II_a}$ or $\mathrm{II_c}$ type. Since $-l_2/l_1 \notin \mathbf{Q}$, being $\mathrm{II_c}$ type is the same as being $\mathrm{II_a}$ type. Hence, we may assume

$$D : |z_2|\,|z_3|^{l_2} < 1,\ |z_1|\,|z_3|^{l_1} < 1.$$

Suppose $(z_1, z_2, z_3) \in D^0$, $z_3 \neq 0$ and $(z_1, z_2) \neq (0, 0)$. Then, depending on whether $z_1 = 0$ or $z_2 = 0$, we have either

$$0 < |z_2|\,|z_3|^{l_2} < 1 \quad \text{or} \quad 0 < |z_1|\,|z_3|^{l_3} < 1.$$

Therefore, set

$$\begin{aligned}
&\Gamma^1_\rho : z_1 = 0,\ |z_2|\,|z_3|^{l_2} = \rho;\\
&\overline{\Gamma^1_\rho} : z_2 = 0,\ |z_1|\,|z_3|^{l_1} = \rho;\\
&\Gamma^e : z_3 = 0 \quad \text{or} \quad (z_1, z_2) = (0, 0),
\end{aligned}$$

where $\rho \in I^*$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\Gamma^1_\rho,\ \overline{\Gamma^1_\rho},\ \Gamma^e \,|\, \rho \in I^*\}.$$

As in case 2, we see that $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. In addition, a Liouville map

$$(|z_2|\,|z_3|^{l_2},\ |z_1|\,|z_3|^{l_3})$$

on D takes a constant value $(\rho, 0)$ on Γ^1_ρ, a constant value $(0, \rho)$ on $\overline{\Gamma^1_\rho}$, and a constant value $(0, 0)$ on Γ^e. Hence, this Liouville map separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 4. Suppose that D is $\mathrm{II_b}$ type. We may assume

$$D : |z_2| < 1,\ |z_1||z_2|^n|z_3|^{l_1} < 1.$$

If $(z_1, z_2, z_3) \in D^0$ and $z_2 \neq 0$, then

$$z_1z_3 = 0,\ 0 < |z_2| < 1.$$

Therefore, set

$$\begin{aligned} &\Gamma^1 : z_2 = 0; \\ &\Gamma^2_\zeta : z_1z_3 = 0,\ z_2 = \zeta, \end{aligned}$$

where $\zeta \in B_1^*$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\Gamma^1,\ \Gamma^2_\zeta \,|\, \zeta \in B_1^*\}.$$

Obviously, $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. In addition, a Liouville map

$$(|z_2|,\ |z_1||z_2|^n|z_3|^{l_1})$$

on D separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Case 5. Suppose that D is III type. We may assume

$$D : |z_2| < 1,\ |z_1||z_3|^{l_1} < 1.$$

Suppose $(z_1, z_2, z_3) \in D^0$. If $z_1z_3 \neq 0$, then

$$z_2 = 0,\ 0 < |z_1||z_3|^{l_1} < 1.$$

If $z_1z_3 = 0$, then we have $|z_2| < 1$. Therefore, set

$$\begin{aligned} &\overline{\Gamma^1_\rho} : z_2 = 0,\ |z_1||z_3|^{l_1} = \rho; \\ &\Gamma^2_\eta : z_1z_3 = 0,\ z_2 = \eta \end{aligned}$$

where $\rho \in I^*$ and $\eta \in B_1$, and define $\mathcal{F}^0$ by

$$\mathcal{F}^0 = \{\,\overline{\Gamma^1_\rho},\ \Gamma^2_\eta \,|\, \rho \in I^*, \eta \in B_1\}.$$

As in case 2, we see that $\mathcal{F}^0$ satisfies conditions (L1), (L2), and (L3) of a sub-Liouville foliation. In addition, a Liouville map

$$(|z_2|,\ |z_1||z_3|^{l_1})$$

on D separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

Theorem 7.2. *Let D be a Reinhart domain in $\mathbf{C}^3$ with $(\ell(D), \delta(D)) = (1, 3)$. Then, D can be identified with fundamental Reinhardt domain*

$$D_{m,n}(\vec{l}) :\ |z_1|^m|z_2|\,|z_3|^\kappa < 1,\ |z_1|\,|z_2|^n|z_3|^\lambda < 1$$

from the viewpoint of the structures of the Liouville foliations defined on them, where $\kappa = l_1m + l_2$, $\lambda = l_1 + l_2n$.

The sub-Liouville foliation $\mathcal{F}^*$ on D^* is given by

$$\mathcal{F}^* := \{\, \Sigma_{\rho_1,\rho_2} \,|\, \rho_1, \rho_2 \in I^*\},$$

and the sub-Liouville foliation $\mathcal{F}^0$ on D^0 is given by the following, according to its type:

If D is O *type, then $\mathcal{F}^0 = \{\Gamma^3\}$;*

If D is $\mathrm{I_a}$ *or* $\mathrm{I_b}$ *type, then $\mathcal{F}^0 = \{\Gamma^1_\rho,\ \Gamma^2 \,|\, \rho \in I^*\}$;*

If D is $\mathrm{II_a}$ *or* $\mathrm{II_c}$ *type, then $\mathcal{F}^0 = \{\Gamma^1_\rho,\ \overline{\Gamma^1_\rho},\ \Gamma^e \,|\, \rho \in I^*\}$;*

If D is $\mathrm{II_b}$ *type, then $\mathcal{F}^0 = \{\Gamma^1,\ \Gamma^2_\zeta \,|\, \zeta \in B_1^*\}$;*

If D is III *type, then $\mathcal{F}^0 = \{\, \overline{\Gamma^1_\rho},\ \Gamma^2_\eta \,|\, \rho \in I^*, \eta \in B_1\}$.*

Regardless of type, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D. Therefore, we have

$$\dim \mathcal{F} = 4.$$

Moreover the dimension and the singularity of an element of $\mathcal{F}^0$ are classified as follows:

	4-dimensional	*3-dimensional*	*2-dimensional*
non-singular	Γ^1	$\Gamma^1_\rho : z_2 = 0,\ \lvert z_1\rvert\,\lvert z_3\rvert^{l_1} = \rho$	
quasi-singular	Γ^2 Γ^3 Γ^e		Γ^2_ζ

7.6 Case of $\delta(D) = 2$

We may assume

$$D: |z_1|^m |z_2|\,|z_3|^\kappa < 1,\ |z_1|\,|z_2|^n |z_3|^\lambda < 1,$$

where $\kappa = l_1 m + l_2$, $\lambda = l_1 + l_2 n$, $mn < 1$.

Since $\dim\{-l_1, -l_2, 1\}_{\mathbf{Q}} = 2$, at least one of l_1 or l_2 is an irrational number. Hence, by noting that replacing z_1 with z_2 implies replacing l_1 with l_2, we may assume

$$l_1 \notin \mathbf{Q},$$

and there exist rational numbers q and r such that

$$l_2 = l_1 q + r. \tag{7.18}$$

As in (7.12), we set

$$\Sigma_{\rho_1,\rho_2}: |z_1|^m |z_2|\,|z_3|^\kappa = \rho_1,\ |z_1|\,|z_2|^n |z_3|^\lambda = \rho_2, \tag{7.19}$$

where ρ_1 and ρ_2 are elements of I^*. From (7.18) and the expression (7.13) of (7.19), we have

$$|z_1|^{-q} |z_2||z_3|^r = (\rho_1^{nq+1} \rho_2^{-q-m})^{\frac{1}{1-mn}}. \tag{7.20}$$

Therefore, if $nq + 1 \neq 0$, then

$$\Sigma_{\rho_1,\rho_2}: |z_1|^{-q}|z_2||z_3|^r = (\rho_1^{nq+1}\rho_2^{-q-m})^{\frac{1}{1-mn}},\ |z_1|\,|z_2|^n|z_3|^\lambda = \rho_2; \tag{7.21}$$

and if $q + m \neq 0$, then

$$\Sigma_{\rho_1,\rho_2}: |z_1|^{-q}|z_2||z_3|^r = (\rho_1^{nq+1}\rho_2^{-q-m})^{\frac{1}{1-mn}},\ |z_1|^m|z_2|\,|z_3|^\kappa = \rho_1. \tag{7.22}$$

First, we construct a sub-Liouville foliation $\mathcal{F}^*$. We start by clarifying the relationship between the case of $nq + 1 \neq 0$ and the case of $q + m \neq 0$: If $nq + 1 \neq 0$, then by increasing n a little to preserve the sign of $nq + 1$ if necessary, we may assume that n is a rational number. Hence,

$$\lambda = l_1 + l_2 n = (nq + 1)l_1 + nr \notin \mathbf{Q}.$$

If $nq + 1 = 0$, then by $q = -1/n$,

$$q + m = -1/n + m \neq 0.$$

Increasing m a little to preserve the sign of $q+m$ if necessary, we may assume that m is a rational number. Hence,

$$\kappa = l_1 m + l_2 = (q+m)l_2 + r \notin \mathbf{Q}.$$

Consequently, each case is the same as the other. Therefore, without loss of generality, we may assume

$$nq + 1 \neq 0 \quad \text{and} \quad \lambda \notin \mathbf{Q}, \tag{7.23}$$

and it suffices that we consider only the case of (7.21).

Next, we look for a condition on the numbers q, r and l_1 such that a function $z_1^{-q} z_2 z_3^r$ induces a bounded holomorphic function on D.

Proposition 7.3. *In order for numbers* m, n $(m \geq 0, n \geq 0, mn < 1)$ *to exist with the properties*
(i) $\kappa = l_1 m + l_2 = l_1 m + l_1 q + r \geq 0,\ \lambda = l_1 + l_2 n = l_1 + l_1 nq + rn \geq 0,$
(ii) $nq + 1 \geq 0,\ -q - m \geq 0,$
the necessary and sufficient condition is

$$(\, q \leq 0,\ r > 0\,) \quad \textit{or} \quad (\, q \leq 0,\ r = 0,\ l_1 > 0\,). \tag{7.24}$$

Proof. Suppose $l_1 > 0$. If $n > 0$, then (i) implies

$$q \geq -m - r/l_1, \quad q \geq -1/n - r/l_1.$$

Since $-1/n < -m$, we have

$$q \geq -m - r/l_1.$$

This is also true when $n = 0$. On the other hand, (ii) implies

$$-1/n < q \leq -m,$$

but if $n = 0$, then this means $q \leq -m$. Hence, we have

$$-1/n < q \leq -m \leq q + r/l_1.$$

For there to exist m, n such that the previous inequality holds, it is necessary and sufficient that

$$q \leq 0,\ r/l_1 \geq 0, \quad \text{hence} \quad q \leq 0, \quad r \geq 0.$$

Suppose $l_1 < 0$. Then, $n > 0$ follows from $\lambda \geq 0$. Hence, (i) implies

$$q \leq -m - r/l_1,\ q \leq -1/n - r/l_1, \quad \text{hence} \quad q \leq -1/n - r/l_1.$$

On the other hand, (ii) implies $-1/n < q \leq -m$. Thus, we have

$$q + r/l_1 \leq -1/n < q \leq -m.$$

For there to exist m, n such that the previous inequality holds, it is necessary and sufficient that

$$q \leq 0,\ r/l_1 < 0, \quad \text{hence} \quad q \leq 0, \quad r > 0.$$

This completes the proof. □

Consider again (7.21). Since q and r are rational numbers, for a suitable positive integer a, raising (7.20) to the ath power gives

$$|z_1|^{a_1}|z_2|^{a_2}|z_3|^{a_3} = \rho,$$

where $\rho = (\rho_1^{nq+1}\rho_2^{-q-m})^{\frac{a}{1-mn}}$ and (a_1, a_2, a_3) is a primitive element of $\mathbf{Z}^3$. Note that since $a_1/a_2 = -q$, from (7.23), $n(-a_1/a_2) + 1 \neq 0$ or

$$a_2 - na_1 \neq 0. \tag{7.25}$$

Hence, we have

$$\Sigma_{\rho_1,\rho_2} : |z_1|^{a_1}|z_2|^{a_2}|z_3|^{a_3} = \rho,\ |z_1|\,|z_2|^n|z_3|^\lambda = \rho_2;$$

and so, for every $\theta \in I$, setting

$$\Sigma_{\rho_1,\rho_2,\theta} : z_1^{a_1}z_2^{a_2}z_3^{a_3} = \rho\, e^{2\pi\theta\sqrt{-1}},\ |z_1|\,|z_2|^n|z_3|^\lambda = \rho_2$$

gives a partition of Σ_{ρ_1,ρ_2}.

Next, we show that any bounded psh function on D takes a constant value on each $\Sigma_{\rho_1,\rho_2,\theta}$. Since (a_1, a_2, a_3) is primitive, there exists an element

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

of $GL(3, \mathbf{Z})$. Therefore,

$$\det A = a_1(b_2c_3 - b_3c_1) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \pm 1. \tag{7.26}$$

Matrix A induces the algebraic automorphism φ of $(\mathbf{C}^*)^3$ given by

$$w_1 = z_1^{a_1} z_2^{a_2} z_3^{a_3},\ w_2 = z_1^{b_1} z_2^{b_2} z_3^{b_3},\ w_3 = z_1^{c_1} z_2^{c_2} z_3^{c_3}.$$

Then, the automorphism φ transforms $z_1^{a_1} z_2^{a_2} z_3^{a_3} = \rho\, e^{2\pi\theta\sqrt{-1}}$ into

$$w_1 = \rho\, e^{2\pi\theta\sqrt{-1}}.$$

Setting

$$A^{-1} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix},$$

where

$$\begin{aligned} p_1 &= b_2c_3 - b_3c_2, & p_2 &= a_3c_2 - a_2c_3, & p_3 &= a_2b_3 - a_3b_2, \\ q_1 &= b_3c_1 - b_1c_3, & q_2 &= a_1c_3 - a_3c_1, & q_3 &= a_3b_1 - a_1b_3, \\ r_1 &= b_1c_2 - b_2c_1, & r_2 &= a_2c_1 - a_1c_2, & r_3 &= a_1b_2 - a_2b_1, \end{aligned}$$

φ^{-1} is given by

$$z_1 = w_1^{p_1} w_2^{p_2} w_3^{p_3},\ z_2 = w_1^{q_1} w_2^{q_2} w_3^{q_3},\ z_3 = w_1^{r_1} w_2^{r_2} w_3^{r_3}.$$

Substituting this into $|z_1|\,|z_2|^n|z_3|^\lambda = \rho_2$ implies

$$\rho^{m_1}|w_2|^{m_2}|w_3|^{m_3} = \rho_2,$$

where

$$m_1 = p_1 + nq_1 + \lambda r_1, m_2 = p_2 + nq_2 + \lambda r_2, m_3 = p_3 + nq_3 + \lambda r_3.$$

Then, we have

$$m_3/m_2 =: c \notin \mathbf{Q}. \tag{7.27}$$

Indeed, if we consider

$$\begin{aligned} \det \begin{pmatrix} p_2 + nq_2 & r_2 \\ p_3 + nq_3 & r_3 \end{pmatrix} &= (p_2 + nq_2)r_3 - (p_3 + nq_3)r_2 \\ &= p_2r_3 - p_3r_2 + n(q_2r_3 - q_3r_2), \end{aligned}$$

we see that the following hold:

$$
\begin{aligned}
p_2r_3 - p_3r_2 &= (a_3c_2 - a_2c_3)(a_1b_2 - a_2b_1) - (a_2b_3 - a_3b_2)(a_2c_1 - a_1c_2) \\
&= a_2\left\{a_1(b_3c_2 - b_2c_3) + a_2(b_2c_3 - b_3c_1) + a_3(b_2c_1 - b_1c_2)\right\}, \\
q_2r_3 - q_3r_2 &= (a_1c_3 - a_3c_1)(a_1b_2 - a_2b_1) - (a_3b_1 - a_1b_3)(a_2c_1 - a_1c_2) \\
&= -a_1\left\{a_1(b_3c_2 - b_2c_3) + a_2(b_2c_3 - b_3c_1) + a_3(b_2c_1 - b_1c_2)\right\}.
\end{aligned}
$$

By (7.25) and (7.26),

$$
\det\begin{pmatrix} p_2 + nq_2 & r_2 \\ p_3 + nq_3 & r_3 \end{pmatrix} = \pm(a_2 - na_1) \neq 0.
$$

Hence, it follows from (7.23) that m_3/m_2 is an irrational number.

Consequently, we have

$$
\varphi(\,\Sigma_{\rho_1,\,\rho_2,\,\theta}\,) : w_1 = \rho\, e^{2\pi\theta\sqrt{-1}},\ |w_2||w_3|^c = (\rho^{-m_1}\rho_2)^{\frac{1}{m_2}},
$$

and the set $\Sigma_{\rho_1,\,\rho_2,\,\theta}$ is a three-dimensional real manifold. By (7.27), as in Section 6.2, case 1, we see that any bounded psh function on D takes a constant value on each $\varphi(\,\Sigma_{\rho_1,\,\rho_2,\,\theta}\,)$, or on each $\Sigma_{\rho_1,\,\rho_2,\,\theta}$.

Next, looking for a sub-Liouville foliation on D^*, set

$$
\mathcal{F}^* = \{\Sigma_{\rho_1,\,\rho_2,\,\theta} \,|\, \rho_1, \rho_2 \in I^*, \theta \in I\}.
$$

Then, the collection $\mathcal{F}^*$ satisfies conditions (L1), (L2), and (L3), and so we shall see that $\mathcal{F}^*$ also satisfies (L4) in the following case (1).

(1) Suppose that l_1 and l_2, with $l_2 = l_1q + r$, satisfy (7.24). Further suppose that the numbers m and n defined by (7.7) satisfy conditions (i) and (ii) of Proposition 7.3. It follows that $nq + 1 > 0, -q - m \geq 0$. This implies

$$
|z_1^{a_1} z_2^{a_2} z_3^{a_3}| = (\,\rho_1^{nq+1}\rho_2^{-q-m})^{\frac{a}{1-mn}} < 1 \quad \text{on every} \quad \Sigma_{\rho_1,\rho_2}.
$$

Since $D^* = \bigsqcup_{\rho_1,\rho_2 \in I^*} \Sigma_{\rho_1,\rho_2}$, by Riemann's removable singularities theorem, we have

$$
|z_1^{a_1} z_2^{a_2} z_3^{a_3}| < 1 \quad \text{on } D.
$$

Therefore, a pair

$$
(\, z_1^{a_1} z_2^{a_2} z_3^{a_3},\ |z_1|\,|z_2|^n |z_3|^\lambda\,)
$$

is a Liouville map on D, and separates every pair of the elements of $\mathcal{F}^*$. Hence, $\mathcal{F}^*$ is a sub-Liouville foliation on D^*.

(2) If the previous case (1) does not hold, the function $z_1^{a_1} z_2^{a_2} z_3^{a_3}$ is not necessarily bounded on D. Therefore, we introduce an equivalence relation on Σ_{ρ_1,ρ_2} so that the two subsets $\Sigma_{\rho_1,\rho_2,\theta_1}$ and $\Sigma_{\rho_1,\rho_2,\theta_2}$ of Σ_{ρ_1,ρ_2} are equivalent if and only if any bounded psh function on D takes the same constant value on both. We denote by

$$\Sigma_\alpha,\ \alpha \in A_{\rho_1,\rho_2}$$

the union of all $\Sigma_{\rho_1,\rho_2,\theta}$ of an equivalence class, where A_{ρ_1,ρ_2} is the index set given by this equivalence relation. Any such set Σ_α is a real manifold, but its dimension is unknown. By noting that every pair $(\,\Sigma_{\rho_1,\rho_2,\theta},\ \Sigma_{\rho'_1,\rho'_2,\theta'}\,)$ with $(\rho_1,\rho_2) \neq (\rho'_1,\rho'_2)$ are separated by a Liouville map

$$(\,|z_1|^m |z_2|\,|z_3|^\kappa,\ |z_1|\,|z_2|^n |z_3|^\lambda\,),$$

we see that a collection

$$\mathcal{F}^* = \{\Sigma_\alpha \,|\, \alpha \in A_{\rho_1,\rho_2},\ \rho_1,\rho_2 \in I^*\}$$

is a sub-Liouville foliation on D^*.

Next, we look for a sub-Liouville foliation $\mathcal{F}^0$ on $D^0 = D \setminus D^*$ type by type, and construct a Liouville foliation $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ on D. Here, we must also consider the case of $l_2 \in \mathbf{Q}$, but by the following argument, we will see that the classification table given in Section 7.2 is still effective.

First, note that, depending on whether D is O, $\mathrm{II_b}$, or III type, D has the same sub-Liouville foliation on $D^0 = D \setminus D^*$ as case 1, case 4, or case 5 of Section 7.5, respectively.

Case 1. Suppose that D is $\mathrm{I_a}$ type. We may assume

$$D : |z_2|\,|z_3|^{l_2} < 1,\ |z_1|\,|z_2|^n |z_3|^\lambda < 1.$$

If l_2 is a rational number, then D is said to be $\mathrm{I_a^1}$ type. Otherwise, D is said to be $\mathrm{I_a^2}$ type.

(1) Suppose that D is $\mathrm{I_a^1}$ type. Then, D may be given by

$$D : |z_2|^{a_2}\,|z_3|^{a_3} < 1,\ |z_1|\,|z_2|^n |z_3|^\lambda < 1,$$

where a_2, a_3 are positive integers. As in Section 7.4, case 2, if we set

$$\Gamma^1_\zeta : z_1 = 0,\ z_2^{a_2} z_3^{a_3} = \zeta;$$
$$\Gamma^2 : z_2 z_3 = 0,$$

where $\zeta \in B_1^*$, and set

$$\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \Gamma^2 | \, \zeta \in B_1^*\},$$

then $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D because the Liouville map

$$(\, z_2^{a_2} z_3^{a_3},\ |z_1||z_2|^n|z_3|^\lambda)$$

on D separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$.
(2) Suppose that D is I^2_a type. As in Section 7.5, case 2, if we set

$$\begin{aligned}&\Gamma^1_\rho : z_1 = 0,\ |z_2|\,|z_3|^{l_2} = \rho;\\ &\Gamma^2 : z_2 z_3 = 0,\end{aligned}$$

where $\rho \in I^*$, and set

$$\mathcal{F}^0 = \{\Gamma^1_\rho,\ \Gamma^2 \,|\, \rho \in I^*\},$$

then $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.

The $\mathrm{I_b}$-type case is the same as the $\mathrm{I_a}$-type case.

Case 2. Suppose that D is $\mathrm{II_a}$ type. We may assume

$$D : |z_2|\,|z_3|^{l_2} < 1,\ |z_1||z_3|^{l_1} < 1.$$

By the assumption mentioned at the beginning of Section 7.6, exponent l_1 is an irrational number. If l_2 is rational, then D is said to be II^1_a type. Otherwise, D is said to be II^2_a type.
(1) Suppose that D is II^1_a type. Then, D may be given by

$$D : |z_2|^{a_2}\,|z_3|^{a_3} < 1,\ |z_1|\,|z_3|^{l_1} < 1,$$

where a_2, a_3 are positive integers. As in Section 7.4, case 3 and Section 7.5, case 3, if we set

$$\begin{aligned}&\Gamma^1_\zeta : z_1 = 0,\ z_2^{a_2} z_3^{a_3} = \zeta;\\ &\overline{\Gamma^1_\rho} : z_2 = 0,\ |z_1||z_3|^{l_1} = \rho;\\ &\Gamma^e : z_3 = 0 \quad \text{or} \quad (z_1, z_2) = (0, 0),\end{aligned}$$

where $\zeta \in B_1^*, \rho \in I^*$, and set

$$\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \overline{\Gamma^1_\rho},\ \Gamma^e \,|\, \zeta \in B_1^*, \rho \in I^*\},$$

then $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D because a Liouville map

$$(z_2^{a_2} z_3^{a_3},\ |z_1||z_3|^{l_1})$$

on D takes a constant value $(\zeta, 0)$ on Γ^1_ζ, a constant value $(0, \rho)$ on $\overline{\Gamma^1_\zeta}$, and a constant value $(0,0)$ on Γ^e, Hence, this Liouville map separates any pair of the elements of $\mathcal{F}^0$, and separates any pair (Σ, Γ) with $\Sigma \in \mathcal{F}^*$ and $\Gamma \in \mathcal{F}^0$. Therefore, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ is a Liouville foliation on D.
(2) Suppose that D is II^2_a type. Since the exponents l_1, l_2 are irrational numbers, this case is the same as the case in Secton 7.5.

The II_c-type case is the same as the II_a-type case.

Theorem 7.4. *Let D be a Reinhart domain in $\mathbf{C}^3$ with $(\ell(D), \delta(D)) = (1,2)$. Then, D can be identified with a fundamental Reinhardt domain*

$$D_{m,n}(\vec{l}) : |z_1|^m |z_2|\, |z_3|^\kappa < 1,\ |z_1|\, |z_2|^n |z_3|^\lambda < 1$$

from the viewpoint of the structures of the Liouville foliations, where $\kappa = l_1 m + l_2$, $\lambda = l_1 + l_2 n$. Since $\delta(D) = 2$, we may assume $l_1 \notin \mathbf{Q}$ and there exist rational numbers q, r such that $l_2 = l_1 q + r$.

If l_1, q, r satisfy (7.24) and the numbers m and n defined by (7.7) satisfy conditions (i) and (ii) of Proposition 7.3, then the sub-Liouville foliation on D^* is given by

$$\mathcal{F}^* = \{\, \Sigma_{\rho_1,\rho_2,\theta} \,|\, \rho_1, \rho_2 \in I^*, \theta \in I\}.$$

Otherwise, we have

$$\mathcal{F}^* = \{\Sigma_\alpha \,|\, \alpha \in A_{\rho_1,\rho_2},\ \rho_1, \rho_2 \in I^*\}.$$

On the other hand, the sub-Liouville foliation $\mathcal{F}^0$ on D^0 is given by the following according to its type:

If D is O *type, then $\mathcal{F}^0 = \{\Gamma^3\}$;*
If D is I^1_a type, then $\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \Gamma^2 |\, \zeta \in B_1^*\}$;
If D is I^2_a type, then $\mathcal{F}^0 = \{\Gamma^1_\rho,\ \Gamma^2 |\, \rho \in I_1^*\}$;
If D is I_b type, then this is the same as the I_a-type case;
If D is II^1_a type, then $\mathcal{F}^0 = \{\Gamma^1_\zeta,\ \overline{\Gamma^1_\rho},\ \Gamma^e \,|\, \zeta \in B_1^*, \rho \in I^*\}$;
If D is II^2_a type, then $\mathcal{F}^0 = \{\Gamma^1_\rho,\ \overline{\Gamma^1_\rho},\ \Gamma^e \,|\, \rho \in I^*\}$;
If D is II_b type, then $\mathcal{F}^0 = \{\Gamma^1,\ \Gamma^2_\zeta \,|\, \zeta \in B_1^*\}$;
If D is II_c type, then this is the same as the II_a-type case;
If D is III *type, then $\mathcal{F}^0 = \{\, \overline{\Gamma^1_\rho},\ \Gamma^2_\eta \,|\, \rho \in I^*, \eta \in B_1\}$.*

Regardless of type, $\mathcal{F} = \mathcal{F}^* \sqcup \mathcal{F}^0$ *is a Liouville foliation on* D*. In addition, if* l_1, q, r *satisfy (7.24) and the numbers* m *and* n *defined by (7.7) satisfy conditions (i) and (ii) of Proposition 7.3, then*

$$\dim \mathcal{F} = 3.$$

Otherwise, the dimension is unknown. Moreover, the dimension and the singularity of an element of $\mathcal{F}^0$ *are classified as follows:*

	4-dimensional	*3-dimensional*	*2-dimensional*
non-singular	Γ^1	Γ^1_ρ	Γ^1_ζ
quasi-singular	Γ^2 Γ^3 Γ^e		Γ^2_ζ

By the arguments of Section 7.3 and Theorem 7.1, 7.2, and 7.4, the proofs of Theorem 5.1 and Corollary 5.2 are complete.

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