Harmonic functions and fundamental solutions of non-local operators generating jump Markov processes

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Contents

Introduction

There is a one to one correspondence between Markov processes and Markov generators. We denote them $\{X_t\}_{t\geq 0}$ and $\mathscr{L}$. When a Markov process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is given, the behavior of its harmonic function $u(x)$ and the estimates of transition density function $p(t,x,y)$ play a crucial role for knowing the properties of the Markov process. Here, the harmonic function is probabilistically defined as a bounded function such that $\{u(X_t)\}_{t\geq 0}$ is a martingale and the transition density function is characterized by

$$\mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy.$$

The harmonic function $u(x)$ is also characterized as $\mathscr{L}u = 0$ and $p(t,x,y)$ is the fundamental solution of equation $\partial u/\partial t = \mathscr{L}u$. A most important Markov process is the standard Brownian motion, whose Markov generator is equal to $(1/2)\Delta$. In this case $p(t,x,y)$ is the fundamental solution of the partial differentiable equation $\partial u/\partial t = (1/2)\Delta u$. We often call $p(t,x,y)$ *heat kernel*.

There are a lot of preceding results on the continuity of harmonic functions and estimates of heat kernels. For example, when Markov generator $\mathscr{L}$ is a uniformly elliptic second order differential operator, the corresponding Markov process is a diffusion process and the continuity of harmonic functions is well known. Moreover, this heat kernel $p(t,x,y)$ admits the Gaussian estimate as follows:

$$C_1 t^{-\frac{d}{2}} \exp\Big(-\frac{C_2|x-y|^2}{t}\Big) \leq p(t,x,y) \leq C_3 t^{-\frac{d}{2}} \exp\Big(-\frac{C_4|x-y|^2}{t}\Big).$$

Here C_i's are positive constants. The studies on the continuity of harmonic functions and estimate of heat kernels for jump Markov processes have been developed for the last decade. Jump Markov processes have discontinuous sample paths. A most typical example is the rotationally invariant α-stable process generated by the fractional Laplacian $-(-\Delta)^{\alpha/2}$, where $0 < \alpha < 2$. Unlike diffusion processes, harmonic functions of jump Markov processes are not necessarily continuous (Barlow, Bass and et. al. [3]). Thus, for the continuity of harmonic functions we need to impose some conditions on the generator. This problem is considered by Bass and Levin [5], Bass and Kassmann [4] and Husseini and Kassmann [14]. Moreover, it is proved that heat kernels of jump Markov processes admit two-sided estimates different from those of diffusion processes. Chen and Kumagai [9] treated α-stable-like processes, which is a generalization of the rotationally invariant α-stable process. In Chen, Kim and Kumagai [7], they gave two-sided estimates for heat kernel of relativistic α-stable-like processes. These processes are symmetric with respect to the Lebesgue measure m. To analyze symmetric Markov processes, the Dirichlet form theory is a powerful tool. We introduce the Dirichlet form $(\mathscr{E},\mathscr{F})$ corresponding to the Markov process $\{X_t\}_{t\geq 0}$:

$$\mathscr{E}(u,v) = (-\mathscr{L}u,v),$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{R}^d)$ and $\mathscr{F}$ is the domain of the form $\mathscr{E}$, the closure of the domain of $\mathscr{L}$. We define a Schrödinger form $\mathscr{E}^\mu$ by

$$\mathscr{E}^\mu(u,v) = \mathscr{E}(u,v) - \int_{\mathbb{R}^d} u(x)v(x)\mu(dx). \tag{0.1}$$

Here μ is a suitable positive measure. We note that the second term of (0.1) contributes not killing but creation of the process. We consider the fundamental solution of $\partial u/\partial t = \mathscr{L}^\mu u$, where $\mathscr{L}^\mu$ is the Schrödinger type operator defined by

$$(-\mathscr{L}^\mu u,v) = \mathscr{E}^\mu(u,v).$$

We denote by $p^\mu(t,x,y)$ the fundamental solution. Since the perturbation in a Schrödinger form is defined by creation, $p^\mu(t,x,y)$ is no longer a transition density of a Markov process. However, we can define the corresponding semigroup by using the positive continuous additive functional A_t^μ. Here A_t^μ is determined uniquely from μ by the Revuz correspondence. Then $p^\mu(t,x,y)$ is the integral kernel of the Feynman-Kac semigroup:

$$\mathbb{E}_x[\exp(A_t^\mu)f(X_t)] = \int_{\mathbb{R}^d} p^\mu(t,x,y)f(y)dy. \tag{0.2}$$

We compare $p^\mu(t,x,y)$ with $p(t,x,y)$. If $p^\mu(t,x,y)$ has the same type two-sided estimate as $p(t,x,y)$ up to positive constants, we call this phenomenon *the stability of fundamental solutions*. Intuitively, if the potential μ is large, $p^\mu(t,x,y)$ has a different estimate from $p(t,x,y)$. Thus we need to formulate the smallness of the measure μ. To this end, we use the bottom of spectrum of the time changed process by A_t^μ. We recall a result for a transient Brownian motion. Assume μ is in a certain class. Takeda [32] showed that under a certain condition on μ, the stability of fundamental solution holds if and only if

$$\lambda(\mu) := \inf\left\{\frac{1}{2}\mathbb{D}(u,u)\ ;\ u\in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} > 1, \tag{0.3}$$

where $\mathbb{D}$ is the Dirichlet integral defined by

$$\mathbb{D}(u,v) = \int_{\mathbb{R}^d} \nabla u(x)\cdot\nabla v(x)dx$$

and $H^1(\mathbb{R}^d)$ is the 1-order Sobolev space. $\lambda(\mu)$ is regarded as the bottom of spectrum of the operator $(1/2\mu)\Delta$, the generator of the time changed process of the Brownian motion by A_t^μ. The formula (0.3) describes the smallness of the measure μ. Indeed, if

$\mu_1 \leq \mu_2$, then $\lambda(\mu_1) \geq \lambda(\mu_2)$. Moreover, the following each statement is equivalent to (0.3):

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x[\exp(A_\infty^\mu)] < \infty, \tag{0.4}$$

$$G^\mu(x,y) := \int_0^\infty p^\mu(t,x,y)dt < \infty \quad \text{for } x \neq y. \tag{0.5}$$

If the measure μ satisfies (0.4) and (0.5), then μ is said to be *gaugeable* and *subcritical* respectively. For the proof of the stability of fundamental solutions, the equivalence of (0.3)–(0.5) is crucial. In this thesis, the measure μ is said to be subcritical if $\lambda(\mu) > 1$. Furthermore,

(1) μ is said to be *critical* if $\lambda(\mu) = 1$,

(2) μ is said to be *supercritical* if $\lambda(\mu) < 1$.

In two cases above, we cannot expect that $p^\mu(t,x,y)$ have the same two-sided estimate as $p(t,x,y)$. We have not been able to give a two-sided estimate of $p^\mu(t,x,y)$. However, there exist some papers which deal with large time asymptotics of the Feynman-Kac semigroups, $\mathbb{E}_x[\exp(A_t^\mu)]$, if μ is critical or supercritical. We see from (0.2) that

$$\mathbb{E}_x[\exp(A_t^\mu)] = \int_{\mathbb{R}^d} p^\mu(t,x,y)dy. \tag{0.6}$$

Since the subcriticality is equivalent to the gaugeability, the Feynman-Kac semigroup diverges as $t \to \infty$. Takeda [28, 30] showed that

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x[\exp(A_t^\mu)] = C(\mu), \tag{0.7}$$

where

$$C(\mu) = -\inf\left\{ \mathscr{E}^\mu(u,u) \; ; \; \int_{\mathbb{R}^d} u^2 dx = 1 \right\},$$

that is, $-C(\mu)$ is the bottom of spectrum of the operator $-\mathscr{L}^\mu = -(\mathscr{L} + \mu)$. Hence we see that if μ is supercritical, then $C(\mu)$ is positive and the expectation (0.6) grows exponentially.

If μ is critical, $C(\mu) = 0$ and the expectation (0.6) seems to have polynomial growth. This conjecture is proved by Simon [25] and Cranston, Kolokoltsov and et. al. [8] in case that μ is absolutely continuous with respect to the Lebesgue measure. More precisely, when $\mu = V(x)dx$ for $V \in C_0^\infty(\mathbb{R}^d)$, the growth of the expectation (0.6) has the following asymptotics:

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim \begin{cases} C_1 t^{\frac{1}{2}} & (d = 3) \\ C_2 t / \log t & (d = 4) \\ C_3 t & (d \geq 5). \end{cases}$$

In this thesis we extend these results to jump Markov processes. The Dirichlet form associated with jump Markov process $\{X_t\}_{t\geq 0}$ is expressed by

$$\mathscr{E}(u,v) = \int_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))(v(y)-v(x))J(x,y)dxdy.$$

Here $J(x,y)$ is a positive symmetric Borel function called *jump intensity measure*. The terms 'α-stable-like process' and 'relativistic α-stable-like process' come from the behavior of jump intensity measure. In this thesis, we introduce three classes of Radon measures: the *Kato-class* $\mathbf{K}$, *Green-tight Kato class* $\mathbf{K}_\infty$ and *conditional Green-tight Kato class* $\mathbf{S}_\infty$ (For the definitions, see Section 2.2). In particular, the class $\mathbf{K}_\infty$ plays a crucial role. For $\mu\in\mathbf{K}_\infty$, we consider the Schrödinger form $\mathscr{E}^\mu$ defined in (0.1) and denote by $\{P_t^\mu\}$ a semigroup generated by $\mathscr{E}^\mu$. Using a harmonic function $h(x)$ of Schrödinger operator, the Schrödigner form can be transformed to a Dirichlet form. This method is called Doob's *h-transform* used intensively in Chen and Zhang [12]. We consider the transformed semigroup $\{P_t^{\mu,h}\}_{t\geq 0} : L^2(h^2m)\to L^2(h^2m)$,

$$P_t^{\mu,h} f(x) = \mathbb{E}_x\left[\frac{h(X_t)}{h(X_0)}\exp(A_t^\mu)f(X_t)\right].$$

The transformed semigroup generates Markov process on $\mathbb{R}^d$ with symmetric measure h^2m. Moreover, we see that the associated Dirichlet form on $L^2(h^2m)$ is expressed as

$$\mathscr{E}^{\mu,h}(u,v) = \int_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))(v(y)-v(x))J(x,y)h(x)h(y)dxdy, \tag{0.8}$$

namely, the jump intensity measure $J(x,y)$ is transformed to $J(x,y)h(x)h(y)$.

First we establish a necessary and sufficient condition on μ for the stability of fundamental solution when the Markov process $\{X_t\}_{t\geq 0}$ is α-stable-like or relativistic α-stable-like. If μ is subcritical, the gauge function $h(x)=\mathbb{E}_x[\exp(A_\infty^\mu)]$ satisfies $1\leq h(x)\leq C_1$ for some positive constant. Hence $J(x,y)h(x)h(y)$ is equivalent to $J(x,y)$ and consequently the transition density function of the transformed process is equivalent to the original one. Noting that $p^\mu(t,x,y)/h(x)h(y)$ is equal to the transition density function of the transformed process, we can conclude the stability of fundamental solutions. For example, let $\{X_t\}_{t\geq 0}$ be a transient α-stable-like process on $\mathbb{R}^d$. In this case, the jump intensity measure $J(x,y)$ satisfies

$$\frac{C_1}{|x-y|^{d+\alpha}} \leq J(x,y) \leq \frac{C_2}{|x-y|^{d+\alpha}}$$

for some positive constants C_1 and C_2. The transformed Dirichlet form (0.8) is also α-stable-like if and only if the measure μ satisfies

$$\inf\left\{\mathscr{E}(u,u)\;\middle|\; u\in\mathscr{F},\ \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} > 1. \tag{0.9}$$

From Chen and Kumagai [9], we know the two-sided estimate for the heat kernel of α-stable-like process:

$$C_1\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\leq p(t,x,y)\leq C_2\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).$$

Therefore, we can conclude that the stability of fundamental solution is equivalent to the condition (0.9), which is an extension of (0.3).

We next consider large time asymptotics of the expectation $\mathbb{E}_x[\exp(A_t^\mu)]$, when μ is critical or supercritical. Since (0.7) is valid for the rotationally invariant α-stable process ([33]), the expectation (0.6) grows exponentially if μ is supercritical.

If μ is critical, we have a concrete growth order of (0.6) for $d = 2$ and $\alpha = 1$, which is the same as that of 4-dimensional Brownian motion. Takeda [34] proved that for the rotationally invariant α-stable process with $d/\alpha > 2$, the growth of (0.6) is proportional to the time t. In this case the function $h(x)$ belongs to $L^2(\mathbb{R}^d)$ and, as a result, the transformed Markov process has the finite invariant measure h^2m. Applying the ergodic theory, we can obtain the growth order of (0.6). But when $d/\alpha \le 2$, we cannot use this argument because $h(x)$ is not in $L^2(\mathbb{R}^d)$. Hence, we apply an analytical methods due to Simon [19].

If μ is critical, the construction of $h(x)$ is based on Takeda and Tsuchida [35], in which they proved that $h(x)$ is continuous and satisfies $h(x) \asymp 1 \wedge |x|^{\alpha-d}$. Hence, in order to estimate the heat kernel of the h-transformed process, we need to treat the jump intensity like $h(x)h(y)/|x-y|^{d+\alpha}$, which depends not only $|x-y|$ but also $|x|$ and $|y|$. The estimate of heat kernels for these type jump processes is an interesting problem.

We finally consider harmonic functions with respect to the Markov generators. Bass and Levin [5] showed the Hölder continuity of harmonic functions for α-stable-like generators. Bass and Kassmann [4] showed the continuity of harmonic functions for more general jump-type Markov generators. They impose two conditions on the jump intensity measure $J(x,y)$: one is *singularity of small jumps*, namely how the amount of jumps with size r grows as r tends to 0. The other is *quasi rotationally invariance*, namely how the process jumps in any direction to some extent. In this paper we prove the continuity of harmonic functions under conditions weaker than those in Bass and Kassmann [4]. For instance, we consider the case $J(x,y)$ satisfies

$$\frac{C_1}{|x-y|^{d+1}} \log \frac{3}{|x-y|} \le J(x,y) \le \frac{C_2}{|x-y|^{d+1}} \log \frac{3}{|x-y|} \qquad (0 < |x-y| \le 2)$$

for some positive constants C_1 and C_2. This jump intensity measure is a bit different from that of α-stable-like processes. For detail, see Example 4.1.

This thesis is organized as follows: In Chapter 1, we prepare the basic material: Dirichlet forms, Hunt processes, smooth measures and additive functionals. In Chapter 2, we establish a necessary and sufficient condition on μ for the stability of fundamental solutions when the process is α-stable-like or relativistic α-stable-like. In Chapter 3, for the rotationally invariant 1-stable process on $\mathbb{R}^2$, we consider the growth order of Feynman-Kac expectations and differentiability of spectral functions associated with critical Schrödinger forms . In Chapter 4, we study the continuity of harmonic functions for a class of jump-type Markov generators containing α-stable-like ones. Our result is an extension of preceding ones in Bass and Kassmann [4], and Husseini and Kassmann [14].

Chapter 1

Preliminaries

In this chapter we introduce basic material that will be used through this paper. In Section 1, we review basic properties of Dirichlet forms and Markov processes. In Section 2, we mainly treat the smooth measures and additive functionals.

1.1 Dirichlet forms and symmetric Hunt processes

Let $(\mathscr{E},\mathscr{F})$ be a Dirichlet form on $L^2(\mathbb{R}^d)$. Here $\mathscr{E}$ is a non-negative definite symmetric bilinear form and $\mathscr{F}$ is an appropriate subspace of $L^2(\mathbb{R}^d)$ called domain. Define the norm on $\mathscr{F}$ by

$$\mathscr{E}_1(u,v) = \mathscr{E}(u,v) + \int_{\mathbb{R}^d} u(x)v(x)dx.$$

Then $\mathscr{F}$ is a Hilbert space with respect to $\mathscr{E}_1^{1/2}$-norm. Let $C_c(\mathbb{R}^d)$ be the family of continuous functions on $\mathbb{R}^d$ with compact supports equipped with the uniform norm. If the domain $\mathscr{F}$ is both dense in $L^2(\mathbb{R}^d)$ with respect to the L^2-norm and in $C_c(\mathbb{R}^d)$ with respect to the uniform norm, the Dirichlet form $(\mathscr{E},\mathscr{F})$ is said to be *regular*.

It is known that a regular Dirichlet form on $L^2(\mathbb{R}^d)$ has a unique representation as follows:

$$\mathscr{E}(u,v) = \mathscr{E}^c(u,v) + \int_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x)) \\ (v(y)-v(x))J(x,y)dxdy + \int_{\mathbb{R}^d} u(x)v(x)k(dx). \qquad (1.1)$$

We call (1.1) *formulae of Beurling-Deny and LeJan*. For further arguments, see §3.2 of [13]. Furthermore, by the general theory of regular Dirichlet forms, there exists a unique Hunt process on $\mathbb{R}^d$ associated with $(\mathscr{E},\mathscr{F})$ (See Chapter 7 of [13]). Hunt processes on $\mathbb{R}^d$ consists of three parts according to the behaviors of sample paths. (1.1) enables us to distinguish these behaviors of sample paths: The first part of (1.1) is called *diffusion part* which satisfies strong local property: i.e. if v is equal to some constant on the support of u, it follows that $\mathscr{E}^c(u,v) = 0$. Diffusion part describes the continuous movement of a particle, such as the Brownian motion. The second part of (1.1) is called *jump part* which satisfies non-local property and describes the discontinuous movement of a particle. The most typical example is the rotationally invariant α-stable process for $0<\alpha<2$ (See Example 1.1). The last part of (1.1) is called *killing part* which describes the disappearance of a particle in the state space.

Next we review definitions and properties of Hunt process. Let $(\Omega,\mathscr{M},\{X_t\}_{t\geq 0},\mathbb{P})$ be a stochastic process with state space $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d))$, where $\mathscr{B}(\mathbb{R}^d)$ stands for the Borel σ-field on $\mathbb{R}^d$, i.e. $(\Omega,\mathscr{M},\mathbb{P})$ is a probability space and each X_t is a measurable map from Ω to $\mathbb{R}^d$. The last condition of the measurability is explicitly indicated by $X_t\in\mathscr{M}/\mathscr{B}(\mathbb{R}^d)$. We say that the family $\{\mathscr{M}_t\}_{t\geq 0}$ of sub-σ-fields of $\mathscr{M}$ is an *admissible filtration* if $\{\mathscr{M}_t\}_{t\geq 0}$ is increasing in t and $X_t\in\mathscr{M}_t/\mathscr{B}$ for each $t\geq 0$. An admissible filtration $\{\mathscr{M}_t\}_{t\geq 0}$ is called *right continuous* if for any $t\geq 0$,

$$\mathscr{M}_t=\mathscr{M}_{t+}:=\cap_{t'>t}\mathscr{M}_{t'}.$$

Adjoining an extra point Δ to a measurable space $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d))$, we set

$$\overline{\mathbb{R}^d}=\mathbb{R}^d\cup\Delta,\qquad\qquad \mathscr{B}(\overline{\mathbb{R}^d})=\mathscr{B}(\mathbb{R}^d)\cup\{B\cup\Delta\,:\,B\in\mathscr{B}(\mathbb{R}^d)\}.$$

A quadruple $\mathbb{M}=(\Omega,\mathscr{M},\{X_t\}_{t\geq 0},\{\mathbb{P}_x\}_{x\in\overline{\mathbb{R}^d}})$ is said to be *normal Markov process* on $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d))$ if the following conditions are satisfied:

(M-1) For each $x\in\overline{\mathbb{R}^d}$, $(\Omega,\mathscr{M},\{X_t\}_{t\geq 0},\mathbb{P}_x)$ is a stochastic process on $\mathbb{R}^d$.

(M-2) $\mathbb{P}_x(X_t\in E)$ is measurable for $t\geq 0$ and $E\in\mathscr{B}(\mathbb{R}^d)$.

(M-3) There exists an admissible filtration $\{\mathscr{M}_t\}_{t\geq 0}$ such that

$$\mathbb{P}_x(X_{t+s}\in E\mid\mathscr{M}_t)=\mathbb{P}_{X_t}(X_s\in E),\qquad \mathbb{P}_x\text{-a.s.}$$

for any $x\in\mathbb{R}^d,t,s\geq 0$ and $E\in\mathscr{B}(\mathbb{R}^d)$.

(M-4) $\mathbb{P}_\Delta(X_t=\Delta)=1$ for any $t\geq 0$.

(M-5) $\mathbb{P}_x(X_0=x)=1$ for any $x\in\mathbb{R}^d$.

Before we define a Hunt process, we introduce stopping time and sub-σ-field defined by stopping time. A $[0,\infty]$-valued function σ on Ω is called an *$\mathscr{M}_t$-stopping time* if $\{\sigma\leq t\}\in\mathscr{M}_t$ for each $t\geq 0$. For a stopping time σ, we define the sub-σ-field $\mathscr{M}_\sigma$ by

$$\mathscr{M}_\sigma=\{\Lambda\in\mathscr{M}:\Lambda\cap\{\sigma\leq t\}\in\mathscr{M}_t\text{ for all }t\geq 0\}.$$

Markov process $\mathbb{M}$ is said to be *Hunt process*, if the following additional conditions are satisfied:

(M-6) (i) $X_\infty(\omega)=\Delta$ for any $\omega\in\Omega$.

(ii) $X_t(\omega)=\Delta$ for $t\geq\zeta(\omega)$, where $\zeta(\omega)$ is the *life time* defined by $\zeta(\omega)=\inf\{t\geq 0:X_t(\omega)=\Delta\}$

(iii) for each $t \in [0,\infty]$, there exists a map θ_t from Ω to Ω such that $X_s \circ \theta_t = X_{t+s}$ for $s \geq 0$.

(iv) For each $\omega \in \Omega$, the sample path $t \to X_t(\omega)$ is right continuous on $[0,\infty)$ and has the left limit on $(0,\infty)$.

(M-7) (i) The admissible filtration $\{\mathscr{M}_t\}_{t\geq 0}$ is right continuous and for any probability measure ν on $\overline{\mathbb{R}^d}$, $E \in \mathscr{B}(\overline{\mathbb{R}^d})$ and $s \geq 0$,

$$\mathbb{P}_\nu(X_{\sigma+s} \in E \mid \mathscr{M}_\sigma) = \mathbb{P}_{X_\sigma}(X_s \in E), \qquad \mathbb{P}_\nu\text{-a.s.},$$

where σ is $\{\mathscr{M}_t\}_{t\geq 0}$-stopping time.

(ii) Let ν be an arbitrary probability measure on $\overline{\mathbb{R}^d}$. For any $\{\mathscr{M}_t\}_{t\geq 0}$-stopping time σ_n increasing to σ, it holds that

$$\mathbb{P}_\nu(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma, \sigma < \infty) = \mathbb{P}_\nu(\sigma < \infty).$$

The properties in (M-7) are called the *strong Markov property* and the *quasi left continuity* respectively.

In the sequel, we abbreviate the Hunt process by $\{X_t\}_{t\geq 0}$ for the convenience. Let $\{P_t\}_{t>0}$ be the semigroup generated by the Hunt process $\{X_t\}_{t\geq 0}$:

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy,$$

where $p(t,x,y)$ is the transition probability density of $\{X_t\}_{t\geq 0}$. If we use the unique operator $\mathscr{L}$ satisfying $\mathscr{E}(u,v) = -(\mathscr{L}u, v)$, $p(t,x,y)$ is the fundamental solution of the equation $\partial u/\partial t = \mathscr{L}u$. For $\beta > 0$, we define the resolvent kernel of β-order as follows:

$$G_\beta(x,y) = \int_0^\infty e^{-\beta t} p(t,x,y) dt.$$

If we admit the case the right hand side is equal to infinity, this formula is valid for $\beta = 0$. We denote by $G(x,y)$ the resolvent kernel of 0-order, which we call the *green kernel*. The Hunt process $\{X_t\}_{t\geq 0}$ or the associated Dirichlet form $(\mathscr{E},\mathscr{F})$ is said to be *transient* if $G(x,y) < \infty$ for $x \neq y$. In order to characterize transience property, we often use the *extended Dirichlet space* $\mathscr{F}_e$. $\mathscr{F}_e$ is defined as a family of measurable functions on $\mathbb{R}^d$ satisfying the following two conditions:

- $|u| < \infty \quad m$-a.e.
- There exists an $\mathscr{E}$-Cauchy sequence $\{u_n\}_{n\in\mathbb{N}}$ of functions in $\mathscr{F}$ such that $\lim_{n\to\infty} u_n = u \quad m$-a.e.

Obviously $\mathscr{F}$ is a subspace of $\mathscr{F}_e$, and we see that $\mathscr{F}_e \cap L^2(\mathbb{R}^d) = \mathscr{F}$ from Theorem 1.5.2 of [13]. The following proposition is taken from Theorem 1.5.3 of [13], which describes necessary and sufficient conditions for the transience of Dirichlet form.

Proposition 1.1. *$\mathscr{F}_e$ is the extended Dirichlet space of transient Dirichlet form $\mathscr{E}$ if and only if the following conditions are satisfied.*

(i) $\mathscr{F}_e$ *is a real Hilbert space with inner product* $\mathscr{E}$.

(ii) There exists an m-integrable bounded function g strictly positive m-a.e. such that $\mathscr{F}_e \subset L^1(g \cdot m)$ *and*

$$\int_{\mathbb{R}^d} |u(x)|g(x)m(dx) \leq \sqrt{\mathscr{E}(u,u)}, \quad \forall u \in \mathscr{F}_e.$$

(iii) $\mathscr{F}_e \cap L^2(\mathbb{R}^d)$ *is dense both in* $L^2(\mathbb{R}^d)$ *and in* $(\mathscr{F}_e, \mathscr{E})$.

(iv) For any $u \in \mathscr{F}_e$ *and its normal contraction v, it follows that* $v \in \mathscr{F}_e$ *and* $\mathscr{E}(v,v) \leq \mathscr{E}(u,u)$. *Here we say v is a normal contraction of u, if* $|v(x)| \leq |u(x)|$ *for all* $x \in \mathbb{R}^d$ *and* $|v(x) - v(y)| \leq |u(x) - u(y)|$ *for all* $x, y \in \mathbb{R}^d$.

In this thesis, we consider the Hunt process which consists of only discontinuous sample paths. These Hunt process is called *jump processes*. (1.1) implies that the jump process $\{X_t\}_{t\geq 0}$ is characterized by a regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(\mathbb{R}^d)$ as follows:

$$\mathscr{E}(u,u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - u(x))^2 J(x,y)dxdy, \qquad \mathscr{F} = \overline{\{C_c(\mathbb{R}^d)\}}^{\mathscr{E}_1^{1/2}},$$

where $J(x,y)$ is a symmetric Borel function. $J(x,y)$ describes the frequency of jump and is called jump intensity measure. We close this section by introducing three important examples of jump processes.

Example 1.1. *If* $J(x,y) = C_\alpha / |x-y|^{d+\alpha}$ *for* $0 < \alpha < 2$ *and positive constant* C_α, $\{X_t\}_{t\geq 0}$ *is called the rotationally invariant* α*-stable process. The characteristic function satisfies*

$$\mathbb{E}_0[\exp(iu \cdot X_t)] = \exp(-t|u|^\alpha).$$

Example 1.2. *We call the associated process* $\{X_t\}_{t\geq 0}$ α*-stable-like, if* $J(x,y)$ *satisfies*

$$\frac{C_1}{|x-y|^{d+\alpha}} \leq J(x,y) \leq \frac{C_2}{|x-y|^{d+\alpha}}$$

for some positive constants C_1 *and* C_2.

Example 1.3. $\{X_t\}_{t\geq 0}$ *is called relativistic* α*-stable-like if* $J(x,y)$ *satisfies*

$$\frac{C_1}{|x-y|^{d+\alpha}} \exp(-m_0|x-y|) \leq J(x,y) \leq \frac{C_2}{|x-y|^{d+\alpha}} \exp(-m_0|x-y|)$$

for some positive constants C_1, C_2 *and* m_0.

1.2 Smooth measures and additive functionals

In this section we define smooth measures and positive continuous additive functionals. In order to define smooth measures, we first review the definition of capacity. Denote by $\mathscr{O}$ the family of all open subset of $\mathbb{R}^d$. For $A \in \mathscr{O}$, we define

$$\mathscr{L}_A = \{u \in \mathscr{F} \,;\, u \geq 1 \quad m\text{-a.e. on } A\},$$
$$\mathrm{Cap}(A) = \begin{cases} \inf_{u \in \mathscr{L}_A} \mathscr{E}_1(u,u), & \mathscr{L}_A \neq \emptyset \\ \infty, & \mathscr{L}_A = \emptyset. \end{cases}$$

For any set $A \subset \mathbb{R}^d$, we set

$$\mathrm{Cap}(A) = \inf_{B \in \mathscr{O}, A \subset B} \mathrm{Cap}(B).$$

We call this the *capacity* of A. The set A is said to be *exceptional* if $\mathrm{Cap}(A) = 0$. We use the term *quasi everywhere (in abbreviation, q.e.)* in order to mention 'except for an exceptional set'.

A positive Radon measure μ on $\mathbb{R}^d$ is said to be *of finite energy integral* if

$$\int_{\mathbb{R}^d} |v(x)| \mu(dx) \leq C\mathscr{E}_1(v,v) \quad (v \in \mathscr{F} \cap C_c(\mathbb{R}^d))$$

for some positive constant C. We denote by $\mathbf{S}_0$ the family of positive Radon measures of finite energy integral. Since $\mathscr{F}$ is a Hilbert space with respect to $\mathscr{E}_1$-norm, Riesz representation theorem implies that there exists a unique $U_1\mu \in \mathscr{F}$ such that

$$\mathscr{E}_1(U_1\mu, v) = \int_{\mathbb{R}^d} v(x)\mu(dx).$$

We call the function $U_1\mu$ *1-potential*. Moreover, we define a subset $\mathbf{S}_{00}$ of $\mathbf{S}_0$ by

$$\mathbf{S}_{00} = \{\mu \in \mathbf{S}_0 \,;\, \mu(\mathbb{R}^d) < \infty, \quad \|U_1\mu\|_\infty < \infty\},$$

where $\|\cdot\|_\infty$ stands for the norm of $L^\infty(\mathbb{R}^d)$.
Using these material, we define *smooth measures* and *smooth measures in the strict sense*.

Definition 1.1. *(i) A positive Borel measure μ is said to be smooth ($\mu \in \boldsymbol{S}$) if μ charges no set of zero capacity and there exists an increasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed sets such that $\mu(F_n) < \infty$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} Cap(K \backslash F_n) = 0$ for any compact set K.*

(ii) A positive Borel measure μ is said to be smooth in the strict sense ($\mu \in \boldsymbol{S}_1$) if there exists a sequence $\{E_n\}_{n\geq 1}$ of Borel sets increasing to $\mathbb{R}^d$ such that $1_{E_n} \cdot \mu \in \boldsymbol{S}_{00}$ for each n and $\mathbb{P}_x(\lim_{n\to\infty} T_{\mathbb{R}^d\backslash E_n} = \infty) = 1$ for any $x \in \mathbb{R}^d$, where $T_{\mathbb{R}^d\backslash E_n} := \inf\{t > 0 \,;\, X_t \in \mathbb{R}^d \backslash E_n\}$.

Next we introduce additive functionals. The $\mathbb{R}$-valued stochastic process $\{A_t\}_{t\geq 0}$ is called *positive continuous additive functional* (PCAF in abbreviation) if it satisfies the following conditions:

(A-1) A_t is $\mathscr{F}_t$-measurable, $\{\mathscr{F}_t\}$ being the minimum completed admissible filtration of the Hunt process.

(A-2) There exist a set $\Lambda \in \mathscr{F}_\infty$ and an exceptional set $N \subset \mathbb{R}^d$ such that $\mathbb{P}_x(\Lambda) = 1$ for any $x \in \mathbb{R}^d \backslash N$, $\theta_t \Lambda \subset \Lambda$ for any $t > 0$, and moreover for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is right continuous and has the left limit on $[0, \zeta(\omega)), A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega)$ for any $t, s \geq 0$.

If we can choose the empty set as an exceptional set in (A-2), $\{A_t\}$ is called *PCAF in the strict sense*. It is known from Theorem 5.1.3 (Theorem 5.1.7) of [13] that there is one to one correspondence between the set of smooth measures (in the strict sense) and the set of PCAFs (in the strict sense). This relation is called *the Revuz correspondence*: for all positive bounded Borel measurable function f and γ-excessive function h,

$$\int_{\mathbb{R}^d} f(x)h(x)\mu(dx) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}_{h \cdot m} \left[\int_0^t f(X_s) dA_s^\mu \right].$$

In particular, if μ is absolutely continuous with respect to the Lebesgue measure m and consequently $\mu = V(x)dx$, it holds that $A_t^\mu = \int_0^t V(X_s)ds$.

Chapter 2

Perturbation of Dirichlet forms and stability of fundamental solutions

In this chapter we assume that the Hunt process $\{X_t\}_{t\geq 0}$ or the associated Dirichlet form $(\mathscr{E},\mathscr{F})$ is α-stable-like or relativistic α-stable-like. Denote by $\{P_t\}$ the associated semigroup and let $p(t,x,y)$ be the transition density function of $\{X_t\}_{t\geq 0}$. It is well known that $\{P_t\}$ admits the integral kernel $p(t,x,y)$.

We consider the perturbation of Dirichlet form defined by

$$\mathscr{E}^{\mu}(u,u) := \mathscr{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu = -(\mathscr{L}u,u) - \int_{\mathbb{R}^d} u^2 d\mu.$$

Here μ is a positive measure in the Kato class satisfying Green tightness (in abbreviation $\mu \in \mathbf{K}_\infty$). Let $\{P_t^{\mu}\}$ be the associated semigroup. This semigroup also admits the integral kernel $p^{\mu}(t,x,y)$ defined on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. We compare $p^{\mu}(t,x,y)$ with $p(t,x,y)$. Chen, Kim and Kumagai [7] and Chen and Kumagai [9] proved that both upper estimates and lower estimates of $p(t,x,y)$ are the same function up to positive constants. If $p^{\mu}(t,x,y)$ has the same estimate as $p(t,x,y)$ up to positive constants, we call this phenomenon *stability of fundamental solution*. Suppose that the measure $\mu \in \mathbf{K}_\infty$ is of 0-order finite energy integral, namely, μ and the Green kernel $G(x,y)$ satisfies

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)\mu(dx)\mu(dy) < \infty.$$

Our goal is to prove that the subcriticality of μ is the necessary and sufficient condition for the stability of fundamental solutions.

For the construction of the necessary condition, we need to check some classes of measures. Takeda [29] showed that there are some conditions equivalent to the subcriticality of the measure μ. However, this equivalence is valid for the class of conditionally Green-tight measures $\mathbf{S}_\infty$, which is in general a subclass of $\mathbf{K}_\infty$. In order to apply this argument to the class $\mathbf{K}_\infty$, we first prove $\mathbf{K}_\infty = \mathbf{S}_\infty$ using 3G-inequality.

For the construction of the sufficient condition, we apply Doob's h-transformation. Here $h(x)$ is a harmonic function of the perturbed operator $\mathscr{L}^{\mu} := \mathscr{L} + \mu$. If $h(x) = \exp(u(x))$ for $u \in \mathscr{F}_e$, we can apply the argument of Chen and Zhang [12], and thus construct a transformed semigroup $\{P_t^{\mu,h}\}$ on $L^2(h^2 dm)$. We can describe the associated Dirichlet form $(\mathscr{E}^{\mu,h}, \mathscr{D}(\mathscr{E}^{\mu,h}))$, which is equivalent to the original $(\mathscr{E}, \mathscr{F})$ and then conclude the stability of fundamental solutions. In order to apply this argument, we assume that μ is of 0-order finite energy integral.

This chapter is organized as follows: in Section 1 we review two properties for α-stable-like processes and relativistic α-stable-like processes: one is the conservativeness of processes given by Masamune and Uemura [21], and the other is the two-sided heat kernel estimates given by Chen, Kim and Kumagai [7] and Chen and Kumagai [9]. We also give the two-sided estimates for Green kernels. In Section 2 we will give the definition of some classes of smooth measures: the Kato class $\mathbf{K}$, the Green-tight Kato class $\mathbf{K}_\infty$ and the conditional Green-tight Kato class $\mathbf{S}_\infty$. In Section 3 we prove the main result following the arguments of Chen and Zhang [12] and Takeda [31].

2.1 Heat kernel estimates for jump Markov processes

We first define the Dirichlet forms associated with α-stable-like process and relativistic α-stable-like process. Let $(\mathscr{E}, \mathscr{F})$ be a jump type regular Dirichlet form on $L^2(\mathbb{R}^d)$ as follows:

$$\mathscr{E}(u,u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - u(x))^2 J(x,y) dxdy, \qquad \mathscr{F} = \overline{\{C_c(\mathbb{R}^d)\}}^{\mathscr{E}_1^{1/2}}, \tag{2.1}$$

where $C_c(\mathbb{R}^d)$ is the family of all continuous functions on $\mathbb{R}^d$ with compact supports, $\mathscr{E}_1(u,u) = \mathscr{E}(u,u) + \int_{\mathbb{R}^d} u^2(x)dx$ and $J(x,y)$ is a symmetric Borel function called jump measure. Here we consider the two cases:

$$\frac{C_1}{|x-y|^{d+\alpha}} \leq J(x,y) \leq \frac{C_2}{|x-y|^{d+\alpha}} \quad (0 < \alpha < 2), \tag{2.2}$$

$$\frac{C_1 \exp(-m_0|x-y|)}{|x-y|^{d+\alpha}} \leq J(x,y) \leq \frac{C_2 \exp(-m_0|x-y|)}{|x-y|^{d+\alpha}} \quad (0 < \alpha < 2, \quad m_0 > 0). \tag{2.3}$$

In the sequel C_i's are unimportant positive constants varying line to line. Denote by $\{X_t\}_{t\geq 0}$ the associated Hunt process. If $J(x,y)$ satisfies (2.2), $\{X_t\}_{t\geq 0}$ is called α-stable-like. If $J(x,y)$ satisfies (2.3), $\{X_t\}_{t\geq 0}$ is called relativistic α-stable-like.

The Hunt process $\{X_t\}_{t\geq 0}$ is said to be *conservative* if

$$\mathbb{P}_x(\zeta = \infty) = 1 \quad \text{q.e. } x \in \mathbb{R}^d,$$

where ζ is the life time of $\{X_t\}_{t\geq 0}$ defined by

$$\zeta := \inf\{t \geq 0,\ X_t = \Delta\}.$$

We first show the conservativeness of $\{X_t\}_{t\geq 0}$. Masamune and Uemura [21] established some sufficient conditions for conservativeness of jump Markov processes on locally compact metric spaces. If we rewrite their theorem in the framework of jump Markov processes on $\mathbb{R}^d$, we have the following assertion.

Theorem 2.1. *Let $(\mathscr{E},\mathscr{F})$ be a jump regular Dirichlet form defined by (2.1). The associated Hunt process $\{X_t\}_{t\geq 0}$ is conservative if the following two conditions are satisfied:*

(i) $\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{x\}}(1\wedge|x-y|^2)J(x,y)dy<\infty.$

(ii) *For any $a>0$, it holds that $e^{-a|x|}\in L^1(\mathbb{R}^d)$.*

If $J(x,y)$ satisfies (2.2) or (2.3), we see that $\{X_t\}_{t\geq 0}$ is conservative from this theorem. Indeed, in the both cases, the upper bound of $J(x,y)$ implies

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\{x\}}(1\wedge|x-y|^2)J(x,y)dy &\leq \int_{\mathbb{R}^d\setminus\{x\}}(1\wedge|x-y|^2)\cdot\frac{C_1}{|x-y|^{d+\alpha}}dy\\ &=\int_{|x-y|\leq 1}\frac{C_1}{|x-y|^{d+\alpha-2}}dy+\int_{|x-y|\geq 1}\frac{C_1}{|x-y|^{d+\alpha}}dy.\end{aligned}$$

Using the spherical coordinates, the last line of the above formula is equal to

$$C_2\left(\int_0^1 r^{1-\alpha}dr+\int_1^\infty r^{-1-\alpha}dr\right).$$

Since $0<\alpha<2$, this value is bounded by some positive constant not depending on $x\in\mathbb{R}^d$. Hence we obtain (i). Moreover, we see that

$$\int_{\mathbb{R}^d}\exp(-a|x|)dx=C_3\int_0^\infty r^{d-1}\exp(-ar)dr<\infty.$$

Thus we obtain (ii).

Next we review the two-sided heat kernel estimates. Chen and Kumagai [9] showed heat kernel estimates for α-stable-like processes.

Theorem 2.2. *Let $\{X_t\}_{t\geq 0}$ be an α-stable-like process on $\mathbb{R}^d$. Then there exist positive constants C_1 and C_2 such that for all $t>0$ and $x,y\in\mathbb{R}^d$,*

$$C_1\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\leq p(t,x,y)\leq C_2\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right). \tag{2.4}$$

Furthermore, Chen, Kim and Kumagai [7] gave heat kernel estimates for relativistic α-stable-like processes.

Theorem 2.3. *Let $\{X_t\}_{t\geq 0}$ be an α-stable-like process on $\mathbb{R}^d$. Then the heat kernel $p(t,x,y)$ has different two-sided estimates according to the sizes of t and $|x-y|$.*

(i) *If $0\leq t\leq 1$ and $|x-y|\leq 1$, $p(t,x,y)$ satisfies the same two-sided estimates as α-stable-like processes:*

$$C_1\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\leq p(t,x,y)\leq C_2\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x-y|^{d+\alpha}}\right). \tag{2.5}$$

(ii) *If* $0 \le t \le 1$ *and* $|x-y| \ge 1$, $p(t,x,y)$ *satisfies*

$$C_1 t \frac{\exp(-C_2|x-y|)}{|x-y|^{d+\alpha}} \le p(t,x,y) \le C_3 t \frac{\exp(-C_4|x-y|)}{|x-y|^{d+\alpha}}. \tag{2.6}$$

(iii) *If* $1 \le t \le |x-y|$, $p(t,x,y)$ *satisfies*

$$C_1 t^{-\frac{d}{2}} \exp(-C_2|x-y|) \le p(t,x,y) \le C_3 t^{-\frac{d}{2}} \exp(-C_4|x-y|). \tag{2.7}$$

(iv) *If* $1 \vee |x-y| \le t$, $p(t,x,y)$ *satisfies*

$$C_1 t^{-\frac{d}{2}} \exp\Big(-\frac{C_2|x-y|^2}{t}\Big) \le p(t,x,y) \le C_3 t^{-\frac{d}{2}} \exp\Big(-\frac{C_4|x-y|^2}{t}\Big). \tag{2.8}$$

In the sequel, we assume that $\{X_t\}_{t\ge0}$ is transient and consequently $\{X_t\}_{t\ge0}$ admits the finite Green kernel. We obtain the two-sided estimates for Green kernels from Theorems 2.2 and 2.3.

Proposition 2.1. *Let* $\{X_t\}_{t\ge0}$ *be the transient Hunt process generated by a Dirichlet form* $(\mathscr{E},\mathscr{F})$ *satisfying* (2.1).

(i) *Suppose* $J(x,y)$ *satisfies* (2.2). *Then the Green kernel* $G(x,y)$ *satisfies*

$$\frac{C_1}{|x-y|^{d-\alpha}} \le G(x,y) \le \frac{C_2}{|x-y|^{d-\alpha}}. \tag{2.9}$$

(ii) *Suppose* $J(x,y)$ *satisfies* (2.3). *Then the Green kernel* $G(x,y)$ *satisfies*

$$\begin{aligned} C_1\Big(\frac{1}{|x-y|^{d-\alpha}} \vee \frac{1}{|x-y|^{d-2}}\Big) &\le G(x,y) \\ &\le C_2\Big(\frac{1}{|x-y|^{d-\alpha}} \vee \frac{1}{|x-y|^{d-2}}\Big). \end{aligned} \tag{2.10}$$

Proof. (i) The transience of $\{X_t\}_{t\ge0}$ implies $\alpha < d$. Since we obtain the heat kernel estimates of α-stable-like processes in Theorem 2.2, it follows that

$$C_1 \int_0^\infty \Big(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big) dt \le G(x,y) \le C_2 \int_0^\infty \Big(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big) dt.$$

We first compare $t^{-\frac{d}{\alpha}}$ with $t/|x-y|^{d+\alpha}$. Since it holds that

$$t^{-\frac{d}{\alpha}} \le \frac{t}{|x-y|^{d+\alpha}} \Leftrightarrow |x-y|^{d+\alpha} \le t^{1+\frac{d}{\alpha}} \Leftrightarrow |x-y|^\alpha \le t,$$

we obtain

$$t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} = \begin{cases} t/|x-y|^{d+\alpha} & (t \le |x-y|^\alpha) \\ t^{-\frac{d}{\alpha}} & (t \ge |x-y|^\alpha) \end{cases}$$

This estimate implies that

$$\int_0^\infty \Big(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big)dt = \int_0^{|x-y|^\alpha} \frac{t}{|x-y|^{d+\alpha}}dt + \int_{|x-y|^\alpha}^\infty t^{-\frac{d}{\alpha}}dt$$
$$= \frac{1}{2|x-y|^{d-\alpha}} + \Big(\frac{d}{\alpha}-1\Big)^{-1}\frac{1}{|x-y|^{d-\alpha}} = \frac{C_3}{|x-y|^{d-\alpha}}.$$

Hence, we conclude (2.9).

(ii) The transience of the process implies $d \geq 3$. First we assume that $|x-y| \leq 1$. We have (2.5) for $0 \leq t \leq 1$ and (2.8) for $t \geq 1$. Thus it follows that

$$\int_0^\infty p(t,x,y)dt = \int_0^1 p(t,x,y)dt + \int_1^\infty p(t,x,y)dt$$
$$\leq \int_0^1 C_1\Big(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big)dt + \int_1^\infty C_2 t^{-\frac{d}{2}} \exp\Big(-\frac{C_3|x-y|^2}{t}\Big)dt. \tag{2.11}$$

Here the first term of (2.11) satisfies

$$\int_0^1 C_1\Big(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big)dt$$
$$\leq \int_0^\infty C_1\Big(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\Big)dt \leq \frac{C_4}{|x-y|^{d-\alpha}}. \tag{2.12}$$

Moreover, the second term of (2.11) satisfies

$$\int_1^\infty C_2 t^{-\frac{d}{2}} \exp\Big(-\frac{C_3|x-y|^2}{t}\Big)dt \leq \int_{|x-y|^2}^\infty C_2 t^{-\frac{d}{2}} \exp\Big(-\frac{C_3|x-y|^2}{t}\Big)dt$$
$$\leq \int_{|x-y|^2}^\infty C_2 t^{-\frac{d}{2}}dt \leq \frac{C_5}{|x-y|^{d-2}} \leq \frac{C_5}{|x-y|^{d-\alpha}}. \tag{2.13}$$

Combining (2.12) and (2.13), we obtain the upper bound:

$$\int_0^\infty p(t,x,y)dt \leq \frac{C_6}{|x-y|^{d-\alpha}}. \tag{2.14}$$

For the lower bound, we have

$$\int_0^\infty p(t,x,y)dt \geq \int_0^{|x-y|^\alpha} p(t,x,y)dt$$
$$\geq \int_0^{|x-y|^\alpha} C_7 \frac{t}{|x-y|^{d+\alpha}}dt = \frac{C_8}{|x-y|^{d-\alpha}}. \tag{2.15}$$

Combining (2.14) and (2.15), we have for $|x-y| \leq 1$

$$\frac{C_1}{|x-y|^{d-\alpha}} \leq G(x,y) \leq \frac{C_2}{|x-y|^{d-\alpha}}. \tag{2.16}$$

We next assume that $|x-y| \geq 1$. We obtained (2.6) for $0 \leq t \leq 1$. However, noting that $|x-y| \leq \exp(|x-y|)$, we can make the left hand side of (2.6) smaller:

$$C_1 t \frac{\exp(-C_2|x-y|)}{|x-y|^{d+\alpha}} \geq C_1 t \exp(-(C_2+d+\alpha)|x-y|)$$
$$\geq C_3 t \exp(-C_4|x-y|). \tag{2.17}$$

Similarly, noting that $|x-y| \geq 1$, we can make the right hand side of (2.6) larger:

$$C_5 t \frac{\exp(-C_6|x-y|)}{|x-y|^{d+\alpha}} \leq C_5 t \exp(-C_6|x-y|). \tag{2.18}$$

Hence (2.17) and (2.18) imply

$$C_1 t \exp(-C_2|x-y|) \leq p(t,x,y) \leq C_3 t \exp(-C_4|x-y|).$$

Moreover, we obtained (2.7) for $1 \leq t \leq |x-y|$ and (2.8) for $t \geq |x-y|$. Thus it follows that

$$\int_0^\infty p(t,x,y)dt \leq \int_0^1 C_3 t \exp(-C_4|x-y|)dt + \int_1^{|x-y|} C_5 t^{-\frac{d}{2}} \exp(-C_6|x-y|)dt$$
$$+ \int_{|x-y|}^\infty C_7 t^{-\frac{d}{2}} \exp\Big(-\frac{C_8|x-y|^2}{t}\Big)dt. \tag{2.19}$$

The first term of (2.19) satisfies

$$\int_0^1 C_3 t \exp(-C_4|x-y|)dt \leq C_9 \exp(-C_{10}|x-y|) \leq \frac{C_{11}}{|x-y|^{d-2}}. \tag{2.20}$$

The second term of (2.19) satisfies

$$\int_1^{|x-y|} C_5 t^{-\frac{d}{2}} \exp(-C_6|x-y|)dt \leq \int_1^\infty C_5 t^{-\frac{d}{2}} \exp(-C_6|x-y|)dt$$
$$\leq C_{12} \exp(-C_{13}|x-y|) \leq \frac{C_{14}}{|x-y|^{d-2}}. \tag{2.21}$$

For the third term of (2.19), we substitute $s = |x-y|^2/t$. It follows that

$$\int_{|x-y|}^\infty C_7 t^{-\frac{d}{2}} \exp\Big(-\frac{C_8|x-y|^2}{t}\Big)dt = \int_0^{|x-y|} C_7 |x-y|^{2-d} s^{\frac{d}{2}-2} \exp(-C_8 s)ds$$
$$\leq \frac{C_7}{|x-y|^{d-2}} \int_0^\infty s^{\frac{d}{2}-2} \exp(-C_8 s)ds \leq \frac{C_{15}}{|x-y|^{d-2}}. \tag{2.22}$$

Thus, combining (2.20)–(2.22), we have the upper bound

$$p(t,x,y) \leq \frac{C_{16}}{|x-y|^{d-2}} \tag{2.23}$$

We can also obtain the lower bound:

$$\begin{aligned}\int_0^\infty p(t,x,y)dt &\geq \int_{|x-y|}^\infty p(t,x,y)dt\\ &\geq \int_{|x-y|}^\infty C_{17}t^{-\frac{d}{2}}\exp\Big(-\frac{C_{18}|x-y|^2}{t}\Big)dt\\ &= \int_0^{|x-y|} C_{17}|x-y|^{2-d}s^{\frac{d}{2}-2}\exp(-C_{18}s)ds. \end{aligned} \tag{2.24}$$

Noting that $|x-y|\geq 1$, we can make the right hand side of (2.24) smaller:

$$\begin{aligned}&\int_0^{|x-y|} C_{17}|x-y|^{2-d}s^{\frac{d}{2}-2}\exp(-C_{18}s)ds\\ &\geq \int_0^1 C_{17}|x-y|^{2-d}s^{\frac{d}{2}-2}\exp(-C_{18}s)ds \geq \frac{C_{19}}{|x-y|^{d-2}}.\end{aligned} \tag{2.25}$$

We thus have for $|x-y|\geq 1$

$$\frac{C_1}{|x-y|^{d-2}} \leq G(x,y) \leq \frac{C_2}{|x-y|^{d-2}} \tag{2.26}$$

from (2.23) and (2.25). On account of (2.16) and (2.26), we obtain (2.10).

□

We close this section introducing a certain function space. Let $\mathscr{G}$ be a function space as follows:

$$\mathscr{G} = \Big\{g\ ;\ g \text{ is a positive decreasing function on } (0,\infty), \exists C_1, C_2\ s.t.\ C_1 \leq \frac{g(2r)}{g(r)} \leq C_2\Big\},$$

where C_1 and C_2 are independent of $r>0$. The following corollary plays a crucial role in the next section.

Corollary 2.1. *Let $\{X_t\}_{t\geq 0}$ be either an α-stable-like process or a relativistic α-stable-like process. Then there exists a function $g\in\mathscr{G}$ such that for all $x,y\in\mathbb{R}^d$ with $x\neq y$,*

$$C_1 g(|x-y|) \leq G(x,y) \leq C_2 g(|x-y|).$$

Proof. By Proposition 2.1, we have only to prove that both $g_1(r) = r^{\alpha-d}$ and $g_2(r) = r^{\alpha-d}\vee r^{2-d}$ belong to the function space $\mathscr{G}$. Since $g_1(2r)/g_1(r) = 2^{\alpha-d}$, we conclude $g_1\in\mathscr{G}$. Similarly, we obtain

$$\frac{g_2(2r)}{g_2(r)} = \begin{cases} 2^{\alpha-d} & (r\leq 1/2)\\ 2^{2-d}\cdot r^{2-\alpha} & (1/2\leq r\leq 1)\\ 2^{2-d} & (r\geq 1)\end{cases}$$

and consequently $2^{\alpha-d}\leq g_2(2r)/g_2(r)\leq 2^{2-d}$. Thus we conclude $g_2\in\mathscr{G}$. □

2.2 Properties of Kato class measures

A set $B \subset \overline{\mathbb{R}^d}$ is called *nearly Borel measurable* if for each probability measure ν on $\overline{\mathbb{R}^d}$, there exist Borel sets $B_1, B_2 \in \mathscr{B}(\overline{\mathbb{R}^d})$ such that $B_1 \subset B \subset B_2$ and $\mathbb{P}_\nu(X_t \in B_2 \backslash B_1,\ \exists t \geq 0) = 0$. We introduce some kinds of 'small sets' for a Markov process $\{X_t\}_{t \geq 0}$.

Definition 2.1. (i) *A set $A \subset \mathbb{R}^d$ is said to be polar if there exists a nearly Borel measurable set B such that $A \subset B$ and $\mathbb{P}_x(\sigma_B = \infty) = 1$ for all $x \in \mathbb{R}^d$. Here $\sigma_B = \inf\{t > 0 \mid X_t \in B\}$.*

(ii) *The subset $A \subset \mathbb{R}^d$ is said to be m-polar if there exists a nearly Borel measurable set B such that $A \subset B$ and $\mathbb{P}_m(\sigma_B = \infty) = 1$. Here, $\mathbb{P}_m(\Lambda) = \int_{\mathbb{R}^d} \mathbb{P}_x(\Lambda) m(dx)$.*

Moreover, the set A is m-polar if and only if A is an exceptional set defined in Section 1.2. Recall that $\{X_t\}_{t \geq 0}$ admits an absolute continuous heat kernel with respect to the Lebesgue measure. It is known that the absolute continuity of heat kernel implies that the polar set or the exceptional set is empty set. Thus, when we consider a smooth measure μ or the associated PCAF A_t^μ, we can strengthen them to the strict ones. We denote by $\mathbf{S}_1$ the family of smooth measure in the strict sense. Now we define some subclasses of $\mathbf{S}_1$.

Definition 2.2. *A smooth measure in the strict sense μ is said to be in the Kato class ($\mu \in \boldsymbol{K}$ in notation), if it holds that*

$$\lim_{\beta \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x,y)\mu(dy) = \lim_{\beta \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty e^{-\beta t} p(t,x,y) dt \mu(dy) = 0. \tag{2.27}$$

The following definition on the Green-tight smooth measures of Kato class is taken from Takeda [29].

Definition 2.3. *A measure $\mu \in \boldsymbol{K}$ is said to be Green-tight ($\mu \in \boldsymbol{K}_\infty$ in notation), if for any $\varepsilon > 0$, there exist a compact set K_ε and a positive constant δ_ε such that*

$$\sup_{x \in \mathbb{R}^d} \int_{K_\varepsilon^c} G(x,y)\mu(dy) \leq \varepsilon \tag{2.28}$$

and for any $B \subset K_\varepsilon$ with $\mu(B) < \delta_\varepsilon$, it holds that

$$\sup_{x \in \mathbb{R}^d} \int_B G(x,y)\mu(dy) \leq \varepsilon. \tag{2.29}$$

The following definition on the conditionally Green-tight smooth measures of Kato class is also taken from Takeda [29].

Definition 2.4. *A measure $\mu \in \boldsymbol{K}$ is said to be in the class $\boldsymbol{S}_\infty$, if for any $\varepsilon > 0$, there exists a compact set K_ε and a positive constant δ_ε such that*

$$\sup_{x,z \in \mathbb{R}^d} \int_{K_\varepsilon^c} \frac{G(x,y)G(y,z)}{G(x,z)} \mu(dy) \leq \varepsilon$$

and for any $B \subset K_\varepsilon$ with $\mu(B) < \delta_\varepsilon$, it holds that

$$\sup_{x,z\in\mathbb{R}^d} \int_B \frac{G(x,y)G(y,z)}{G(x,z)} \mu(dy) \le \varepsilon.$$

There are different definitions for $\mathbf{K}, \mathbf{K}_\infty$ and $\mathbf{S}_\infty$ in [6, 36]. We first make sure that these definitions are equivalent each other.

Proposition 2.2. *The following assertions are equivalent each other.*

(i) $\mu \in \boldsymbol{K}$. *i.e.* μ *satisfies (2.27).*

(ii) $\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_t^\mu] = 0.$

(iii) $\lim_{a\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\le a} G(x,y)\mu(dy) = 0.$

Proof. As a definition of $\mathbf{K}$, [6] used (ii) and [36] used (iii) instead of (2.27). Note that

$$\mathbb{E}_x[A_t^\mu] = \int_{\mathbb{R}^d} \int_0^t p(s,x,y)ds\mu(dy) \tag{2.30}$$

by [1]. Thus the formula (ii) is rewritten as follows:

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t p(s,x,y)ds\mu(dy) = 0. \tag{2.31}$$

We see from Lemma 3.1 of [20] that (2.27) and (2.31) are equivalent, which implies the equivalence between (i) and (ii).

As for the equivalence between (ii) and (iii), Kuwae and Takahashi proved in Theorem 3.2 of [20] for more general Markov processes, but we give another proof here by checking some conditions in Zhao [38]. Let

$$\tau_{B(x,r)} := \inf\{t > 0\,;\, X_t \notin B(x,r)\},$$
$$T_{B(x,r)} := \inf\{t > 0\,;\, X_t \in B(x,r)\}.$$

$\tau_{B(x,r)}$ and $T_{B(x,r)}$ are called *first exiting time* and *first hitting time* of $B(x,r)$ respectively. On account of Theorem 1 of [38], it is sufficient to prove the following three formulae:

$$\alpha_0 := \sup_{t>0} \inf_{r>0} \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,r)} > t) < 1, \tag{2.32}$$

$$\beta_0 := \sup_{r>0} \inf_{t>0} \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,r)} < t) < 1, \tag{2.33}$$

$$\lambda_0 := \sup_{u>0} \inf_{r>0} \sup_{|x-y|\ge u} \mathbb{P}_y(T_{B(x,r)} < \infty) < 1. \tag{2.34}$$

Define the function $\phi(r)$ as follows:

$$\phi(r) = \begin{cases} r^\alpha & (\{X_t\}_{t\geq 0}\text{is } \alpha\text{-stable-like}) \\ r^\alpha e^{m_0 r} & (\{X_t\}_{t\geq 0}\text{is relativistic } \alpha\text{-stable-like}). \end{cases}$$

We first show that there exists a positive constant C_1 such that for all $x \in \mathbb{R}^d$ and $0 < r < 1/4$,

$$\mathbb{E}_x[\tau_{B(x,r)}] \leq C_1 \phi(r). \tag{2.35}$$

Indeed, using the Lèvy system formula and the lower bound of jump measure, we have

$$\begin{aligned} 1 &\geq \mathbb{P}_x(X_{\tau_{B(x,r)}} \notin \overline{B(x,2r)}) = \mathbb{E}_x\left[\int_0^{\tau_{B(x,r)}} \int_{\overline{B(x,2r)}^c} J(X_s,u)duds\right] \\ &\geq \mathbb{E}_x\left[\int_0^{\tau_{B(x,r)}} \int_{\overline{B(x,2r)}^c} \frac{C_2}{|X_s-u|^d\phi(|X_s-u|)}duds\right]. \end{aligned} \tag{2.36}$$

Note that $X_s \in B(x,r)$ for $0 \leq s < \tau_{B(x,r)}$ and for fixed $w \in B(x,r)$, $\{u : |w-u| \geq 3r\} \subset \overline{B(x,2r)}^c$. Applying this to the right hand side of (2.36) and using the spherical coordinates, we obtain

$$\begin{aligned} \mathbb{E}_x\left[\int_0^{\tau_{B(x,r)}} \int_{\overline{B(x,2r)}^c} \frac{C_2}{|X_s-u|^d\phi(|X_s-u|)}duds\right] &\geq C_3\int_{3r}^\infty \frac{1}{\rho\phi(\rho)}d\rho\,\mathbb{E}_x[\tau_{B(x,r)}] \\ &\geq C_4\int_{3r}^1 \rho^{-\alpha-1}d\rho\,\mathbb{E}_x[\tau_{B(x,r)}] = C_5((3r)^{-\alpha}-1)\mathbb{E}_x[\tau_{B(x,r)}]. \end{aligned} \tag{2.37}$$

Since we assumed that $0 < r < 1/4$, it follows that

$$(3r)^{-\alpha} - 1 = (3r)^{-\alpha}(1-(3r)^\alpha) \geq C_6 r^{-\alpha}.$$

Hence the right hand side of (2.37) is estimated as follows:

$$C_5((3r)^{-\alpha}-1)\mathbb{E}_x[\tau_{B(x,r)}] \geq C_6 r^{-\alpha}\mathbb{E}_x[\tau_{B(x,r)}] \geq \frac{C_6}{\phi(r)}\mathbb{E}_x[\tau_{B(x,r)}]. \tag{2.38}$$

Thus (2.36)–(2.38) imply $1 \geq C_6/\phi(r)\cdot\mathbb{E}_x[\tau_{B(x,r)}]$ and we have (2.35).This is an extension of Theorem 5.1 in [9]. It is clear that (2.35) implies

$$\mathbb{P}_x(\tau_{B(x,r)} > t) \leq \frac{C_8\phi(r)}{t}$$

and we obtain (2.32) with $\alpha_0 = 0$.

Applying Proposition 4.9 of Chen and Kumagai [10], we see that for arbitrary $\varepsilon > 0$ there exists $\gamma_\varepsilon > 0$ such that for $0 < r < 1$ and $x \in \mathbb{R}^d$

$$\mathbb{P}_x(\tau_{B(x,r)} < \gamma_\varepsilon\phi(r)) \leq \varepsilon.$$

Thus, for $0 < r < 1$ we obtain

$$\inf_{t>0}\sup_{x\in\mathbb{R}^d}\mathbb{P}_x(\tau_{B(x,r)} < t) \leq \sup_{x\in\mathbb{R}^d}\mathbb{P}_x(\tau_{B(x,r)} < \gamma_\varepsilon\phi(r)) \leq \varepsilon.$$

For $r \geq 1$, we obtain

$$\begin{aligned}\inf_{t>0} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,r)} < t) &\leq \sup_{x \in \mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,r)} < \gamma_\varepsilon \phi(1/2)) \\ &\leq \sup_{x \in \mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,1/2)} < \gamma_\varepsilon \phi(1/2)) \leq \varepsilon.\end{aligned}$$

Hence, we obtain (2.33) with $\beta_0 = 0$.

We can prove (2.34) in the same way as is used in Lemma 5 of [38], which deals with Lévy processes. Corollary 2.1 implies that there exist positive constants C_1, C_2 and $g \in \mathscr{G}$ such that

$$C_1 g(|x-y|) \leq G(x,y) \leq C_2 g(|x-y|).$$

Fix $u > 0$ and $b \geq 1$. If $|x-y| \geq u$, $0 < r \leq u/(2b+1)$ and $|z-x| \leq r$, it follows that $|y-z| \geq 2br$ and consequently

$$C_2 g(2br) \geq C_2 g(|y-z|) \geq G(y,z).$$

Thus, we have

$$\begin{aligned}C_3 g(2br) r^d \geq \int_{B(x,r)} G(y,z)dz &\geq \mathbb{E}_y\left[\int_{T_{B(x,r)}}^\infty 1_{B(x,r)}(X_t)dt\right] \\ &= \mathbb{E}_y\left[\mathbb{E}_{X_{T_{B(x,r)}}}\left[\int_0^\infty 1_{B(x,r)}(X_t)dt\right] \,:\, T_{B(x,r)} < \infty\right], \qquad (2.39)\end{aligned}$$

where we used the strong Markov property in the last equality. Note that $X_{T_{B(x,r)}} \in \overline{B(x,r)}$ and

$$\mathbb{E}_w\left[\int_0^\infty 1_{B(x,r)}(X_t)dt\right] = \int_{B(x,r)} G(w,z)dz.$$

Hence the right hand side of (2.39) satisfies

$$\begin{aligned}&\mathbb{E}_y\left[\mathbb{E}_{X_{T_{B(x,r)}}}\left[\int_0^\infty 1_{B(x,r)}(X_t)dt\right] \,:\, T_{B(x,r)} < \infty\right] \\ &\geq \mathbb{P}_y(T_{B(x,r)} < \infty) \cdot \inf_{|w-x| \leq r} \int_{B(x,r)} G(w,z)dz \\ &\geq \mathbb{P}_y(T_{B(x,r)} < \infty) \cdot \inf_{|w-x| \leq r} \int_{B(x,r)} C_1 g(|w-z|)dz. \qquad (2.40)\end{aligned}$$

Moreover, $|w-x| \leq r$ and $z \in B(x,r)$ imply $|w-z| \leq 2r$, and using the monotone decreasing property of g, we obtain

$$\mathbb{P}_y(T_{B(x,r)} < \infty) \cdot \inf_{|w-x| \leq r} \int_{B(x,r)} C_1 g(|w-z|)dz \geq C_4 g(2r) r^d \mathbb{P}_y(T_{B(x,r)} < \infty). \qquad (2.41)$$

From (2.39)–(2.41), we see that

$$\mathbb{P}_y(T_{B(x,r)} < \infty) \leq \frac{C_3 g(2br)}{C_4 g(2r)}$$

for $|x-y| \geq u$, $0 < r \leq u/(2b+1)$ and $u > 0$. We thus see that

$$\begin{aligned}
\inf_{r>0} \sup_{|x-y|\geq u} \mathbb{P}_y(T_{B(x,r)} < \infty) &\leq \inf_{0<r\leq u/(2b+1)} \sup_{|x-y|\geq u} \mathbb{P}_y(T_{B(x,r)} < \infty) \\
&\leq \inf_{0<r\leq u/(2b+1)} \sup_{|x-y|\geq u} \frac{C_3 g(2br)}{C_4 g(2r)} \\
&= \inf_{0<r\leq u/(2b+1)} \frac{C_3 g(2br)}{C_4 g(2r)} \leq \limsup_{r\to 0} \frac{C_3 g(2br)}{C_4 g(2r)}.
\end{aligned}$$

For a sufficiently large b, the right hand side of the above formula is smaller than 1. Hence, we obtain (2.34). □

In the sequel, we assume $\mu \in \mathbf{K}$. The following proposition says that the two definitions on the Green-tight smooth measures of Kato class from [6] and [29] coincide with each other.

Proposition 2.3. *For $\mu \in \boldsymbol{K}$, the following assertions are equivalent each other.*

(i) *$\mu \in \boldsymbol{K}_\infty$. i.e. μ satisfies (2.28)–(2.29).*

(ii) *For any $\varepsilon > 0$ there exist a set F_ε of μ-finite measure and a positive constant $\tilde{\delta}_\varepsilon$ such that*

$$\sup_{x\in\mathbb{R}^d} \int_{F_\varepsilon^c} G(x,y)\mu(dy) \leq \varepsilon$$

and for any $B \subset F_\varepsilon$ with $\mu(B) < \tilde{\delta}_\varepsilon$,

$$\sup_{x\in\mathbb{R}^d} \int_B G(x,y)\mu(dy) \leq \varepsilon.$$

(iii) *It holds that*

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^d} \int_{|y|>r} G(x,y)\mu(dy) = 0.$$

Proof. We assume (i) and let K_ε and δ_ε be a compact set and a positive constant in (2.28)–(2.29) respectively. Since $\mu \in \mathbf{K}$ and $g(|x-y|) \to \infty$ as $|x-y| \to 0$, (iii) of Proposition 2.2 implies that $\mu(B(x,a)) \to 0$ as $a \to 0$ uniformly in $x \in \mathbb{R}^d$. In particular, there exist positive constants a_0 and C_0 such that $\mu(B(x_0,a_0)) \leq C_0$ for all $x \in \mathbb{R}^d$. Since K_ε is compact, K_ε is covered by finite subset of $\{B(x,a_0)\}_{x\in\mathbb{R}^d}$ and thus we obtain $\mu(K_\varepsilon) < \infty$. Hence (ii) follows for $F_\varepsilon = K_\varepsilon$ and $\tilde{\delta}_\varepsilon = \delta_\varepsilon$.

We next show (ii) implies (i). We follow the proof of Theorem 2.1 (3) in [6]. Let F_ε and $\tilde{\delta}_\varepsilon$ be a set of μ-finite measure and a positive constant satisfying (ii). Since

$$\mu(\overline{B(0,R)}^c \cap F_\varepsilon) \to 0 \quad (R \to \infty),$$

there exists a positive constant R_ε such that

$$\sup_{x\in\mathbb{R}^d}\int_{\overline{B(0,R_\varepsilon)}^c\cap F_\varepsilon} G(x,y)\mu(dy)\le\varepsilon.$$

We thus obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d}\int_{\overline{B(0,R_\varepsilon)}^c} G(x,y)\mu(dy)\\ &\le\sup_{x\in\mathbb{R}^d}\Big(\int_{F_\varepsilon^c} G(x,y)\mu(dy)+\int_{\overline{B(0,R_\varepsilon)}^c\cap F_\varepsilon} G(x,y)\mu(dy)\Big)\le 2\varepsilon.\end{aligned}$$

It holds that for $B\subset\overline{B(0,R_\varepsilon)}$ with $\mu(B)<\tilde{\delta}_\varepsilon$

$$\sup_{x\in\mathbb{R}^d}\int_B G(x,y)\mu(dy)\le\sup_{x\in\mathbb{R}^d}\Big(\int_{B\cap F_\varepsilon} G(x,y)\mu(dy)+\int_{F_\varepsilon^c} G(x,y)\mu(dy)\Big)\le 2\varepsilon.$$

Hence, (i) follows for $K_\varepsilon=\overline{B(0,R_{\frac{\varepsilon}{2}})}$ and $\delta_\varepsilon=\tilde{\delta}_{\frac{\varepsilon}{2}}$.

We easily see that (iii) follows from (i) by choosing the empty set as B and $\overline{B(0,r)}$ as K respectively.

If (iii) is valid, it follows that for arbitrary $\varepsilon>0$ there exists a sufficient large $r_\varepsilon>0$ such that

$$\sup_{x\in\mathbb{R}^d}\int_{|y|>r_\varepsilon} G(x,y)\mu(dy)\le\frac{\varepsilon}{2}.$$

Set $K_\varepsilon=\overline{B(0,r_\varepsilon)}$. Since $\mu\in\mathbf{K}$, (iii) of Proposition 2.2 implies that there exists a sufficient small positive constant a_ε such that

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le a_\varepsilon} G(x,y)\mu(dy)\le\frac{\varepsilon}{2}.$$

Thus, it holds that for a measurable set $A\subset K_\varepsilon$

$$\begin{aligned}\int_A G(x,y)\mu(dy)&\le\int_{B(x,a_\varepsilon)} G(x,y)\mu(dy)+\int_{A\cap B(x,a_\varepsilon)^c} G(x,y)\mu(dy)\\ &\le\frac{\varepsilon}{2}+C_1 g(a_\varepsilon)\mu(A).\end{aligned}\tag{2.42}$$

If we choose a sufficient small positive constant δ_ε, the second term of (2.42) can be smaller than $\varepsilon/2$ for any set A with $\mu(A)<\delta_\varepsilon$. Hence we have (i). □

Denote by $\hat{\mathbf{K}}_\infty$ the class of measures satisfying the conditions in (ii) of Proposition 2.3. This is the definition of Green-tight smooth measures of Kato class taken from [6]. Next we compare the definition of $\mathbf{S}_\infty$ with that in [6]. The following definition on the conditionally Green-tight smooth measures of Kato class is taken from Chen [6].

Definition 2.5. *The measure μ belongs to the class $\hat{\mathbf{S}}_\infty$ if for any $\varepsilon > 0$ there exist a set F_ε of μ-finite measure and $\tilde{\delta}_\varepsilon > 0$ such that*

$$\sup_{x,z\in\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta} \int_{F_\varepsilon^c} \frac{G(x,y)G(y,z)}{G(x,z)}\mu(dy) \le \varepsilon,$$

$\Delta = \{(x,x)\,;\, x\in\mathbb{R}^d\}$ *and for any $B\subset F_\varepsilon$ with $\mu(B) < \tilde{\delta}_\varepsilon$*

$$\sup_{x,z\in\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta} \int_{B} \frac{G(x,y)G(y,z)}{G(x,z)}\mu(dy) \le \varepsilon.$$

The following proposition can be proved in the same way as is used in the proof of (i) $\Rightarrow$ (ii) in Proposition 2.3.

Proposition 2.4. *It holds that $\mathbf{S}_\infty \subset \hat{\mathbf{S}}_\infty$.*

Proof. Let μ is in the class $\mathbf{S}_\infty$. Since $\mu\in\mathbf{K}$ from Definition 2.4, we can refrain the argument of the beginning of the proof for Proposition 2.3. In particular, we can choose K_ε and δ_ε in Definition 2.4 as F_ε and $\tilde{\delta}_\varepsilon$ respectively. □

The next theorem is proved by means of 3G-theorem.

Theorem 2.4. *It holds that $\mathbf{K}_\infty = \mathbf{S}_\infty = \hat{\mathbf{S}}_\infty$.*

Proof. Let g be a function in Corollary 2.1. Note that either $|x-y| \ge |x-z|/2$ or $|y-z| \ge |x-z|/2$ holds. If $|x-y| \ge |x-z|/2$, it follows that

$$\begin{aligned}\frac{G(x,y)G(y,z)}{G(x,z)} &\le \frac{C_2 g(|x-y|)}{C_1 g(|x-z|)}G(y,z)\\ &\le \frac{C_2 g(|x-z|/2)}{C_1 g(|x-z|)}G(y,z) \le C_3 G(y,z) = C_3 G(z,y)\end{aligned} \tag{2.43}$$

by Proposition 2.1. If $|y-z| \ge |x-z|/2$ we can similarly obtain

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le C_4 G(x,y). \tag{2.44}$$

Combining (2.43) and (2.44) we see that

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le C_5(G(x,y)+G(z,y)),$$

and thus $\mathbf{K}_\infty \subset \mathbf{S}_\infty$. Moreover, we see $\hat{\mathbf{S}}_\infty \subset \hat{\mathbf{K}}_\infty$ from Corollary 3.1 of [11] and p.4663 of [6]. It is also proved that $\hat{\mathbf{K}}_\infty \subset \mathbf{K}$ by Proposition 2.3 of [6]. Thus we obtain $\hat{\mathbf{S}}_\infty \subset \mathbf{K}_\infty$. Combining with Proposition 2.4, we obtain $\mathbf{K}_\infty \subset \mathbf{S}_\infty \subset \hat{\mathbf{S}}_\infty \subset \mathbf{K}_\infty$ and this is the desired assertion. □

We close this section introducing an important property of the measure in $\mathbf{K}_\infty$.

Corollary 2.2. *For $\mu \in \boldsymbol{K}_\infty$,*

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x[A_\infty^\mu] < \infty. \tag{2.45}$$

Proof. The left hand side of (2.45) is rewritten as follows:

$$\mathbb{E}_x[A_\infty^\mu] = \int_{\mathbb{R}^d} G(x,y)\mu(dy).$$

We see that the right hand side of the above formula is equal to

$$\int_{|x-y| \le a_\varepsilon} G(x,y)\mu(dy) + \int_{K_\varepsilon^c} G(x,y)\mu(dy) + \int_{K_\varepsilon \cap |x-y| > a_\varepsilon} G(x,y)\mu(dy), \tag{2.46}$$

where a_ε is a positive constant such that the first term of (2.46) is smaller than ε uniformly in $x \in \mathbb{R}^d$ and K_ε is a compact set taken from Definition 2.3. Thus the first and the second terms of (2.46) are uniformly bounded. Since K_ε is of finite μ-measure from Proposition 2.3 and $G(x,y)$ is bounded on $|x-y| \ge a_\varepsilon$, the third term of (2.46) is also uniformly bounded. Hence, we have the desired result. □

Remark 2.1. *In [6], the property (2.45) is proved for the wider class of measures $\boldsymbol{K}_1$. μ is said to be in the class $\boldsymbol{K}_1$ if there is a Borel set F of finite μ-measure and a constant $\tilde{\delta}$ such that*

$$\sup_{B \subset F : \mu(B) < \tilde{\delta}} \sup_{x \in \mathbb{R}^d} \int_{F^c \cup B} G(x,y)\mu(dy) < 1.$$

If $\mu \in \boldsymbol{K}_\infty$, we can make the left hand side arbitrarily small.

2.3 Stability of fundamental solutions

In this section we assume $\mu \in \mathbf{K}_\infty$ is positive and satisfies

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} G(x,y)\mu(dx)\mu(dy) < \infty. \tag{2.47}$$

μ is said to be of 0-order finite energy integral if μ satisfies (2.47). Consider the Schrödinger form $(\mathscr{E}^\mu, \mathscr{F})$ defined by

$$\mathscr{E}^\mu(u,u) := \mathscr{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu. \tag{2.48}$$

Denote by $\{P_t^\mu\}_{t \ge 0}$ the corresponding semigroup. It is known that P_t^μ is written by

$$P_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)],$$

where A_t^μ is a positive continuous additive functional in the Revuz correspondence with μ. Following [1, 2], we see that $\{P_t^\mu\}_{t \ge 0}$ admits the integral kernel $p^\mu(t,x,y)$ defined on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Our goal is to prove the following theorem:

Theorem 2.5. *Let $(\mathscr{E},\mathscr{F})$ be a transient Dirichlet form associated with α-stable-like process or relativistic α-stable-like process. Suppose μ is a Green-tight measure of 0-order finite energy integral. Then the stability of fundamental solution is valid if and only if μ satisfies*

$$\inf\left\{\mathscr{E}(u,u)\,;\,u\in\mathscr{F},\ \int_{\mathbb{R}^d}u^2d\mu=1\right\}>1. \tag{2.49}$$

For the proof of Theorem 2.5, the following proposition plays a crucial role.

Proposition 2.5. *For $\mu\in\boldsymbol{K}_\infty$, the following assertions are equivalent.*

(i) $G^\mu(x,y):=\int_0^\infty p^\mu(t,x,y)dt<\infty$ *for* $x\neq y$;

(ii) $\inf\{\mathscr{E}(u,u)\,;\,u\in\mathscr{F},\ \int_{\mathbb{R}^d}u^2d\mu=1\}>1$;

(iii) $\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[\exp(A^\mu_\infty)]<\infty$.

Proof. This proposition is proved for $\mu\in\mathbf{S}_\infty$ in Theorem 2.4 and Theorem 3.9 of [29]. Since we see that $\mathbf{K}_\infty=\mathbf{S}_\infty$ from Theorem 2.4, we obtain the desired result. □

We can prove the 'only if' part of Theorem 2.5 from equivalence between (i) and (ii) in Proposition 2.5. Suppose the stability of fundamental solution holds. Using the Green kernel estimates similarly obtained as in Proposition 2.1, we see that $G^\mu(x,y)<\infty$. This is equivalent to (2.49).
Before proving the 'if' part, we introduce the definition of gaugeability.

Definition 2.6. *The Green-tight measure μ is called* gaugeable *if A^μ_t satisfies (iii) of Proposition 2.5.*

Now we prove the 'if' part of Theorem 2.5. The equation (2.49) and Proposition 2.5 imply that

$$1\leq h(x):=\mathbb{E}_x[\exp(A^\mu_\infty)]\leq C_1<\infty.$$

In order to apply Theorem 3.4 in [12], we need to show that there exists $u\in\mathscr{F}_e$ such that $h(x)=\exp(u(x))$, where $\mathscr{F}_e$ is the extended Dirichlet space, namely the closure of $\mathscr{F}$ with respect to the $\mathscr{E}^{1/2}$-norm.

Lemma 2.1. *For $\mu\in\boldsymbol{K}_\infty$, define*

$$G\mu(x)=\int_{\mathbb{R}^d}G(x,y)\mu(dy).$$

Under assumption (2.47), $G\mu\in\mathscr{F}_e$.

Proof. This lemma is an extension of Lemma 3.1 in [31]. Let $\mu \in \mathbf{K}_\infty$. Then it holds that

$$\int_{\mathbb{R}^d} u^2 d\mu \leq \|G\mu\|_\infty \mathscr{E}(u,u) \tag{2.50}$$

for $u \in \mathscr{F}_e$. This is the modification of Theorem 3.1 in Stollmann and Voigt [27]. Define $\mu_K(\cdot) = \mu(K \cap \cdot)$ for a set K of μ-finite measure. Applying (2.50), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \psi d\mu_K &\leq (\mu(K))^{1/2}(\int_{\mathbb{R}^d} \psi^2 d\mu_K)^{1/2} \\ &\leq (\mu(K))^{1/2}\|G\mu_K\|_\infty^{1/2}\mathscr{E}(\psi,\psi)^{1/2}.\end{aligned}$$

By (2.45), we see $\|G\mu_K\|_\infty \leq \|G\mu\|_\infty = \sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_\infty^\mu] < \infty$ and consequently μ_K is of 0-order finite energy integral in the sense of [13]. We thus have

$$\begin{aligned}\int_{\mathbb{R}^d} \psi d\mu_K &\leq \mathscr{E}(G\mu_K, G\mu_K)^{1/2}\mathscr{E}(\psi,\psi)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)d\mu_K(x)d\mu_K(y)\right)^{1/2}\mathscr{E}(\psi,\psi)^{1/2}.\end{aligned}$$

We see that μ is also of 0-order finite energy integral and $G\mu \in \mathscr{F}_e$ by letting K to $\mathbb{R}^d$. □

Lemma 2.2. *Suppose $\mu \in \boldsymbol{K}_\infty$ is gaugeable. Then it holds that*

$$h(x) = 1 + G(h\mu)(x).$$

Proof. This is an extension of Lemma 3.2 of [31]. Let $\{\mathscr{M}_t\}_{t\geq 0}$ be the filtration equipped with the Hunt process $\{X_t\}_{t\geq 0}$. Set $\overline{M}_t = \mathbb{E}_x[\exp(A_\infty^\mu) \mid \mathscr{M}_t]$. By the Markov property, we have

$$h(X_t) = \mathbb{E}_{X_t}[\exp(A_\infty^\mu)] = \mathbb{E}_x[\exp(A_\infty^\mu \circ \theta_t)|\mathscr{M}_t]$$

$\mathscr{M}_t$-measurability of A_t^μ and the property of additive functionals imply

$$\begin{aligned}\mathbb{E}_x[\exp(A_\infty^\mu \circ \theta_t)|\mathscr{M}_t] &= \exp(-A_t^\mu)\mathbb{E}_x[\exp(A_t^\mu + A_\infty^\mu \circ \theta_t)|\mathscr{M}_t] \\ &= \exp(-A_t^\mu)\mathbb{E}_x[\exp(A_\infty^\mu) \mid \mathscr{M}_t] = \exp(-A_t^\mu)\overline{M}_t,\end{aligned}$$

where θ_t is the shift operator satisfying $X_{t+s} = X_s \circ \theta_t$ for all $s \geq 0$. Hence, we have

$$\mathbb{E}_x\left[\int_0^t h(X_s)dA_s^\mu\right] = \mathbb{E}_x\left[\int_0^t \exp(-A_s^\mu)\overline{M}_s dA_s^\mu\right]. \tag{2.51}$$

Noting that

$$\exp(-A_s^\mu)dA_s^\mu = -d(\exp(-A_s^\mu))$$

and using the integral by parts, the right hand side of (2.51) is equal to

$$\mathbb{E}_x[\overline{M}_0] - \mathbb{E}_x[\exp(-A_t^\mu)\overline{M}_t] + \mathbb{E}_x\left[\int_0^t \exp(-A_s^\mu)d\overline{M}_s\right] = h(x) - \mathbb{E}_x[h(X_t)],$$

where we use the martingale property of $\{\overline{M}_t\}_{t\geq 0}$ in the last equality. Thus, we have

$$\mathbb{E}_x\left[\int_0^t h(X_s)dA_s^\mu\right] = h(x) - \mathbb{E}_x[h(X_t)] \tag{2.52}$$

and note that

$$\lim_{t\to\infty} h(X_t) = \lim_{t\to\infty} \exp(-A_t^\mu)\overline{M}_t = \exp(-A_\infty^\mu)\exp(A_\infty^\mu) = 1.$$

We then have the desired result by letting $t \to \infty$ in (2.52). $\square$

Recall that $\mathscr{F}_e$ is a Hilbert space with inner product $\mathscr{E}^{1/2}$ if $\mathscr{E}$ is transient. Since h is a positive bounded function and $G\mu \in \mathscr{F}_e$, $G(h\mu) \in \mathscr{F}_e$. Moreover, define

$$u(x) := \log h(x) = \log(1 + G(h\mu)(x)).$$

Since $u(x)$ is a normal contraction of $G(h\mu)(x)$, $u \in \mathscr{F}_e$ and hence we conclude that $h(x) = \exp(u(x))$ for $u \in \mathscr{F}_e$.

Next, consider Fukushima's decomposition of $G(h\mu)$:

$$G(h\mu)(X_t) - G(h\mu)(X_0) = M_t^{[G(h\mu)]} + N_t^{[G(h\mu)]}, \tag{2.53}$$

where $M_t^{[G(h\mu)]}$ is a martingale additive functional of finite energy and $N_t^{[G(h\mu)]}$ is a continuous additive functional of zero energy. Since the left hand side of (2.53) equals $h(X_t) - h(X_0)$ by Lemma 2.2, $M_t^{[G(h\mu)]}$ equals $M_t^{[h]}$. Moreover we see from Lemma 5.4.1 of [13] that

$$N_t^{[G(h\mu)]} = -\int_0^t h(X_s)dA_s^\mu.$$

We thus have

$$h(X_t) - h(X_0) = M_t^{[h]} - \int_0^t h(X_s)dA_s^\mu.$$

Furthermore, we define a martingale by

$$M_t = \int_0^t \frac{1}{h(X_{s-})}dM_s^{[h]}$$

and denote by L_t the unique solution of Doleans-Dade equation: $L_t = 1 + \int_0^t L_{s-}dM_s$. From Theorem 9.39 of [15], we know that L_t is expressed as

$$L_t = \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right)\prod_{0<s\leq t}(1 + \Delta M_s)\exp(-\Delta M_s).$$

where M^c is the continuous part of martingale M and $\langle M^c \rangle$ is the quadratic variation of M^c. Noting that

$$\Delta M_s = M_s - M_{s-} = \frac{1}{h(X_{s-})}(h(X_s) - h(X_{s-})) = \frac{h(X_s)}{h(X_{s-})} - 1,$$

we obtain

$$L_t = \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t\right) \prod_{0<s\le t} \frac{h(X_s)}{h(X_{s-})} \exp\left(1 - \frac{h(X_s)}{h(X_{s-})}\right),$$

In order to calculate L_t, we apply Itô formula to the semimartingale $h(X_t)$ and the function $\log x$. Thus, we have

$$\begin{aligned} \log h(X_t) &= \log h(X_0) + \int_0^t \frac{1}{h(X_{s-})}(dM_s^{[h]} - h(X_s)dA_s^\mu) \\ &+ \sum_{0<s\le t}\left(\log h(X_s) - \log h(X_{s-}) - \frac{1}{h(X_{s-})}\Delta M_s^{[h]}\right) + \frac{1}{2}\int_0^t \frac{-1}{h(X_{s-})^2} d\langle h(X)^c \rangle_s. \end{aligned} \tag{2.54}$$

The right hand side of (2.54) is equal to

$$\begin{aligned} &\log h(X_0) + M_t - \int_0^t \frac{h(X_s)}{h(X_{s-})} dA_s^\mu \\ &\quad + \sum_{0<s\le t}\left\{\log \frac{h(X_s)}{h(X_{s-})} + \left(1 - \frac{h(X_s)}{h(X_{s-})}\right)\right\} - \frac{1}{2}\int_0^t \frac{1}{h(X_{s-})^2} d\langle M^{[h],c} \rangle_s. \end{aligned}$$

Noting that the set $\{s\,;\, X_s \neq X_{s-}\}$ is at most countable, the above formula is equal to

$$\log h(X_0) + M_t - A_t^\mu + \sum_{0<s\le t}\left\{\log \frac{h(X_s)}{h(X_{s-})} + \left(1 - \frac{h(X_s)}{h(X_{s-})}\right)\right\} - \frac{1}{2}\langle M^c \rangle_t$$

Hence we obtain

$$L_t = \frac{h(X_t)}{h(X_0)} \exp(A_t^\mu).$$

We consider the transformed semigroup $\{P_t^{\mu,h}\}_{t\ge 0}$ by L_t,

$$P_t^{\mu,h} f(x) = \mathbb{E}_x[L_t f(X_t)] = \frac{1}{h(x)}\mathbb{E}_x[h(X_t)\exp(A_t^\mu) f(X_t)].$$

We then know from Theorem 3.4 of [12] that the Dirichlet form generated by $\{P_t^{\mu,h}\}_{t\ge 0}$ is identified.

Proposition 2.6. *There exists a Dirichlet form $(\mathscr{E}^{\mu,h}, \mathscr{D}(\mathscr{E}^{\mu,h}))$ on $L^2(h^2dx)$ corresponding to the semigroup $\{P_t^{\mu,h}\}$ and it has the representation as follows:*

$$\mathscr{E}^{\mu,h}(u,v) = \int_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))(v(y)-v(x))J(x,y)h(x)h(y)dxdy, \tag{2.55}$$

$$\mathscr{D}(\mathscr{E}^{\mu,h}) = \mathscr{F}. \tag{2.56}$$

Using this proposition, we can prove the 'if' part of Theorem 2.5. Note that $P_t^{\mu,h}$ admits the integral kernel $h(x)^{-1}p^{\mu}(t,x,y)h(y)^{-1}$ with respect to the measure $h^2(y)dy$. Since $1 \le h(x) \le C_1$ for some positive constant C_1, we see that the form defined by (2.55)–(2.56) is the regular Dirichlet form on $L^2(\mathbb{R}^d)$ with jump measure $J_1(x,y) := J(x,y)h(x)h(y)$. Moreover, there exists positive constants C_2 and C_3 such that

$$\frac{C_2}{|x-y|^d\phi(|x-y|)} \le J_1(x,y) \le \frac{C_3}{|x-y|^d\phi(|x-y|)},$$

where $\phi(r) = r^{\alpha}$ ($\phi(r) = r^{\alpha}\exp(m_0 r)$) if $\{X_t\}_{t\ge 0}$ is α-stable-like (relativistic α-stable-like). Hence, we see that $h(x)^{-1}p^{\mu}(t,x,y)h(y)^{-1}$ has the same two sided estimates as those given in Theorems 2.2 and 2.3 for $p(t,x,y)$. Since $1 \le h(x) \le C_1$, so does $p^{\mu}(t,x,y)$.

The following example is constructed by Takeda and Uemura [36].

Example 2.1. *Let σ_r be a surface measure of sphere $\partial B_r = \{|x| = r\}$. Since the symmetric α-stable process hits the sphere ∂B_r if $1 < \alpha \le 2$(see* e.g., *E44.19 in [23]), the measure σ_r is in K_∞. The measure σ_r is then gaugeable if and only if*

$$\inf\{\mathscr{E}(u,u) : \int_{\{|x|=r\}} u^2 d\sigma_r = 1\} > 1.$$

Since the measure σ_r is spherically symmetric, the infimum is attained by the function

$$u(x) = c\,\mathbb{P}_x(\sigma_{\partial B_r} < \infty),\ x \in \mathbb{R}^d,$$

where $c = 1/\sqrt{\sigma_r(\partial B_r)}$. Let $\mathrm{Cap}^{(\alpha)}(\cdot)$ be the 0*-order capacity with respect to the symmetric α-stable process. Then the infimum above equals to*

$$\frac{\mathrm{Cap}^{(\alpha)}(\partial B_r)}{\sigma_r(\partial B_r)},$$

because

$$\mathscr{E}^{(\alpha)}(\mathbb{P}_\cdot(\sigma_{\partial B_r} < \infty), \mathbb{P}_\cdot(\sigma_{\partial B_r} < \infty)) = \mathrm{Cap}^{(\alpha)}(\partial B_r).$$

It is known (see e.g., *Corollary 2.2 in [22]) that*

$$\mathrm{Cap}^{(\alpha)}(\partial B_r) = \frac{2\pi^{(d+1)/2}\Gamma\left(\frac{\alpha+d}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)} r^{d-\alpha}.$$

Therefore, the measure σ_r is gaugeable if and only if

$$\left\{\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)}\right\}^{\frac{1}{\alpha-1}} > r.$$

We close this chapter introducing the extension of the stability by Kim and Kuwae.

Remark 2.2. *The assumption that μ is of 0-order finite energy integral is unnecessary. Kim and Kuwae [16] extends the argument of h-transformation to more general class of function. Moreover, Kim and Kuwae [17] proved the stability of fundamental solutions for perturbations containing non-local parts.*

Chapter 3

Critical Schrödinger forms and spectral functions

Let $\{X_t\}_{t\geq 0}$ be a transient jump Markov process on $\mathbb{R}^d$ and μ be a green-tight measure in the Kato class. In the previous chapter, we established the necessary and sufficient condition on the measure μ for the fundamental solution to satisfy the stability i.e. $p^\mu(t,x,y)$ has the same two sided estimates as $p(t,x,y)$ up to positive constants. This condition is said to be subcriticality of μ, which describes the smallness of measure μ. If μ is not subcritical, there are two cases to be considered: critical case and supercritical case. In these two cases, we expect that $p^\mu(t,x,y)$ has different behavior from $p(t,x,y)$. We have not determined the estimates of $p^\mu(t,x,y)$ however, there are some papers mentioning the estimates of the integral $\int_{\mathbb{R}^d} p^\mu(t,x,y)dy$. Note that this integral is equal to the expectation $\mathbb{E}_x[\exp(A_t^\mu)]$. Furthermore, we see from the previous section that μ is subcritical if and only if $\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[\exp(A_\infty^\mu)] < \infty$. Hence, the expectation $\mathbb{E}_x[\exp(A_t^\mu)]$ diverges as $t\to\infty$ if μ is critical or supercritical. To know the growth of $\mathbb{E}_x[\exp(A_t^\mu)]$, we first consider the limit defined by

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x[\exp(A_t^\mu)].$$

Takeda [30, 33] and Tsuchida [37] proved this limit is equal to the spectral bound as follows:

$$C(\mu) := -\inf\left\{\mathscr{E}^\mu(u,u)\ u\in\mathscr{F},\ \int_{\mathbb{R}^d} u^2(x)dx = 1\right\}.$$

[30] treats the standard Brownian motion, [33] treats the rotationally invariant α-stable process and [37] treats the relativistic α-stable process. It is known that $C(\mu)>0$ if μ is supercritical, and hence $\mathbb{E}_x[\exp(A_t^\mu)]$ grows exponentially. If μ is critical, $C(\mu)=0$ and the growth is slower than the exponential one. Simon [25] and Cranston, Kolokoltsov and et. al. [8] considered the same problem when $\{X_t\}_{t\geq 0}$ is the transient Brownian motion and $\mu = V\cdot m$ for non-negative $V\in C_0^\infty(\mathbb{R}^d)$. The growth order of the expecta-

tion $\mathbb{E}_x[\exp(A_t^\mu)]$ depends on the dimension d and satisfies

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim \begin{cases} C_1 h(x) \cdot t^{1/2} & (d=3) \\ C_2 h(x) \cdot t/\log t & (d=4) \\ C_3 h(x) \cdot t & (d \geq 5), \end{cases}$$

where $h(x)$ is the harmonic function of $(-\Delta - V)$. In the sequel, we assume that μ is critical and $\{X_t\}_{t\geq 0}$ is the rotationally invariant α-stable process. Takeda [34] proved that $\mathbb{E}_x[\exp(A_t^\mu)]$ is proportional to t if $d/\alpha > 2$. This is an analogy of the d-dimensional Brownian motion for $d \geq 5$. The outline of the proof is based on the probabilistic theory and as follows:

First we rewrite the expectation as follows:

$$\mathbb{E}_x[\exp(A_t^\mu)] = 1 + \mathbb{E}_x\left[\int_0^t \exp(A_s^\mu) dA_s^\mu\right].$$

Next we consider the transformation of Schrödinger semigroup by the harmonic function $h(x)$. In the critical case, we can also construct the harmonic function following the argument of Takeda and Tsuchida [35]. Using the h-transformed semigroup defined by

$$\mathbb{E}_x^h[f(X_t)] = \mathbb{E}_x\left[\frac{h(X_t)}{h(X_0)} \exp(A_t^\mu) f(X_t)\right],$$

we conclude that

$$\mathbb{E}_x[\exp(A_t^\mu)] = 1 + \mathbb{E}_x\left[\int_0^t \exp(A_s^\mu) dA_s^\mu\right] = 1 + h(x)\mathbb{E}_x^h\left[\int_0^t \frac{dA_s^\mu}{h(X_s)}\right].$$

Moreover, [35] proved that the harmonic function $h(x)$ satisfies

$$C_1(1 \wedge |x|^{\alpha-d}) \leq h(x) \leq C_2(1 \wedge |x|^{\alpha-d}).$$

The function $h(x)$ is in $L^2(\mathbb{R}^d)$ for $d/\alpha > 2$. Thus, the transformed process is an ergodic process with the finite invariant measure $h^2 \cdot m$ and consequently, we obtain

$$\frac{1}{t}\mathbb{E}_x^h\left[\int_0^t \frac{dA_s^\mu}{h(X_s)}\right] \to \int_{\mathbb{R}^d} h(x)\mu(dx) \qquad (t \to \infty).$$

Hence we determine the growth of $\mathbb{E}_x[\exp(A_t^\mu)]$.

However, we cannot apply this method for the case $d/\alpha \leq 2$ because $h(x)$ is not in $L^2(\mathbb{R}^d)$. Thus, we use other analytical method based on the argument of Simon [25]. In this chapter, we consider the case $\{X_t\}_{t\geq 0}$ is the rotationally invariant 1-stable process on $\mathbb{R}^2$ and μ is absolutely continuous with respect to the Lebesgue measure. This chapter is organized as follows: In Section 1, we calculate the transition density function of $\{X_t\}_{t\geq 0}$ and give the asymptotic expansion of the resolvent $\{G_\beta\}$ when β tends to 0. In Section 2 we consider the asymptotic behavior of $\{G_\beta^\mu\}$, namely the resolvent of Schrödinger form. Furthermore we apply the Tauberian theorem, which describes the relation between the asymptotic behavior of resolvent and that of semigroup. In Section 3 we consider the behavior of the spectral function. For fixed measure μ, the spectral function $C(\lambda)$ is defined as $C(\lambda) = C(\lambda\mu)$, where $C(\lambda\mu)$ stands for the spectral bound of $\lambda\mu$. Takeda and Tsuchida [35] established the criterion of differentiability of $C(\lambda)$. Using the asymptotic expansion obtained in Section 2, we can determine the precise behavior of $C(\lambda)$.

3.1 The asymptotic behavior of the resolvent

We first calculate concrete transition density function and resolvent of the Markov process. Let $\{X_t\}_{t\geq 0}$ be the rotationally invariant α-stable process. The characteristic function of $\{X_t\}_{t\geq 0}$satisfies

$$\mathbb{E}_0[\exp(iu \cdot X_t)] = \exp(-t|u|^{\alpha}). \tag{3.1}$$

Using the transition density function, (3.1) is rewritten as follows:

$$\int_{\mathbb{R}^d} \exp(iu \cdot (y-x))p(t,x,y)dy = \exp(-t|u|^{\alpha}).$$

Furthermore, we obtain the explicit transition density function by the inverse Fourier transform:

$$p(t,x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(-t|u|^{\alpha} - iu \cdot (y-x))du. \tag{3.2}$$

Proposition 3.1. *Let $p(t,x,y)$ be the heat kernel of 1-stable process on $\mathbb{R}^2$. Then it follows that*

$$p(t,x,y) = \frac{t}{2\pi(t^2 + |x-y|^2)^{\frac{3}{2}}}. \tag{3.3}$$

Proof. We have only to substitute $d = 2$ and $\alpha = 1$ in (3.2). Using the polar coordinates, we obtain

$$\begin{aligned} p(t,x,y) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(-t|u| + i(x-y) \cdot u)du \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} r \exp(-tr + ir|x-y| \cos\theta)drd\theta \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \frac{d\theta}{(t-i|x-y|\cos\theta)^2} = \frac{1}{2\pi^2} \int_0^{\pi} \frac{d\theta}{(t-i|x-y|\cos\theta)^2}. \end{aligned}$$

Moreover we can calculate the integral of the right hand side as follows:

$$\int_0^{\pi} \frac{d\theta}{(t-i|x-y|\cos\theta)^2} = \int_0^{\frac{\pi}{2}} \Big(\frac{1}{(t+i|x-y|\cos\theta)^2} + \frac{1}{(t-i|x-y|\cos\theta)^2} \Big) d\theta$$

$$
\begin{aligned}
&= 2\int_0^{\frac{\pi}{2}} \frac{t^2-|x-y|^2\cos^2\theta}{(t^2+|x-y|^2\cos^2\theta)^2}d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \frac{(C-B\cos 2\theta)}{(A+B\cos 2\theta)^2}d\theta = \int_0^{\pi} \frac{(C-B\cos\theta)}{(A+B\cos\theta)^2}d\theta, \qquad (3.4)
\end{aligned}
$$

where $A=t^2+|x-y|^2/2, B=|x-y|^2/2$ and $C=t^2-|x-y|^2/2$. Here we consider the integrals as follows:

$$
\int_0^{\pi} \frac{1}{(A+B\cos\theta)^2}d\theta, \qquad \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(A+B\cos\theta)^2}d\theta.
$$

The former is calculated as follows:

$$
\begin{aligned}
\int_0^{\pi} \frac{1}{(A+B\cos\theta)^2}d\theta &= \int_0^{\infty} \Big(A+B\cdot\frac{1-t^2}{1+t^2}\Big)^{-2}\cdot\frac{2}{1+t^2}dt \\
&= \int_0^{\infty} \frac{2(1+t^2)}{((A+B)+(A-B)t^2)^2}dt \\
&= \int_0^{\infty} \Big(\frac{2/(A-B)}{(A+B)+(A-B)t^2} - \frac{4B/(A-B)}{((A+B)+(A-B)t^2)^2}\Big)dt.
\end{aligned}
$$

Here we used the substitution of $\tan(\theta/2)=t$ in the first equality. Since both $A+B$ and $A-B$ are positive, the right hand side of the above formula is equal to

$$
\frac{2}{(A-B)^2}\cdot\Big(\frac{A-B}{A+B}\Big)^{\frac{1}{2}}\cdot\frac{\pi}{2} - \frac{4B}{(A-B)^3}\cdot\Big(\frac{A-B}{A+B}\Big)^{\frac{3}{2}}\cdot\frac{\pi}{4} = \pi\cdot\frac{A}{(A-B)^{\frac{3}{2}}(A+B)^{\frac{3}{2}}}. \qquad (3.5)
$$

The latter is calculated as follows:

$$
\begin{aligned}
\int_0^{\pi} \frac{\cos\theta}{(A+B\cos\theta)^2}d\theta &= \int_0^{\infty} \Big(A+B\cdot\frac{1-t^2}{1+t^2}\Big)^{-2}\cdot\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}dt \\
&= \int_0^{\infty} \frac{2(1-t^2)}{((A+B)+(A-B)t^2)^2}dt \\
&= \int_0^{\infty} \Big(\frac{-2/(A-B)}{(A+B)+(A-B)t^2} + \frac{4A/(A-B)}{((A+B)+(A-B)t^2)^2}\Big)dt \\
&= -\frac{2}{(A-B)^2}\cdot\Big(\frac{A-B}{A+B}\Big)^{\frac{1}{2}}\cdot\frac{\pi}{2} + \frac{4A}{(A-B)^3}\cdot\Big(\frac{A-B}{A+B}\Big)^{\frac{3}{2}}\cdot\frac{\pi}{4} \\
&= \pi\cdot\frac{-B}{(A-B)^{\frac{3}{2}}(A+B)^{\frac{3}{2}}}. \qquad (3.6)
\end{aligned}
$$

(3.5) and (3.6) imply that the right hand side of (3.4) is equal to

$$
\pi\cdot\frac{AC+B^2}{(A-B)^{\frac{3}{2}}(A+B)^{\frac{3}{2}}} = \pi\cdot\frac{t^4}{t^3(t^2+|x-y|^2)^{\frac{3}{2}}} = \pi\cdot\frac{t}{(t^2+|x-y|^2)^{\frac{3}{2}}}.
$$

Thus, we obtain (3.3). □

We next consider the behavior of the β-order resolvent when β tends to 0.

Lemma 3.1. *The β-order resolvent kernel $G_\beta(x,y)$ has the asymptotic expansion as follows:*

$$G_\beta(x,y) = G_0(x,y) + \frac{1}{2\pi}\beta\log\beta + \frac{\gamma - \log 2 + \log|x-y|}{2\pi}\beta + \mathscr{O}(\beta^2). \tag{3.7}$$

Proof. (3.3) implies that

$$G_\beta(x,y) = \int_0^\infty p(t,x,y)e^{-\beta t}dt = \frac{1}{2\pi}\int_0^\infty \frac{te^{-\beta t}}{(t^2+|x-y|^2)^{\frac{3}{2}}}dt.$$

Substituting $t = |x-y|\sinh z$, we have

$$\begin{aligned} G_0(x,y) &= \frac{1}{2\pi}\int_0^\infty \frac{t}{(t^2+|x-y|^2)^{\frac{3}{2}}}dt = \frac{1}{2\pi}\int_0^\infty \frac{|x-y|\sinh z}{|x-y|^3\cosh^3 z}\cdot|x-y|\cosh z dz \\ &= \frac{1}{2\pi|x-y|}\int_0^\infty \frac{\sinh z}{\cosh^2 z}dz = \frac{1}{2\pi|x-y|} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} G_\beta(x,y) &= \frac{1}{2\pi}\int_0^\infty \frac{te^{-\beta t}}{(t^2+|x-y|^2)^{\frac{3}{2}}}dt = \frac{1}{2\pi|x-y|}\int_0^\infty \frac{\sinh z}{\cosh^2 z}e^{-\beta|x-y|\sinh z}dz \\ &= \frac{1}{2\pi|x-y|}\left\{\left[\frac{-1}{\cosh z}e^{-\beta|x-y|\sinh z}\right]_{z=0}^\infty - \int_0^\infty \beta|x-y|e^{-\beta|x-y|\sinh z}dz\right\} \\ &= \frac{1}{2\pi|x-y|} - \frac{\beta}{2\pi}\int_0^\infty e^{-\beta|x-y|\sinh z}dz = G_0(x,y) - \frac{\beta}{2\pi}\int_0^\infty e^{-\beta|x-y|\sinh z}dz. \end{aligned} \tag{3.9}$$

Now we evaluate the second term of (3.9). We see that

$$\begin{aligned} \int_0^\infty e^{-\beta|x-y|\sinh z}dz &= \int_0^\infty \exp\Big(-\frac{\beta|x-y|}{2}e^z\Big)\cdot\exp\Big(\frac{\beta|x-y|e^{-z}}{2}\Big)dz \\ &= \int_0^\infty \exp\Big(-\frac{\beta|x-y|}{2}e^z\Big)\cdot\sum_{n=0}^\infty \frac{1}{n!}\Big(\frac{\beta|x-y|e^{-z}}{2}\Big)^n dz \end{aligned}$$

Using the Fubini's theorem, the right hand side of the above formula is equal to

$$\sum_{n=0}^\infty \int_0^\infty \frac{(\beta|x-y|)^n}{2^n\cdot n!}\exp\Big(-nz - \frac{\beta|x-y|}{2}e^z\Big)dz =: \sum_{n=0}^\infty I_n,$$

First we consider the integral I_0. Substituting $w = \beta|x-y|e^z/2$, we see that

$$I_0 = \int_{\frac{\beta|x-y|}{2}}^\infty \frac{e^{-w}}{w}dw = -\log\Big(\frac{\beta|x-y|}{2}\Big) - \gamma - \sum_{n=1}^\infty \frac{(-\beta|x-y|)^n}{2^n\cdot n\cdot n!}, \tag{3.10}$$

where γ is Euler's Gamma and we use the residue integral as follows:

$$\int_x^\infty \frac{e^{-w}}{w} dw = -\log x - \gamma - \sum_{n=1}^\infty \frac{(-x)^n}{n \cdot n!}.$$

Furthermore, for $n \geq 1$, I_n has the upper bound as follows:

$$I_n \leq \frac{(\beta|x-y|)^n}{2^n \cdot n!} \int_0^\infty e^{-nz} dz = \frac{(\beta|x-y|)^n}{2^n \cdot n \cdot n!}. \tag{3.11}$$

Thus, for fixed $x, y \in \mathbb{R}^d$, we obtain $\sum_{n=1}^\infty I_n = \mathcal{O}(\beta)$ and

$$\int_0^\infty e^{-\beta|x-y|\sinh z} dz = -\log\Big(\frac{\beta|x-y|}{2}\Big) - \gamma + \mathcal{O}(\beta) \quad (\beta \to 0). \tag{3.12}$$

Hence we obtain (3.7). □

3.2 Growth of Feynman-Kac semigroups

Let $(\mathcal{E}, \mathcal{F})$ be the symmetric jump Dirichlet form associated with $\{X_t\}_{t\geq 0}$, i.e.

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} (u(y) - u(x))^2 \frac{\kappa_0}{|x-y|^3} dxdy, \quad \mathcal{F} = H^{\frac{1}{2}}(\mathbb{R}^2),$$

where κ_0 is an appropriate positive constant and $H^{\frac{1}{2}}(\mathbb{R}^2)$ is the Sobolev space with order $1/2$. Let H is a non-local operator satisfying

$$\mathcal{E}(u,v) = (Hu, v),$$

where the right hand side is the inner product of $L^2(\mathbb{R}^2)$.

Let μ be a Green-tight measure in Definition 2.3 and define the Schrödinger form by

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu.$$

We divide the class of Green-tight measures into three subclasses according to the smallness of measures.

Definition 3.1. *For $\mu \in \boldsymbol{K}_\infty$, define*

$$\lambda(\mu) := \inf\{\mathcal{E}(u,u) \,;\, u \in \mathcal{F}_e, \int_{\mathbb{R}^d} u^2 d\mu = 1\}.$$

(1) *μ is said to be subcritical if $\lambda(\mu) > 1$.*

(2) *μ is said to be critical if $\lambda(\mu) = 1$.*

(3) *μ is said to be supercritical if $\lambda(\mu) < 1$.*

We proved that the stability of fundamental solution is equivalent to the subcriticality of μ in Theorem 2.5. Thus we see that the behavior of $p^{\mu}(t,x,y)$ is different from that of $p(t,x,y)$ when μ is critical or supercritical. We have not determined the behavior of $p^{\mu}(t,x,y)$. Instead, we consider the behavior of

$$\mathbb{E}_x[\exp(A_t^{\mu})] = \int_{\mathbb{R}^d} p^{\mu}(t,x,y)dy. \tag{3.13}$$

Note that the subcriticality of μ is equivalent to the gaugeability of A_t^{μ}, namely,

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[\exp(A_{\infty}^{\mu})] < \infty$$

Thus, the value $\mathbb{E}_x[\exp(A_t^{\mu})]$ diverges as t tends to ∞ in the other cases. First we define the spectral bound $C(\mu)$ by

$$C(\mu) = -\inf\{\mathscr{E}^{\mu}(u,u)\ ;\ u \in \mathscr{F}, \int_{\mathbb{R}^d} u^2(x)dx = 1\}. \tag{3.14}$$

The following theorem is given in Theorem 5.2 of [33].

Theorem 3.1. *For Green-tight measure μ, it follows that*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x[\exp(A_t^{\mu})] = C(\mu).$$

Moreover we can characterize supercriticality of μ by using the spectral bound $C(\mu)$. The following proposition is proved in Lemma 2.2 of [35].

Proposition 3.2. *The measure μ is supercritical if and only if $C(\mu) > 0$.*

Combining Theorem 3.1 and Proposition 3.2, we can obtain the following corollary.

Corollary 3.1. *Suppose the Green-tight measure μ is supercritical. $\mathbb{E}_x[\exp(A_t^{\mu})]$ has exponential growth as $t \to \infty$.*

If μ is critical, $C(\mu) = 0$ and thus $\mathbb{E}_x[\exp(A_t^{\mu})]$ does not have the exponential growth. We first introduce two items: one is the ground state, equivalently the harmonic function with respect to Schrödinger operator $H-\mu$ and the other is the result for Brownian motion.

The following theorem is proved in Theorem 3.4 of Takeda and Tsuchida [35].

Theorem 3.2. *For $\mu \in \boldsymbol{K}_{\infty}$, the extended Dirichlet space $\mathscr{F}_e$ is compactly embedded into $L^2(\mu)$*

Thus we see that there exists a $h_0 \in \mathscr{F}_e$ such that

$$\mathscr{E}(h_0,h_0) = \lambda(\mu), \qquad \int_{\mathbb{R}^d} h_0^2 d\mu = 1.$$

We call h_0 *ground state of the Schrödinger form $\mathscr{E}^{\mu}$*. Moreover, they showed the precise behavior of h_0 as follows:

Lemma 3.2. *The function $h_0(x)$ is continuous and there exist positive constants C_1 and C_2 such that*

$$C_1\left(1\wedge\frac{1}{|x|}\right)\le h_0(x)\le C_2\left(1\wedge\frac{1}{|x|}\right).$$

Proof. We know from Lemma 4.7 and Proposition 4.14 of [35] that the ground state $h_0(x)$ satisfies

$$C_1\left(1\wedge\frac{1}{|x|^{d-\alpha}}\right)\le h_0(x)\le C_2\left(1\wedge\frac{1}{|x|^{d-\alpha}}\right)$$

for the rotationally invariant α-stable process on $\mathbb{R}^d$. Since $d=2$ and $\alpha=1$, the assertion follows. □

Our goal is to prove the following theorem:

Theorem 3.3. *Suppose $\{X_t\}_{t\ge 0}$ is the 1-stable process on $\mathbb{R}^2$ and μ is critical. Assume $\mu=V\cdot m$ for positive $V\in C_0^\infty(\mathbb{R}^2)$. Then there exists a positive constant C_1 such that*

$$\lim_{t\to\infty}\frac{\log t}{t}\mathbb{E}_x[\exp(A_t^\mu)]=C_1h_0(x), \tag{3.15}$$

where $h_0(x)$ is the ground state of the Schrödinger operator $H-V$.

In the sequel, we consider the growth of $\mathbb{E}_x[\exp(A_t^\mu)]$ as $t\to\infty$ when V is critical. Denote the semigroup and the resolvent associated with $\mathscr{E}^\mu$ by $\{P_t^\mu\}$ and $\{G_\beta^\mu\}$ respectively. Recall that

$$P_t^\mu f(x)=\mathbb{E}_x[\exp(A_t^\mu)f(X_t)]$$

and consequently $P_t^\mu 1(x)=\mathbb{E}_x[\exp(A_t^\mu)]$.

Proposition 3.3. *The semigroup $\{P_t^\mu\}$ satisfies the following formula:*

$$P_t^\mu 1=1+\int_0^t P_s^\mu V ds. \tag{3.16}$$

Proof. Noting that the absolute continuity of the measure μ, we have

$$\exp(A_t^\mu)-1=\int_0^t\exp(A_s^\mu)dA_s^\mu=\int_0^t\exp(A_s^\mu)V(X_s)ds.$$

Thus, it follows that

$$\begin{aligned}P_t^\mu f(x)&=\mathbb{E}_x\left[\left(1+\int_0^t\exp(A_s^\mu)V(X_s)ds\right)f(X_t)\right]\\&=P_tf(x)+\mathbb{E}_x\left[\int_0^t\exp(A_s^\mu)V(X_s)f(X_t)ds\right]\end{aligned}$$

$$= P_t f(x) + \int_0^t \mathbb{E}_x[\exp(A_s^\mu)V(X_s)f(X_t)]ds.$$

Using the Markov property, we can rewrite the above formula as follows:

$$\begin{aligned} & P_t f(x) + \int_0^t \mathbb{E}_x[\mathbb{E}_x[\exp(A_s^\mu)V(X_s)f(X_t)|\mathscr{F}_s]] \\ =& P_t f(x) + \int_0^t \mathbb{E}_x[\exp(A_s^\mu)V(X_s)\mathbb{E}_x[f(X_t)|\mathscr{F}_s]] \\ =& P_t f(x) + \int_0^t P_s^\mu (VP_{t-s}f)(x)ds. \end{aligned}$$

Hence, we obtain

$$P_t^\mu f(x) = P_t f(x) + \int_0^t P_s^\mu (VP_{t-s}f)(x)ds.$$

If we substitute f by the constant 1, the conservativeness of $\{X_t\}_{t\geq 0}$ implies (3.16). $\square$

In order to evaluate $P_s^\mu V$, we consider its Laplace transform, $G_\beta^\mu V$. The following proposition is proved by the resolvent equation.

Proposition 3.4. *It follows that*

$$G_\beta^\mu V(x) = (1 - L_\beta)^{-1}(G_\beta V)(x), \tag{3.17}$$

where $G_\beta V$ is defined by

$$G_\beta V(x) = \int_{\mathbb{R}^2} G_\beta(x,y)V(y)dy$$

and L_β is an operator defined by $L_\beta f(x) = G_\beta(Vf)(x)$.

Proof. Since $G_\beta = (H+\beta)^{-1}$ and $G_\beta^\mu = (H-V+\beta)^{-1}$, we obtain

$$(H-V+\beta)^{-1} - (H+\beta)^{-1} = (H+\beta)^{-1}V(H-V+\beta)^{-1}$$

by the resolvent equation. If we solve this equation with respect to $(H-V+\beta)^{-1}$, we have

$$(1-(H+\beta)^{-1}V)(H-V+\beta)^{-1} = (H+\beta)^{-1}$$

and consequently

$$(H-V+\beta)^{-1} = (1-(H+\beta)^{-1}V)^{-1}(H+\beta)^{-1}.$$

If we operate the function V on both sides, we can have the desired result. $\square$

For the calculation of the resolvent G_β^μ, we have to consider the behavior of the function $G_\beta V$ and operator L_β. We begin with function $G_\beta V$.

Lemma 3.3. *For $\beta \geq 0$ and $V \in C_0^\infty(\mathbb{R}^2)$, $G_\beta V$ belongs to the class $L^\infty(\mathbb{R}^2)$ and converges uniformly to $G_0 V$ as $\beta \to 0$.*

Proof. Since the integral kernels $\{G_\beta(x,y)\}_{\beta\geq 0}$ is increasing as $\beta \to 0$, it suffices to prove $G_0 V \in L^\infty(\mathbb{R}^2)$. We know from (3.8) that

$$G_0(x,y) = \frac{C_1}{|x-y|}$$

Set $R > 0$ satisfying $supp(V) \subset B(0,R)$, where $B(0,R)$ is a ball in $\mathbb{R}^2$ with center O and radius R. If $|x| > 2R$, it follows that

$$G_0 V(x) \leq \frac{C_1}{|x|-R}\int_{\mathbb{R}^2} V(y)dy \leq \frac{C_2}{|x|-R}. \tag{3.18}$$

As for $|x| \leq 2R$, we obtain

$$G_0 V(x) \leq \int_{|x-y|\leq 3R} \frac{V(y)}{|x-y|} \leq \|V\|_\infty \int_{|x-y|\leq 3R} \frac{dy}{|x-y|} \leq C_3. \tag{3.19}$$

Thus we can conclude $G_0 V \in L^\infty(\mathbb{R}^d)$ and so is $G_\beta V$.
Next we prove $\lim_{\beta\to 0} \|G_0 V - G_\beta V\|_\infty = 0$. We may assume that $|x|$ is bounded. Indeed, for any $\varepsilon > 0$, (3.18) implies that there exists $R_\varepsilon > 0$ such that $G_0 V(x) < \varepsilon$ for $|x| > R_\varepsilon$. We may also assume that the support of V is contained in $B_0(R_\varepsilon)$. We reconsider the error term in asymptotic expansion of the resolvent kernel as follows:

$$G_0(x,y) - G_\beta(x,y) = \frac{\beta}{2\pi}\int_0^\infty e^{-\beta|x-y|\sinh z}dz. \tag{3.20}$$

Estimations in (3.10) and (3.11) imply

$$\beta\int_0^\infty e^{-\beta|x-y|\sinh z}dz \leq -\beta\log\Big(\frac{\beta|x-y|}{2}\Big) - \beta\gamma + 2\beta\exp\Big(\frac{\beta|x-y|}{2}\Big) \tag{3.21}$$

Noting that the third term in (3.12) is uniformly for $x, y \in \mathbb{R}^2$ satisfying $|x-y| \leq 2R_\varepsilon$, we obtain

$$\begin{aligned}
|G_0 V(x) - G_\beta V(x)| &\leq \int_{B_0(R_\varepsilon)} \Big(-\beta\log\Big(\frac{\beta|x-y|}{2}\Big) + 2\beta\exp\Big(\frac{\beta|x-y|}{2}\Big)\Big)V(y)dy \\
&\leq -\beta\log\beta\int_{B_0(R_\varepsilon)} V(y)dy \\
&\quad + \beta\int_{B_0(R_\varepsilon)} \Big(\Big|\log\Big(\frac{|x-y|}{2}\Big)\Big| + 2\exp\Big(\frac{\beta|x-y|}{2}\Big)\Big)V(y)dy
\end{aligned}$$

If we choose β sufficiently small according to R_ε, the right hand side of the previous formula is bounded by ε. Therefore we have proved that $\|G_0 V - G_\beta V\|_\infty \to 0$ as $\beta \to 0$. □

We next consider the behavior of operators $\{L_\beta\}_{\beta\geq 0}$.

Lemma 3.4. *The operator L_β defined by $f \to G_\beta(Vf)$ is a compact operator from $L^\infty(\mathbb{R}^2)$ to $L^\infty(\mathbb{R}^2)$. Furthermore it is continuous with respect to norm when $\beta \to 0$.*

Proof. Since the multiple operator V is compact and the resolvent operator $G_\beta(\beta > 0)$ is bounded, L_β is a compact operator. Furthermore, L_β converges to L_0 in the space of bounded operators on $L^\infty(\mathbb{R}^2)$. Indeed, for any $f \in L^2(\mathbb{R}^2)$ with $\|f\|_\infty = 1$, $V(x)f(x)$ has a compact support. Thus, using the same argument of Lemma 3.3 we obtain

$$\|L_\beta f - L_0 f\|_\infty = \|G_\beta(Vf) - G_0(Vf)\|_\infty \to 0 \qquad (\beta \to 0).$$

Moreover, the space of compact operators is a closed subspace in the space of bounded operators. Hence, L_0 is also a compact operator and we obtain the desired result. □

Next we consider the operator defined on $L^2(\mathbb{R}^2)$.

Lemma 3.5. *Define the operator $K_\beta : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by*

$$K_\beta f(x) = V^{\frac{1}{2}} G_\beta(V^{\frac{1}{2}} f)(x)$$

Then e_β is an eigenvalue of L_β if and only if it is an eigenvalue of K_β.

Proof. Suppose the function g satisfies $L_\beta g = e_\beta g$ for some $e_\beta > 0$. For $h = V^{\frac{1}{2}} g$, it follows that $K_\beta h = e_\beta h$. □

The following lemma is based on Theorem 6.1 of Klaus and Simon [19].

Lemma 3.6. *Suppose μ is critical and $\mu = V \cdot m$ for positive $V \in C_0^\infty(\mathbb{R}^2)$. Denote by e_β the principal eigenvalue of K_β. Then e_β has the expansion as follows:*

$$e_\beta = 1 + C_1 \beta \log \beta + \dots \quad (C_1 > 0). \tag{3.22}$$

Proof. Note that $\lambda\mu$ is supercritical for $\lambda > 1$. For $\lambda > 1$ and $\beta > 0$, consider the equation

$$(H - \lambda V) u_\lambda = -\beta u_\lambda. \tag{3.23}$$

(3.23) is rewritten as follows:

$$(H + \beta) u_\lambda = \lambda V u_\lambda$$

Noting that $(H+\beta)^{-1}$ is the resolvent operator with order β, we have

$$\lambda^{-1} u_\lambda = G_\beta(V u_\lambda)$$

Moreover, if we substitute $w_\lambda = V^{\frac{1}{2}} u_\lambda$, we have

$$\lambda^{-1} w_\lambda = V^{1/2} G_\beta(V^{1/2} w_\lambda),$$

Thus (3.23) is equivalent to the equation $K_\beta w_\lambda = \lambda^{-1} w_\lambda$. Since $G_\beta(x,y)$ satisfies asymptotic expansion (3.7), the operator K_β has asymptotic expansion as follows:

$$K_\beta = K_0 + \beta \log \beta D_1 + \beta D_2 + \dots \tag{3.24}$$

where D_1 and D_2 are operators defined by

$$D_1 f(x) = \frac{V^{\frac{1}{2}}(x)}{2\pi} \cdot \int_{\mathbb{R}^d} V^{\frac{1}{2}}(y) f(y) dy$$
$$D_2 f(x) = \frac{\gamma - \log 2}{2\pi} V^{\frac{1}{2}}(x) \cdot \int_{\mathbb{R}^d} V^{\frac{1}{2}}(y) f(y) dy + \frac{V^{\frac{1}{2}}(x)}{2\pi} \cdot \int_{\mathbb{R}^d} \log|x-y| V^{\frac{1}{2}}(y) f(y) dy.$$

The expansion (3.24) is the same form as that of 4-dimensional Brownian motion treated in Klaus and Simon [19]. Moreover $V^{1/2}h_0$ satisfies $K_0(V^{1/2}h_0) = V^{1/2}h_0$ where h_0 is the ground state of $\mathscr{E}^\mu$ defined in Lemma 3.2. Since h_0 is positive, $(V^{1/2}, V^{1/2}h_0) \neq 0$ and the proof of Theorem 6.1 in [19] implies

$$e_\beta = 1 + C_1 \beta \log \beta + \dots$$

for $C_1 \neq 0$. Moreover, noting that $e_\beta = \lambda^{-1} < 1$, we see $C_1 > 0$. □

Now we obtain the behavior of the principal eigenvalue of the compact operator L_β. We consider the decomposition of the operator $(1 - L_\beta)^{-1}$ as follows:

Lemma 3.7. *Denote by P_β the projection operator associated with L_β. Then it holds that*

$$(1 - L_\beta)^{-1} = (1 - e_\beta)^{-1} P_\beta + Q_\beta (1 - P_\beta)$$

where Q_β is norm continuous with finite limit as $\beta \to 0$.

Proof. Since L_β is a compact operator for $\beta > 0$, L_β has a spectral decomposition. Denote by e'_β the second largest eigenvalue of L_β. Then $e'_0 < e_0 = 1$ and $\|Q_\beta\|_\infty$ is smaller than $2(1 - e'_0)^{-1}$ if β is sufficiently small. □

The previous lemma and (3.17) imply that

$$\lim_{\beta \to 0} \|G^\mu_\beta V - (1 - e_\beta)^{-1} P_\beta G_\beta V\|_\infty < \infty.$$

Moreover $P_0 \psi = h_0 l(\psi)$, where l is a suitable linear functional and h_0 is the harmonic function satisfying $(H - V)h_0 = 0$. Thus, combining (3.22), we have

$$\lim_{\beta \to 0} -\beta \log \beta G^\mu_\beta(V) = C_1 h_0(x) \tag{3.25}$$

where C_1 is some positive constant. Note that $C_1 \neq 0$ follows because $(V, h_0) \neq 0$. Now we obtain the behavior of $G^\mu_\beta V$ as $\beta \to 0$. Applying the Tauberian theorem in [24], we can prove Theorem 3.3:

Proof of Theorem 3.3 First we fix $x \in \mathbb{R}^2$ and define the class of measures $\{\nu_\beta^x\}_{\beta>0}$ on $\mathbb{R}_+ := [0,\infty)$ by

$$\nu_\beta^x(A) = -\beta\log\beta \int_{\beta^{-1}A} P_s^\mu V(x)ds = -\beta\log\beta \int_A P_{\beta^{-1}s}^\mu V(x)\cdot\beta^{-1}ds.$$

Denote the Lebesgue measure on $\mathbb{R}_+$ by ν. We have only to prove that $\nu_\beta^x([0,1])$ converges to $C_1h_0(x)\nu([0,1])$ as $\beta\to 0$ up to positive constant multiple. Indeed, it follows that

$$\nu_\beta^x[0,1] = -\beta\log\beta\int_0^{\beta^{-1}} P_s^\mu V(x)ds$$

and substituting $\beta^{-1}=t$, we obtain the desired result. First, (3.25) implies

$$\begin{aligned}-\beta\log\beta\int_0^\infty e^{-\beta t}P_t^\mu V(x)dt &= -\beta\log\beta\int_0^\infty e^{-s}P_{\beta^{-1}s}^\mu V(x)\cdot\beta^{-1}ds\\ &= \int_0^\infty e^{-s}d\nu_\beta^x(s)\to C_1h_0(x). \quad (\beta\to 0)\end{aligned}$$

Moreover we see that the $C_1h_0(x) = C_1h_0(x)\int_0^\infty e^{-s}\nu(ds)$ and thus it follows that

$$\lim_{\beta\to 0}\int_0^\infty e^{-s}d\nu_\beta^x(s) = C_1h_0(x)\int_0^\infty e^{-s}d\nu(s).$$

For integer n, we consider

$$\begin{aligned}\lim_{\beta\to 0}\int_0^\infty e^{-ns}d\nu_\beta^x(s) &= \lim_{\beta\to 0}\int_0^\infty e^{-ns}\cdot(-\beta\log\beta)P_{\beta^{-1}s}^\mu V(x)\cdot\beta^{-1}ds\\ &= \lim_{\beta\to 0}\int_0^\infty e^{-t}\cdot(-\beta\log\beta)P_{(n\beta)^{-1}t}^\mu V(x)\cdot(n\beta)^{-1}dt. \qquad (3.26)\end{aligned}$$

Noting that

$$\lim_{\beta\to 0}\int_0^\infty e^{-t}\cdot(-n\beta\log n\beta)P_{(n\beta)^{-1}t}^\mu V(x)\cdot(n\beta)^{-1}dt = \lim_{\beta\to 0}\int_0^\infty e^{-t}d\nu_{n\beta}^x(t) = C_1h_0(x),$$

we obtain

$$\lim_{\beta\to 0}\int_0^\infty e^{-t}\cdot(-\beta\log\beta)P_{(n\beta)^{-1}t}^\mu V(x)\cdot(n\beta)^{-1}dt = \frac{C_1h_0(x)}{n} = C_1h_0(x)\int_0^\infty e^{-ns}d\nu(s).$$

Thus, for $f\in\{e^{-ns}\}_{n\in\mathbb{N}}$, it follows that

$$\int_0^\infty f(s)d\nu_\beta^x(s)\to C_1h_0(x)\int_0^\infty f(s)d\nu(s) \quad (\beta\to 0). \qquad (3.27)$$

Since polynomials in $\{e^{-ns}\}_{n\in\mathbb{N}}$ are dense in $C_\infty(\mathbb{R}_+)$ by the Stone-Weierstrass theorem, (3.27) is valid for $f\in C_\infty(\mathbb{R}_+)$. Finally we prove (3.27) for $f(s) = 1_{[0,1]}(s)$, namely the main assertion $\nu_\beta^x[0,1] \to C_1 h_0(x)\nu[0,1]$. Here $1_{[0,1]}(s)$ is the indicator function of the set $[0,1]$. Set the sequences of functions $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ by

$$f_n(s) = \begin{cases} 1 \quad (0 \leq s \leq 1) \\ 1 - n(s-1) \quad (1 \leq s \leq 1 + 1/n) \\ 0 \quad (1 + 1/n \leq s), \end{cases}$$

$$g_n(s) = \begin{cases} 1 \quad (0 \leq s \leq 1 - 1/n) \\ n(1-s) \quad (1 - 1/n \leq s \leq 1) \\ 0 \quad (1 \leq s) \end{cases}$$

Note that these functions satisfies

$$\begin{aligned} 0 \leq g_n(s) \leq 1_{[0,1]}(s) \leq f_n(s) \leq 1, \quad (s \in \mathbb{R}_+) \\ \lim_{n\to\infty} f_n(s) = 1_{[0,1]}(s) \quad (s \in \mathbb{R}_+) \\ \lim_{n\to\infty} g_n(s) = 1_{[0,1]}(s) \quad (s \in \mathbb{R}_+ \backslash \{1\}). \end{aligned}$$

Then it follows that

$$\nu_\beta^x[0,1] = \int_{\mathbb{R}_+} 1_{[0,1]}(s)\nu_\beta^x(ds) \leq \int_{\mathbb{R}_+} f_n(s)\nu_\beta^x(ds)$$

Noting that $f_n \in C_\infty(\mathbb{R}_+)$ and $\lim_{\beta\to 0}\int_0^\infty f_n d\nu_\beta^x = C_1 h_0(x)\int_0^\infty f_n d\nu$, we obtain for arbitrary $n \in \mathbb{N}$

$$\limsup_{\beta\to 0} \nu_\beta^x[0,1] \leq C_1 h_0(x)\int_0^\infty f_n d\nu. \tag{3.28}$$

Letting $n \to \infty$ in (3.28), the dominated convergence theorem implies the right hand side converges to $\nu[0,1]$ and thus we have

$$\limsup_{\beta\to 0} \nu_\beta^x[0,1] \leq C_1 h_0(x)\nu[0,1]. \tag{3.29}$$

Conversely, it follows that

$$\nu_\beta^x[0,1] \geq \int_0^\infty g_n(s)\nu_\beta^x(ds)$$

and thus we obtain

$$\liminf_{\beta\to 0} \nu_\beta^x[0,1] \geq C_1 h_0(x)\int_0^\infty g_n(s)\nu(ds). \tag{3.30}$$

Note that $\lim_{n\to\infty} g_n(s) = 1_{[0,1]}(s)$ for $s \neq 1$. However, using $\nu(\{1\}) = 0$, this convergence is valid for ν-a.e. s. Thus, applying the dominated convergence theorem in the right hand side of (3.30), we have

$$\liminf_{\beta\to 0} \nu_\beta^x[0,1] \geq C_1 h_0(x)\nu[0,1]. \tag{3.31}$$

Combining (3.29) and (3.31), we have $\lim_{\beta\to 0} \nu_\beta^x[0,1] = C_1 h_0(x)\nu[0,1]$. □

Remark 3.1. *(i) Theorem 3.3 is an extension of the result for 4-dimensional Brownian motion. The result by Takeda [34] is an extension of d-dimensional Brownian motion for $d \geq 5$. Thus the growth of $\mathbb{E}_x[\exp(A_t^\mu)]$ seems to depend on d/α.*

(ii) Since we use an analytic method in the proof, we assume the absolute continuity of μ. However, Takeda [34] treated general measures. Thus, it seems to extend the result to the case of general measures.

3.3 Differentiability of spectral functions

In the previous section we define the spectral bound of μ by formula (3.14). For $\lambda \geq 0$, we define the spectral function $C(\lambda)$ by

$$C(\lambda) = -\inf\left\{\mathscr{E}(u,u) - \lambda \int_{\mathbb{R}^d} u^2 d\mu \,|\, \int_{\mathbb{R}^d} u^2(x)dx = 1\right\}. \tag{3.32}$$

Suppose μ is critical and $\mu = V \cdot m$ for $V \in C_0^\infty(\mathbb{R}^2)$. Lemma 3.6 shows that (3.22) is valid for $e_\beta = \lambda^{-1}$ and $\beta = C(\lambda)$ for $\lambda \geq 1$. Thus, if we apply the inverse function theorem, we can determine the behavior of the spectral function. In [35], Takeda and Tsuchida gave the criterion for the differentiability of the spectral function as follows:

Theorem 3.4. *Let $\{X_t\}_{t\geq 0}$ be the rotationally invariant α-stable process on $\mathbb{R}^d$ and μ be a Green-tight, smooth measure in the Kato class. The spectral function $C(\lambda)$ is differentiable if and only if $1 < d/\alpha \leq 2$.*

We justify their result considering the concrete behavior of $C(\lambda)$ in the case of $d = 2, \alpha = 1$.

Theorem 3.5. *The spectral function $C(\lambda)$ satisfies*

$$C(\lambda) \asymp c\frac{1-\lambda^{-1}}{-\log(1-\lambda^{-1})} \qquad \lambda \to 1. \tag{3.33}$$

In particular, it follows

$$\lim_{\lambda\to 1} \frac{1-\lambda^{-1}}{-(\lambda-1)\log(1-\lambda^{-1})} = \lim_{\lambda\to 1} \frac{\lambda^{-1}}{\log(1-\lambda^{-1})} = 0$$

and hence $C(\lambda)$ is differentiable at $\lambda = 1$, which has consistency with Theorem 3.4.

In order to prove this theorem, we begin with the inverse function theorem containing logarithm.

Proposition 3.5. *Let f and g be analytic function near $\xi = 0$ and real-valued for ξ real. Suppose $f(0) = f'(0) = 0$ and $g(0) = g'(0) = 0$ and $g''(0) < 0$ and $\eta = f(\xi) + g(\xi)\log\xi$. Then $\xi(\eta)$ satisfies*

$$\xi(\eta) = \sum_{n\geq 1, m,k\geq 0} C_{n,m,k}\sigma^n\tau^m\omega^k,$$

$$\sigma = \left(\frac{\eta}{\log\eta^{-1}}\right)^{\frac{1}{2}}, \quad \tau = \frac{1}{\log\eta^{-1}}, \quad \omega = \frac{\log(\log\eta^{-1})}{\log\eta^{-1}}$$

Proof. Without loss of generality we may assume that

$$f(\xi)=\sum_{n=2}^{\infty}a_n\xi^n, \qquad g(\xi)=-2\xi^2\log\xi+\log\xi\sum_{n=3}^{\infty}b_n\xi^n$$

It follows that

$$\eta=\sum_{n=2}^{\infty}a_n\xi^n+\log\xi\sum_{n=3}^{\infty}b_n\xi^n \tag{3.34}$$

and substitute $\xi=\sigma(1+z)$. (3.34) is rewritten as follows:

$$\begin{aligned}\eta&=\sum_{n=2}^{\infty}a_n\sigma^n(1+z)^n-2\sigma^2(1+z)^2\log(\sigma(1+z))+\log(1+z)\sum_{n=3}^{\infty}b_n\sigma^n(1+z)^n\\&=\sigma^2\Big(\sum_{n=0}^{\infty}a_{n+2}\sigma^n(1+z)^{n+2}-2(1+z)^2\log(\sigma(1+z))\\&+\sigma\log(\sigma(1+z))\sum_{n=0}^{\infty}b_{n+3}\sigma^{n-3}(1+z)^n\Big)\end{aligned} \tag{3.35}$$

Noting that $\sigma^2=\eta/(\log\eta^{-1})$, we have

$$\begin{aligned}\log\eta^{-1}&=\sum_{n=0}^{\infty}a_{n+2}\sigma^n(1+z)^{n+2}-2(1+z)^2\log(\sigma(1+z))\\&+\sigma\log(\sigma(1+z))\sum_{n=0}^{\infty}b_{n+3}\sigma^{n-3}(1+z)^n\end{aligned} \tag{3.36}$$

Multiplying τ in the both side of (3.36), we obtain

$$\begin{aligned}1&=\tau\sum_{n=0}^{\infty}a_{n+2}\sigma^n(1+z)^{n+2}-2(1+z)^2\tau\log(\sigma(1+z))\\&+\sigma\tau\log(\sigma(1+z))\sum_{n=0}^{\infty}b_{n+3}\sigma^n(1+z)^{n+3}\end{aligned} \tag{3.37}$$

$\log(\sigma(1+z))$ is rewritten as follows:

$$\begin{aligned}\log\sigma+\log(1+z)&=\frac{1}{2}(\log\eta-\log(\log\eta^{-1}))+\log(1+z)\\&=-\frac{1+\omega}{2\tau}+\log(1+z)\end{aligned}$$

Thus, we obtain

$$\begin{aligned}1&=\tau\sum_{n=0}^{\infty}a_{n+2}\sigma^n(1+z)^{n+2}+(1+z)^2(1+\omega)\\&-2(1+z)^2\tau\log(1+z)-\Big(\frac{\sigma}{2}+\frac{\sigma\omega}{2}-\sigma\tau\log(1+z)\Big)\sum_{n=0}^{\infty}b_{n+3}\sigma^n(1+z)^{n+3}\end{aligned}$$

Set $F(z,\sigma,\tau,\omega)$ by

$$F(z,\sigma,\tau,\omega) = z^2+2z+(1+z)^2\omega-2\tau(1+z)^2\log(1+z)$$
$$+\tau\sum_{n=0}^{\infty}a_{n+2}\sigma^n(1+z)^{n+2}-\Big(\frac{\sigma}{2}+\frac{\sigma\omega}{2}-\sigma\tau\log(1+z)\Big)\sum_{n=0}^{\infty}b_{n+3}\sigma^n(1+z)^{n+3}$$

Then we see that $F(z,\sigma,\tau,\omega)=0$ and $\partial F/\partial z(0,0,0,0)\neq 0$, and thus we conclude

$$z=\sum_{n,m,k\geq 0}d_{n,m,k}\sigma^n\tau^m\omega^k$$

for z sufficiently near 0. Noting that $\xi=\sigma(1+z)$, we have the desired result. □

Now we prove Theorem 3.5.

Proof of Theorem 3.5 In order to apply the inverse function theorem, we need to substitute $\beta=\xi^2$ and $e_\beta=\eta$ in (3.22). Thus we obtain

$$\eta=1+C_1\xi^2\log\xi+C_2\xi^2+\dots$$

and $C_1>0$. Applying Proposition 3.5, we obtain the expansion of ξ as follows:

$$\xi=\sum_{n\geq 1,m,k\geq 0}d_{n,m,k}\sigma^n\tau^m\omega^k \tag{3.38}$$

where

$$\sigma=\Big(\frac{1-\eta}{\log(1-\eta)^{-1}}\Big)^{\frac{1}{2}}\qquad \tau=\frac{1}{\log(1-\eta)^{-1}}\qquad \omega=\frac{\log(\log(1-\eta)^{-1})}{\log(1-\eta)^{-1}}$$

Since $\eta\to 1$ as $\xi\to 0$, the principal term of (3.38) is $d_{1,0,0}\sigma$. Noting that $\eta=e_\beta=\lambda^{-1}$ and $C(\lambda)=\xi^2$, we obtain the desired result. □

Chapter 4

Continuity of harmonic functions for non-local Markov generators

We treated harmonic functions for Schrödinger operator in Chapters 2 and 3. Here we consider the harmonic function for Markov generator $\mathscr{L}$. We assume that $\mathscr{L}$-martingale problem is well-posed. We denote the associated jump process by $\{X_t\}_{t\geq 0}$. That is to say, the following two conditions are satisfied;

- For all $x \in \mathbb{R}^d$, $\mathbb{P}_x(X_0 = x) = 1$.
- For all $f \in C_b^2$, $\{f(X_t) - f(x) - \int_0^t \mathscr{L} f(X_s)ds\}_{t\geq 0}$ is a $\mathbb{P}_x$-martingale.

We define $\mathscr{L}$-harmonic function as a bounded function u such that $\{u(X_t)\}_{t\geq 0}$ is a martingale. Note that u satisfies $\mathscr{L}u = 0$. If $\mathscr{L}$ is a uniform elliptic second order operator, $\{X_t\}_{t\geq 0}$ is the diffusion process and harmonic function u is continuous. In this chapter, we consider the same problem for non-local Markov generators. Thus, $\mathscr{L}$ is given by

$$\mathscr{L}u(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+h) - u(x) - h \cdot \nabla u(x) 1_{\{|h|\leq 1\}}\big) n(x,h) dh,$$

where $n(x,h)$ is a non-negative measurable function on $\mathbb{R}^d \times \mathbb{R}^d$. The function $n(x,h)$ is called a jump measure or a jump density from x to $x+h$. When $n(x,h)$ satisfies

$$\frac{C_1}{|h|^{d+\alpha}} \leq n(x,h) \leq \frac{C_2}{|h|^{d+\alpha}}$$

for some constants $0 < C_1 \leq C_2$ and $0 < \alpha < 2$, the associated jump process is called the α-stable-like process in Chen and Kumagai [9]. Bass and Levin [5] is one of the earliest papers which deals with this class of jump processes with non-smooth $n(x,h)$. They establish elliptic Harnack inequalities and the Hölder continuity for harmonic functions. In [26], Song and Vondraček consider a wider class of jump processes, such as a sum of symmetric stable processes with different order.

In general, the exponent α depends on x and h, and the associated operator is of variable order. One of the difficulties of variable order case is that we cannot use scaling property, unlike the stable-like case. It is a very delicate problem whether harmonic functions for operators of variable order are continuous or not. Indeed, Barlow and coauthors [3] prove that harmonic functions are not necessarily continuous in general. To guarantee the continuity of harmonic functions, we impose some conditions on $n(x,h)$. Bass and Kassmann [4] established sufficient conditions for the continuity of harmonic functions. These conditions are divided to two parts:

(SSJ) Singularity of small jumps, i.e. how the amount of jumps with size r grows as r tends to 0.

(QRI) Quasi rotationally invariance, i.e. how the process jumps in any direction to some extent.

Husseini and Kassmann [14] treated weaker conditions than those of Bass and Kassmann [4]. In this chapter we reconsider conditions (SSJ) and (QRI) to extend their results. This chapter is organized as follows: In Section 1, we review a main result by Bass and Kassmann [4] which describes the sufficient condition for Hölder continuity of harmonic functions. We refer where the conditions (SSJ) and (QRI) contributes to the Hölder continuity. In Section 2, we consider the condition (SSJ) and extend the result of Bass and Kassmann [4]. In Section 3, we weaken the condition (QRI) and prove the continuity of harmonic functions.

4.1 Preceding results on continuous harmonic functions

Let $\mathscr{L}$ be a non-local operator of form (4.1) given by

$$\mathscr{L}u(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+h) - u(x) - h\cdot\nabla u(x) 1_{\{|h|\le 1\}}\big) n(x,h)dh, \tag{4.1}$$

and suppose the jump measure $n(x,h)$ satisfies

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (|h|^2\wedge 1) n(x,h)dh < \infty. \tag{4.2}$$

Let A be a Borel set. For the process $\{X_t\}_{t\ge 0}$, denote the first exiting time of A by τ_A and the first hitting time of A by T_A respectively; i.e.

$$\tau_A := \inf\{t>0 : X_t \notin A\}, \qquad T_A := \inf\{t>0 : X_t \in A\}.$$

Now we give the definition of harmonic functions.

Definition 4.1. *The bounded function u on $\mathbb{R}^d$ is called harmonic with respect to $\mathscr{L}$ on $D\subset\mathbb{R}^d$ if $\{u(X_{t\wedge\tau_D})\}_{t\ge 0}$ is a martingale.*

Roughly speaking, a harmonic function u satisfies $\mathscr{L}u(x) = 0$ for $x\in D$.

The following definition of $S(x,r)$, $L(x,r)$ and $N(x,r)$ are taken from Bass and Kassmann [4].

$$S(x,r) = \int_{|h|\geq r} n(x,h)dh,$$
$$L(x,r) = S(x,r) + \frac{1}{r}\Big|\int_{r\leq|h|\leq 1} h\, n(x,h)dh\Big| + \frac{1}{r^2}\int_{|h|\leq r} |h|^2 n(x,h)dh,$$
$$N(x,r) = \inf\left\{\int_{h\in A-x} n(x,h)dh \ :\ A\subset B(x,2r) \text{ and } |A| \geq \frac{|B(x,r)|}{3\cdot 2^d}\right\},$$

where $B(x,r)$ and $|A|$ stand for an open ball in $\mathbb{R}^d$ centered at x with radius r and the Lebesgue measure of A respectively. The following assumption is also taken from Bass and Kassmann [4].

Assumption 4.1. *Let R_0 be a constant such that $0 < R_0 < 1$. Suppose the following conditions hold.*

(SSJ-1) There exist $\kappa_1 > 0$ and $\sigma > 0$ such that for all $x \in \mathbb{R}^d$, $r \in (0,R_0)$ and $1 < \lambda < 1/r$,

$$\frac{S(x,\lambda r)}{S(x,r)} \leq \kappa_1 \lambda^{-\sigma}.$$

(QRI-1) There exists $\kappa_2 > 0$ such that for all $x \in \mathbb{R}^d$, $r \in (0,R_0)$ and $y \in B(x,2r)$,

$$N(x,r) \geq \kappa_2 L(y,r/2).$$

In [4], Bass and Kassmann prove the Hölder continuity of harmonic functions under the assumptions (SSJ-1) and (QRI-1). Precise statement of their theorem is the following.

Theorem 4.1. *([4] Theorem2.2.) Let R_0 be a constant such that $0 < R_0 < 1$. Suppose $0 < R < R_0$ and let u be bounded in $\mathbb{R}^d$ and harmonic in $B(z_0,R)$ with respect to $\mathscr{L}$. Then, under the assumptions (SSJ-1) and (QRI-1), there exist $\nu \in (0,1)$ and $C > 0$ depending only on κ_1, κ_2, and σ such that*

$$|u(z) - u(y)| \leq C\|u\|_\infty \Big(\frac{|z-y|}{R}\Big)^\nu, \qquad z,\, y \in B(z_0,R/3). \tag{4.3}$$

Define $L_0(x_0,r)$ and $N_0(x_0,r)$ as follows:

$$L_0(x_0,r) := \sup_{x\in B(x_0,r)} L(x,r). \qquad N_0(x_0,r) := \inf_{x\in B(x_0,r)} N(x,r)$$

For the proof of Theorem 4.1, we begin with three lemmas.

Proposition 4.1. *([4] Proposition 3.1.) There exists C_1 such that*

$$\mathbb{P}_{x_0}(\tau_{B(x_0,r)} < t) \leq C_1 t L_0(x_0,r).$$

Proof. Let u be a $C^2(\mathbb{R}^d)$ function with bounded first and second partial derivatives that satisfies the following;

- $u(x) = |x-x_0|^2/r^2$ for $|x-x_0| \le r$.
- $1 \le u(x) \le C_2$ for $|x-x_0| > r$.
- $|\nabla u| \le C_3/r$.
- The second partial derivatives are bounded by C_4/r^2.

Since $\mathbb{P}_{x_0}$ is a solution to the martingale problem for $\mathscr{L}$ started x_0,

$$\mathbb{E}_{x_0}[u(X_{t\wedge\tau_{B(x_0,r)}})] - u(x_0) = \mathbb{E}_{x_0}\left[\int_0^{t\wedge\tau_{B(x_0,r)}} \mathscr{L}u(X_s)ds\right].$$

The left hand side is larger than $\mathbb{P}_{x_0}(\tau_{B(x_0,r)} \le t)$, while the right hand side is bounded from above by

$$t \sup_{x\in B(x_0,r)} |\mathscr{L}u(x)| \le (C_2+C_3+C_4)tL_0(x_0,r).$$

We thus obtain the assertion. □

Denote the left limit of X_t by X_{t-} and set $\Delta X_t := X_t - X_{t-}$. Let Q be a Borel set and $U := \inf\{t : |\Delta X_t| \ge r\}$ for $r < R_0$. We then have the following.

Proposition 4.2. *([4] Proposition 3.3.) Under the condition (SSJ-1), we have*

$$\mathbb{P}_x(|\Delta X_{U\wedge\tau_Q}| \ge \lambda r) \le \kappa_1 \lambda^{-\sigma} \tag{4.4}$$

for $r < R_0, 1 < \lambda < 1/r$ and $x \in \mathbb{R}^d$.

Proof. By the Lévy system formula, for every bounded stopping time S and disjoint Borel sets B and C, we have

$$\mathbb{E}_y\left[\sum_{s\le S} 1_{(X_{s-}\in B, X_s\in C)}\right] = \mathbb{E}_y\left[\int_0^S 1_B(X_s)n(X_s, C-X_s)ds\right]. \tag{4.5}$$

Using this formula and the condition (SSJ-1), for arbitrary $t > 0$ we have

$$\begin{aligned}\mathbb{P}_x(|\Delta X_{U\wedge\tau_Q\wedge t}| \ge \lambda r) &= \mathbb{E}_x\left[\sum_{s\le U\wedge\tau_Q\wedge t} 1_{(|\Delta X_s|\ge\lambda r)}\right] \\ &= \mathbb{E}_x\left[\int_0^{U\wedge\tau_Q\wedge t}\int_{|h|\ge\lambda r} n(X_s,h)dhds\right] \\ &= \mathbb{E}_x\left[\int_0^{U\wedge\tau_Q\wedge t} S(X_s,\lambda r)ds\right] \le \kappa_1\lambda^{-\sigma}\mathbb{E}_x\left[\int_0^{U\wedge\tau_Q\wedge t} S(X_s,r)ds\right] \quad (4.6)\end{aligned}$$

By the Lévy system formula, the right hand side of (4.6) is estimated as follows:

$$\kappa_1\lambda^{-\sigma}\mathbb{E}_x\left[\int_0^{U\wedge\tau_Q\wedge t} S(X_s,r)ds\right] = \kappa_1\lambda^{-\sigma}\mathbb{E}_x\left[\int_0^{U\wedge\tau_Q\wedge t}\int_{|h|\ge r} n(X_s,h)dhds\right]$$

$$= \kappa_1 \lambda^{-\sigma} \mathbb{E}_x \left[\sum_{s \le U \wedge \tau_Q \wedge t} 1_{(|\Delta X_s| \ge r)} \right] = \kappa_1 \lambda^{-\sigma} \mathbb{P}_x(|\Delta X_{U \wedge \tau_Q \wedge t}| \ge r) \le \kappa_1 \lambda^{-\sigma}.$$

Thus, we obtain

$$\mathbb{P}_x(|\Delta X_{U \wedge \tau_Q \wedge t}| \ge \lambda r) \le \kappa_1 \lambda^{-\sigma}.$$

Now letting $t \to \infty$ and using the dominated convergence theorem, the assertion (4.4) follows. □

Proposition 4.3. *([4] Proposition 3.2.) Suppose the condition (QRI-1) holds. Suppose* $0 < r < R_0, A \subset B(x_0, r), y \in B(x_0, r/2)$*, and* $|A|/|B(x_0, r)| \ge 1/(3 \cdot 2^d)$*. Then, there exists* $\kappa_5 > 0$ *not depending on* x_0, r*,or* A *such that*

$$\mathbb{P}_y(T_A < \tau_{B(x_0,r)}) \ge \kappa_5.$$

Proof. Set $\tau = \tau_{B(x_0,r)}$. If $\mathbb{P}_y(T_A < \tau) \ge 1/4$, there is nothing to prove. Therefore, we may assume $\mathbb{P}_y(T_A < \tau) \le 1/4$ without loss of generality. Note that $B(x_0, r) \supset B(y, r/2)$. Combining with Proposition 4.1, we have

$$\mathbb{P}_y(\tau \le t_0) \le \mathbb{P}_y(\tau_{B(y,r/2)} \le t_0) \le C_1 t_0 L_0(y, r/2).$$

Therefore, for sufficiently small C_2 if we set

$$t_0 = \frac{C_2}{L_0(y, r/2)},$$

then $\mathbb{P}_y(\tau \le t_0) \le 1/2$.
Moreover, if $x \in B(x_0, r)$, we have

$$n(x, A - x) := \int_{A-x} n(x, h) dh \ge N_0(x_0, r). \tag{4.7}$$

Combining this estimate with the Lévy system formula (4.5), we obtain

$$\begin{aligned} \mathbb{P}_y(T_A \le \tau) &\ge \mathbb{E}^y \left[\sum_{s \le T_A \wedge \tau \wedge t_0} 1_{\{X_{s-} \ne X_s,\, X_s \in A\}} \right] \\ &\ge \mathbb{E}_y \left[\int_0^{T_A \wedge \tau \wedge t_0} n(X_s, A - X_s) ds \right] \ge N_0(x_0, r) \mathbb{E}_y(T_A \wedge \tau \wedge t_0). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \mathbb{E}_y(T_A \wedge \tau \wedge t_0) &\ge t_0 \mathbb{P}_y(T_A \ge \tau \ge t_0) \\ &\ge t_0[1 - \mathbb{P}_y(T_A < \tau) - \mathbb{P}_y(\tau < t_0)] \ge \frac{t_0}{4}. \end{aligned}$$

Therefore, we conclude

$$\mathbb{P}_y(T_A < \tau) \ge \frac{t_0}{4} N_0(x_0, r) = \frac{C_2}{4} \cdot \frac{N_0(x_0, r)}{L_0(y, r/2)} \ge \frac{C_2 \kappa_2}{4}$$

and the assertion follows, where in the last inequality we used (QRI-1). □

These three propositions are the keys to prove Theorem 4.1.

Proof of Theorem 4.1. Suppose $\|u\|_\infty \le K$ and choose $0 < \rho_0 < \frac{1}{4}$ appropriately. Set

$$r_n := \frac{R}{12}\rho_0^{n-1} \quad (n \in \mathbb{N}).$$

Then, we have $B(z_1, 2r_1) \subset B(z_0, R/2)$ for arbitrary $z_1 \in B(z_0, R/3)$. Set B_n, M_n and m_n as follows.

$$B_n := B(z_1, r_n), \qquad M_n := \sup_{x \in B_n} u(x), \qquad m_n := \inf_{x \in B_n} u(x).$$

Let $a < 1$ and $\theta_1 > 2K$ be positive constants chosen later appropriately. Set

$$s_n = \theta_1 a^n \quad (n \in \mathbb{N}).$$

We will prove by induction that

$$M_n - m_n \le s_n \qquad (\forall n \in \mathbb{N}). \tag{4.8}$$

We assume that $M_k - m_k \le s_k$ for $1 \le k \le n$ and consider the case of $k = n+1$. Choose $y, z \in B_{n+1}$ such that

$$u(z) \ge M_{n+1} - \varepsilon, \qquad u(y) \le m_{n+1} + \varepsilon,$$

where ε is an arbitrary positive number. Set

$$A_n = \{z \in B_n \,:\, u(z) \le (M_n + m_n)/2\}.$$

Without loss of generality, we may assume $|A_n|/|B_n| \ge 1/2$. (If $|A_n|/|B_n| < 1/2$, consider the function $-u(x)$ instead of $u(x)$.)
Choose a compact set D_n such that

$$D_n \subset A_n, \qquad |D_n|/|B_n| \ge 1/3.$$

Note that Proposition 4.3 implies $\mathbb{P}_z(T_{D_n} < \tau_{B_n}) \ge \kappa_5$. Let $p_n := \mathbb{P}_z(T_{D_n} < \tau_{B_n})$ and define $\tau_n := \tau_{B_n}$. Harmonic property of u and the optional sampling theorem imply that

$$\begin{aligned} u(z) - u(y) &= \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y)] \\ &= \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \,:\, T_{D_n} < \tau_n, X_{\tau_n} \in B_{n-1}] \\ &\quad + \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \,:\, T_{D_n} \ge \tau_n, X_{\tau_n} \in B_{n-1}] \\ &\quad + \sum_{i=1}^{n-2} \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \,:\, X_{\tau_n} \in B_{n-i-1} \backslash B_{n-i}] \\ &\quad + \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \,:\, X_{\tau_n} \notin B_1] \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.9}$$

Here we define $u(X_\infty) := \lim_{t \to \infty} u(X_t)$ and assume $X_{\tau_n} \in B_{n-1}$ when $\tau_n = \infty$. Now we estimate each term of (4.9). First, we consider I_1 and I_2.

$$I_1 \le \frac{M_n - m_n}{2}\mathbb{P}_z(T_{D_n} < \tau_n) \le \frac{s_n}{2} p_n, \tag{4.10}$$

$$I_2 \le (M_{n-1} - m_{n-1})\mathbb{P}_z(\tau_n \le T_{D_n}) \le s_{n-1}(1-p_n). \tag{4.11}$$

In order to estimate I_3 and I_4, we first define

$$F_i := \mathbb{P}_z(X_{\tau_n} \notin B_{n-i})$$

and consider this value. If $X_{\tau_n} \notin B_{n-i}$, the process cannot have a jump larger than $2r_n$ before the time τ_n and $|X_{\tau_n} - X_{\tau_n-}| \ge r_{n-i} - r_n$. Note that $\rho_0 < \frac{1}{4}$ and therefore $2r_n \le r_{n-i} - r_n$. Then we can apply Proposition 4.2 and conclude that

$$F_i \le \kappa_1 \Big(\frac{2r_n}{r_{n-i} - r_n}\Big)^{\sigma} = \kappa_1 \Big(\frac{2\rho_0^i}{1-\rho_0^i}\Big)^{\sigma} \le C_2 \rho_0^{i\sigma}.$$

Then we can obtain the following estimate;

$$\begin{aligned} I_3 &\le \sum_{i=1}^{n-2}(M_{n-i-1} - m_{n-i-1})(F_i - F_{i+1}) \le \sum_{i=1}^{n-2} s_{n-i-1}(F_i - F_{i+1}) \\ &= s_{n-2}F_1 - s_1 F_{n-1} + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i \le s_{n-2}F_1 + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i \\ &\le s_{n-2}C_2\rho_0^{\sigma} + s_n C_2(a^{-1} - 1)\sum_{i=2}^{n-2}\Big(\frac{\rho_0^{\sigma}}{a}\Big)^i, \end{aligned} \tag{4.12}$$

$$I_4 \le 2K \cdot F_{n-1} \le \theta_1 C_2 \rho_0^{(n-1)\sigma}. \tag{4.13}$$

Combining the estimates (4.10)–(4.13), we conclude that

$$\begin{aligned} u(z) - u(y) &\le \frac{1}{2}s_n p_n + s_{n-1}(1-p_n) + s_{n-2}F_1 + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i + 2K \cdot F_{n-1} \\ &\le \frac{1}{2}s_n p_n + s_{n-1}(1-p_n) + s_{n-2}C_2\rho_0^{\sigma} \\ &\quad + s_n C_2(a^{-1} - 1)\sum_{i=2}^{n-2}\Big(\frac{\rho_0^{\sigma}}{a}\Big)^i + \theta_1 C_2 \rho_0^{(n-1)\sigma}. \end{aligned} \tag{4.14}$$

In the rest of the proof, we assume n is large enough. Choose a such that $\rho_0^{\frac{\sigma}{2}} < a < 1$, then we have

$$\sum_{i=2}^{n-2}\Big(\frac{\rho_0^{\sigma}}{a}\Big)^i \le C_3, \quad \text{and} \quad \theta_1 \rho_0^{(n-1)\sigma} \le C_4 s_n \rho_0^{n\sigma/3}.$$

Moreover, recall that there exists $\kappa_5 > 0$ such that $p_n \ge \kappa_5$ for arbitrary $n \in \mathbb{N}$. Combining these facts, we obtain that

$$\text{R.H.S. of (4.14)} \le s_n\big(a^{-1} - \frac{\kappa_5}{2} + a^{-2}C_2\rho_0^{\sigma} + C_5(a^{-1} - 1) + C_6\rho_0^{n\sigma/3}\big)$$

$$\leq s_n(1-\frac{\kappa_5}{2}+a^{-2}C_2\rho_0^{\sigma}+C_7(a^{-1}-1)+C_6\rho_0^{n\sigma/3}). \tag{4.15}$$

If we choose a sufficiently close to 1 and ρ_0 sufficiently small, there exists $n_0 \in \mathbb{N}$ such that for arbitrary $n \geq n_0$,

$$\text{R.H.S. of (4.15)} \leq s_n(1-\frac{\kappa_5}{4}) \leq s_{n+1}. \tag{4.16}$$

Furthermore, if we choose θ_1 such that $\theta_1 a^{n_0} \geq 2K$, then the induction hypothesis is satisfied for $1 \leq n \leq n_0$. Therefore, we have proved (4.8) and this implies the Hölder continuity of u. Indeed, set $z = z_1$ and choose $y \in B(z_0, \frac{R}{3})$. If there exists $n \in \mathbb{N}$ such that

$$\frac{R}{12}\rho_0^n \leq |z-y| < \frac{R}{12}\rho_0^{n-1}, \tag{4.17}$$

then, we have

$$|u(z)-u(y)| \leq \theta_1 a^n = C_8K(\exp(n\log\rho_0))^{\frac{\log a}{\log \rho_0}} \leq C_9\|u\|_\infty\Big(\frac{|z-y|}{R}\Big)^{\frac{\log a}{\log \rho_0}}.$$

If there is no n that satisfies (4.17), then $|z-y| \geq R/12$. Therefore, there exists $C_{10} > 0$ such that

$$|u(z)-u(y)| \leq 2\|u\|_\infty \leq C_{10}\|u\|_\infty\Big(\frac{|z-y|}{R}\Big)^{\frac{\log a}{\log \rho_0}}.$$

Noting that $z = z_1 \in B(z_0, R/3)$ is arbitrary, this completes the proof of the Hölder continuity of u in $B(z_0, R/3)$. □

Remark 4.1. *There is a minor error in the proof of Theorem 2.2 in [4] by Bass and Kassmann. One of the keys to prove this theorem is the oscillation arguments; i.e. we choose appropriate sequences $\{r_n\}$ and $\{s_n\}$ decreasing to 0 as $n \to \infty$ and show that*

$$|u(z)-u(y)| \leq s_n \qquad (|z-y| \leq r_n) \tag{4.18}$$

by induction. Where the choice $r_n = \theta_2/4^n$ is specified, it is necessary to take $r_n = \theta_2\rho^n$ for some $0 < \rho < 1/4$. They choose $r_n = \theta_2/4^n$, where θ_2 is a sufficiently small positive constant. With this choice, the proof does not necessarily work well. We may resolve this by choice $r_n = \theta_2\rho^n$ with appropriate $0 < \rho < 1/4$.

4.2 Condition (SSJ) and continuous harmonic functions

We consider the following assumption instead of (SSJ-1);

(SSJ-2) There exist $\kappa_3 > 0$ and $\gamma > 1$ such that for all $x \in \mathbb{R}^d$, $r \in (0, R_0)$ and $1 < \lambda < 1/r$,

$$\frac{S(x,\lambda r)}{S(x,r)} \leq \kappa_3(\log\lambda)^{-\gamma}.$$

It is clear that (SSJ-1) implies (SSJ-2). In this section, we will prove the equivalence of the conditions (SSJ-1) and (SSJ-2) with $\gamma > 0$. The following proposition plays a crucial role in proving this equivalence.

Theorem 4.2. *Let $f : (0,\infty) \to (0,\infty)$ be a non-increasing function satisfying*

$$\frac{f(\lambda r)}{f(r)} \le g(\lambda) \quad \text{for all} \quad r \in (0,1) \quad \text{and} \quad \lambda \in \left(1, \frac{1}{r}\right) \tag{4.19}$$

for a function $g : (1,\infty) \to (0,\infty)$ such that $\liminf_{\lambda\to\infty} g(\lambda) < 1$. There exist constants $c > 0, \sigma > 0$ such that (4.19) holds for $g(\lambda) = c\lambda^{-\sigma}$, i.e.

$$\frac{f(\lambda r)}{f(r)} \le c\lambda^{-\sigma} \quad \text{for all} \quad r \in (0,1) \quad \text{and} \quad \lambda \in \left(1, \frac{1}{r}\right)$$

Proof. Let $r \in (0,1)$ and $\lambda \in (1, 1/r)$. Choose $b > 1$ so that $g(b) < 1$. Let $n \in \mathbb{Z}_+$ be such that $b^n \le \lambda < b^{n+1}$. Assume first that $n \ge 1$.
Since $b^n < 1/r$, we see that $b^{n-1}r < b^{-1} < 1$ and so we can apply (4.19) with $b^{n-1}r \in (0,1)$ and $b \in (1, 1/b^{n-1}r)$ to get

$$\frac{f(b^n r)}{f(b^{n-1}r)} = \frac{f(b \cdot b^{n-1}r)}{f(b^{n-1}r)} \le g(b). \tag{4.20}$$

Iterating (4.20), it follows that

$$\begin{aligned}\frac{f(b^n r)}{f(r)} &\le g(b)^n = g(b)\exp((n-1)\log g(b)) = g(b)\exp\left((n-1)\log b^{-1} \cdot \frac{\log g(b)}{\log b^{-1}}\right) \\ &= g(b) \cdot (b^{-(n-1)\sigma}) = g(b) \cdot b^{2\sigma} \cdot b^{-(n+1)\sigma},\end{aligned}$$

where $\sigma := -\log g(b)/\log b > 0$. Since f is non-increasing, we finally obtain

$$\frac{f(\lambda r)}{f(r)} \le \frac{f(b^n r)}{f(r)} \le b^{2\sigma} g(b) b^{-(n+1)\sigma} \le b^{2\sigma} g(b) \lambda^{-\sigma}.$$

The case $n = 0$ follows from the fact that f is non-increasing:

$$\frac{f(\lambda r)}{f(r)} \le 1 < b^{\sigma}\lambda^{-\sigma}.$$

Thus, it is enough to choose $c := g(b)b^{2\sigma} \vee b^{\sigma}$. □

Applying Theorem 4.2 with $g(\lambda) = \kappa_3(\log\lambda)^{-\gamma}$, (SSJ-2) implies (SSJ-1) and thus we can see the equivalence of (SSJ-1) and (SSJ-2) with $\gamma > 0$. Consequently, we can obtain the Hölder continuity of harmonic functions even if we change the condition (SSJ-1) into (SSJ-2) with $\gamma > 0$. Thus, combining this equivalence and Theorem 4.1, we can obtain the following corollary.

Corollary 4.1. *Suppose $0 < R < R_0 < 1$ and let u be bounded in $\mathbb{R}^d$ and harmonic in $B(z_0,R)$ with respect to $\mathscr{L}$. Then, under the assumptions (SSJ-2) with $\gamma > 0$ and (QRI-1), there exist $\nu \in (0,1)$ and $C > 0$ depending only on R_0, κ_2, κ_3 and γ such that*

$$|u(z) - u(y)| \le C\|u\|_\infty \left(\frac{|z-y|}{R}\right)^{\nu}, \qquad z, y \in B(z_0, R/3). \tag{4.21}$$

4.3 Condition (QRI) and continuous harmonic functions

We next consider the following condition which is weaker than (QRI-1);

(QRI-2) There exists $\kappa_4 > 0$ such that for all $x \in \mathbb{R}^d, r \in (0, R_0)$ and $y \in B(x, 2r)$,

$$N(x,r) \geq \frac{\kappa_4}{|\log r|} L(y, r/2).$$

If we change the condition (QRI-1) into (QRI-2), we can prove the uniform continuity of harmonic functions. The precise statement is as follows;

Theorem 4.3. *Let $0 < R < R_0 < 1$ and suppose (SSJ-1) or (SSJ-2) with $\gamma > 0$ and in addition (QRI-2). Then there exists a monotone increasing continuous function P : $\mathbb{R}_+ \to \mathbb{R}_+$ with $P(0) = 0$ such that for any $\mathscr{L}$-harmonic function on $B(z_0, R)$*

$$|u(z) - u(y)| \leq \|u\|_\infty P(|z - y|), \qquad z, y \in B(z_0, R/2).$$

Although Husseini and Kassmann [14] also obtain similar theorem under the assumptions (SSJ-1) and (QRI-2), it seems that there is a gap in their proof. (See Remark 4.3 for details.) Thus, we will give a full proof there. We first consider an example where the assumption (SSJ-1) and (QRI-2) are satisfied.

Example 4.1. *Let κ_0 be a given positive constant. If the jump density function $n(x,h)$ satisfies for $0 < |h| \leq 2$*

$$\frac{\kappa_0}{|h|^{d+\alpha}} \log \frac{3}{|h|} \leq n(x,h) \leq \frac{{\kappa_0}^{-1}}{|h|^{d+\alpha}} \log \frac{3}{|h|} \tag{4.22}$$

with $1 \leq \alpha < 2$ and $\sup_{x \in \mathbb{R}^d} S(x,1) < \infty$, then the uniform continuity of harmonic function holds.

Proof. Due to Theorem 4.3, it is enough to verify (4.2), (SSJ-1) and (QRI-2). (4.2) is easy to verify. By (4.22), we have

$$\begin{aligned}\int_{|h|\leq 1} |h|^2 n(x,h) dh &\leq \int_{|h|\leq 1} |h|^2 \cdot \frac{{\kappa_0}^{-1}}{|h|^{d+\alpha}} \log \frac{3}{|h|} dh \\ &= C_1 \int_0^1 \frac{1}{\psi^{\alpha-1}} \log \frac{3}{\psi} d\psi = C_2 \int_0^\infty (\chi + \log 3) e^{(\alpha-2)\chi} d\chi < \infty,\end{aligned}$$

where we use $\psi = e^{-\chi}$ in the last equality. Thus, combining with the assumption $\sup_{x\in\mathbb{R}^d} S(x,1) < \infty$, we obtain

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (|h|^2 \wedge 1) n(x,h) dh \leq \sup_{x\in\mathbb{R}^d} S(x,1) + \int_{|h|\leq 1} \frac{{\kappa_0}^{-1}}{|h|^{d+\alpha-2}} \log \frac{3}{|h|} dh < \infty.$$

Let r be $0<r<R_0<1$. To see (SSJ-1) follow, we begin with the estimate of

$$\int_{r\le|h|\le 1}\frac{1}{|h|^{d+\alpha}}\log\frac{3}{|h|}dh.$$

Using the spherical coordinate, we can obtain that

$$\begin{aligned}\int_{r\le|h|\le 1}\frac{1}{|h|^{d+\alpha}}\log\frac{3}{|h|}dh &= C_2\int_r^1\frac{1}{\psi^{\alpha+1}}\log\frac{3}{\psi}d\psi\\ &= C_2\Big(\frac{1}{\alpha}r^{-\alpha}\log\frac{1}{r}+\frac{\alpha\log 3-1}{\alpha^2}(r^{-\alpha}-1)\Big)\\ &= C_3\Big(r^{-\alpha}\log\frac{1}{r}+(\log 3-\frac{1}{\alpha})(r^{-\alpha}-1)\Big).\end{aligned}$$

We now estimate $S(x,r)$ from below. The above formula and (4.22) imply

$$\begin{aligned}S(x,r) &= \int_{|h|\ge r}n(x,h)dh \ge \int_{r\le|h|\le 1}n(x,h)dh\\ &\ge C_4\Big(r^{-\alpha}\log\frac{1}{r}+(\log 3-\frac{1}{\alpha})(r^{-\alpha}-1)\Big)\ge C_4r^{-\alpha}\log\frac{1}{r},\end{aligned}\tag{4.23}$$

where we used the assumption $1\le\alpha<2$ in the last inequality. Moreover, if we estimate $S(x,r)$ from above, we obtain that

$$\begin{aligned}S(x,r) =& S(x,1)+\int_{r\le|h|\le 1}n(x,h)dh\\ &\le C_5+C_6\Big(r^{-\alpha}\log\frac{1}{r}+(\log 3-\frac{1}{\alpha})(r^{-\alpha}-1)\Big)\\ &\le C_5+C_6r^{-\alpha}\log\frac{3}{r}\le C_7r^{-\alpha}\log\frac{3}{r}.\end{aligned}\tag{4.24}$$

Combining (4.23) and (4.24), we see that

$$\frac{S(x,\lambda r)}{S(x,r)}\le\frac{C_3(\lambda r)^{-\alpha}\log(3/\lambda r)}{C_1r^{-\alpha}\log(1/r)}\le C_4\lambda^{-\alpha}\frac{\log(3/\lambda r)}{\log(1/r)}\le C_5\lambda^{-\alpha},$$

where we used the assumption $r<R_0<1$ in the last inequality. We thus obtain the condition (SSJ-1).
We next prove (QRI-2). Using (4.22) and (4.24), we have for $0<r<R_0$,

$$N(x,r)\ge\frac{\kappa_0}{(2r)^{d+\alpha}}\log\frac{3}{2r}\cdot\frac{1}{3\cdot 2^d}|B(x,r)|\ge C_6\frac{1}{r^\alpha}\log\frac{1}{r}.\tag{4.25}$$

and

$$\begin{aligned}L(y,r/2) &= S(y,r/2)+\frac{2}{r}\Big|\int_{\frac{r}{2}\le|h|\le 1}h\,n(x,h)dh\Big|+\frac{4}{r^2}\int_{|h|\le\frac{r}{2}}|h|^2n(x,h)dh\\ &\le C_3\Big(\frac{r}{2}\Big)^{-\alpha}\log\frac{6}{r}+\frac{C_7}{r}\int_{\frac{r}{2}}^1\frac{1}{\psi^\alpha}\log\frac{3}{\psi}d\psi+\frac{C_8}{r^2}\int_0^{\frac{r}{2}}\frac{1}{\psi^{\alpha-1}}\log\frac{3}{\psi}d\psi\end{aligned}$$

$$=: L_1 + L_2 + L_3$$

It is clear that $L_1 \le C_{10} r^{-\alpha} \log(1/r)$. L_3 satisfies the upper estimate as follows:

$$L_3 \le \frac{C_{13}}{r^2} \int_0^{\frac{r}{2}} \frac{1}{\psi^{\alpha-1}} \log \frac{1}{\psi} d\psi \le C_{14} r^{-\alpha} \log \frac{1}{r}.$$

Moreover, L_2 satisfies the upper estimate as follows:
Case1. $1 < \alpha < 2$.

$$L_2 \le \frac{C_{11}}{r} \left(\frac{-\log(r/2) + \log 3}{\alpha - 1} \left(\frac{2}{r}\right)^{\alpha-1} + \frac{1}{(\alpha-1)^2} \right) \le C_{15} r^{-\alpha} \log \frac{1}{r}.$$

Case2. $\alpha = 1$.

$$L_2 \le \frac{C_{11}}{r} \left(\left(\log \frac{r}{2}\right)^2 + \log 3 \cdot \left(-\log \frac{r}{2}\right) \right) \le \frac{C_{16}}{r} \left(\log \frac{1}{r}\right)^2.$$

Consequently, we conclude that

$$L(y, r/2) \le \begin{cases} C_{11} r^{-\alpha} \log(1/r) & (\alpha \ne 1) \\ C_{12} r^{-1} (\log(1/r))^2 & (\alpha = 1). \end{cases} \tag{4.26}$$

Combining (4.25) and (4.26), we see that there exists κ_4 such that the condition (QRI-2) holds. □

Remark 4.2. *As we see in the proof above, when $1 < \alpha < 2$, (QRI-1) holds. Consequently, we have the Hölder continuity of harmonic functions.*

Now we prove Theorem 4.3. We use the similar method as the case of Theorem 4.1. Note that Proposition 4.1 and 4.2 are valid under the assumption of Theorem 4.3. Moreover, the following lemma is obtained by easy modification of Proposition 4.3

Proposition 4.4. *([14] Lemma 2.5.) Suppose (QRI-2) holds. Then there exists $\kappa_6 > 0$ such that*

$$\mathbb{P}_y(T_A < \tau_{B(x_0,r)}) \ge \frac{\kappa_6}{|\log r|}$$

for $r \in (0, R_0), y \in B(x_0, r/2)$, and $A \subset B(x_0, r)$ such that $|A| \ge |B(x_0, r)|/(3 \cdot 2^d)$.

We now prove Theorem 4.3.

Proof of Theorem 4.3. It is sufficient to prove the assertion under the assumptions (SSJ-1) and (QRI-2). The idea of the proof is the same as Theorem 4.1. Suppose $\|u\|_\infty = K$, $z_1 \in B(z_0, R/2)$ and define

$$s_n := \frac{\theta_1}{(\log(n+1))^{\beta_1}}, \qquad r_n := \frac{\theta_2}{n^{\beta_2 n}},$$

where θ_1 is chosen large enough to satisfy $\theta_1 > 2K$, and θ_2 is chosen small enough to satisfy $\theta_2 < R/4$. In addition, suppose $0 < \beta_1 \le 1$ and $\beta_2 \ge 1$ which are chosen appropriately later. Note that $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} s_n = 0$. Set B_n, M_n and m_n as follows.

$$B_n := B(z_1, r_n), \qquad M_n := \sup_{z\in B_n} u(z), \qquad m_n := \inf_{z\in B_n} u(z).$$

As in the proof of Theorem 4.1, we will prove by induction that

$$M_n - m_n \le s_n \qquad (\forall n \in \mathbb{N}).$$

We assume that $M_k - m_k \le s_k$ for $1 \le k \le n$ and consider the case of $k = n+1$. Choose $y, z \in B_{n+1}$ such that

$$u(z) \ge M_{n+1} - \varepsilon, \qquad u(y) \le m_{n+1} + \varepsilon,$$

where ε is an arbitrary positive number. Define

$$A_n = \{z \in B_n \ : \ u(z) \le (M_n + m_n)/2\}$$

and assume that $|A_n|/|B_n| \ge 1/2$. Let D_n be a compact set that satisfies

$$D_n \subset A_n, \qquad |D_n|/|B_n| \ge 1/3.$$

Let

$$p_n := \mathbb{P}_z(T_{D_n} < \tau_n), \qquad\qquad F_i := \mathbb{P}_z(X_{\tau_n} \notin B_{n-i}),$$

where $\tau_n := \tau_{B_n}$. The harmonic property of u and the optional sampling theorem imply that

$$\begin{aligned} u(z) - u(y) &= \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y)] \\ &= \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \ : \ T_{D_n} < \tau_n, X_{\tau_n} \in B_{n-1}] \\ &\quad + \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \ : \ T_{D_n} \ge \tau_n, X_{\tau_n} \in B_{n-1}] \\ &\quad + \sum_{i=1}^{n-2} \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \ : \ X_{\tau_n} \in B_{n-i-1} \backslash B_{n-i}] \\ &\quad + \mathbb{E}_z[u(X_{\tau_n \wedge T_{D_n}}) - u(y) \ : \ X_{\tau_n} \notin B_1] \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.27}$$

Here we define $u(X_\infty) := \lim_{t\to\infty} u(X_t)$ and assume $X_{\tau_n} \in B_{n-1}$ when $\tau_n = \infty$. Now we estimate each term of (4.27). First, we consider I_1 and I_2.

$$I_1 \le \frac{M_n - m_n}{2} \mathbb{P}_z(T_{D_n} < \tau_n) \le \frac{s_n}{2} p_n, \tag{4.28}$$

$$I_2 \le (M_{n-1} - m_{n-1}) \mathbb{P}_z(\tau_n \le T_{D_n}) \le s_{n-1}(1 - p_n). \tag{4.29}$$

In order to estimate I_3 and I_4, we use F_i.

$$I_3 \le \sum_{i=1}^{n-2} (M_{n-i-1} - m_{n-i-1})(F_i - F_{i+1}) \le \sum_{i=1}^{n-2} s_{n-i-1}(F_i - F_{i+1})$$

$$
\begin{aligned}
&= s_{n-2}F_1 - s_1F_{n-1} + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i \\
&\le s_{n-2}F_1 + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i. && (4.30)
\end{aligned}
$$

$$
I_4 \le 2K \cdot F_{n-1}. \qquad (4.31)
$$

Combining the estimates from (4.28) to (4.31), we can obtain

$$
\begin{aligned}
u(z) - u(y) &\le \frac{1}{2}s_n p_n + s_{n-1}(1-p_n) + s_{n-2}F_1 + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i + 2K \cdot F_{n-1} \\
&\le s_{n-1} - \frac{1}{2}s_n p_n + s_{n-2}F_1 + \sum_{i=2}^{n-2}(s_{n-i-1} - s_{n-i})F_i + 2K \cdot F_{n-1} \\
&= s_{n+1}\Big(\frac{s_{n-1}}{s_{n+1}} - \frac{s_n p_n}{2s_{n+1}} + \frac{s_{n-2}}{s_{n+1}}F_1 + \sum_{i=2}^{n-2}\frac{s_{n-i-1} - s_{n-i}}{s_{n+1}}F_i + \frac{2K}{s_{n+1}} \cdot F_{n-1}\Big) \\
&=: s_{n+1}(J_1 + J_2 + J_3 + J_4 + J_5). && (4.32)
\end{aligned}
$$

Our goal is to prove that

$$
J_1 + J_2 + J_3 + J_4 + J_5 \le 1. \qquad (4.33)
$$

Now, we estimate each J_i. In the rest of the proof, we assume that n is large enough. J_1 and J_2 are easily estimated.

$$
\begin{aligned}
J_1 = \frac{s_{n-1}}{s_{n+1}} &= \Big(\frac{\log(n+2)}{\log n}\Big)^{\beta_1} = \Big(1 + \frac{\log(1+\frac{2}{n})}{\log n}\Big)^{\beta_1} \\
&\le \Big(1 + \frac{2}{n\log n}\Big)^{\beta_1} \le 1 + \frac{2\beta_1}{n\log n}, && (4.34)
\end{aligned}
$$

where we used the assumption $0 < \beta_1 \le 1$ in the last inequality. Note that Lemma 4.4 implies $p_n \ge \kappa_6/|\log r_n|$. Then, we obtain

$$
\begin{aligned}
J_2 = -\frac{s_n p_n}{2s_{n+1}} \le -\frac{1}{2}p_n &\le -\frac{\kappa_6}{2|\log r_n|} \\
&= -\frac{\kappa_6}{2(|\log\theta_2| + \beta_2 n\log n)} \le -\frac{\kappa_7}{\beta_2 n\log n}, && (4.35)
\end{aligned}
$$

where κ_7 is a constant that does not depend on β_1, β_2 or n. In order to estimate J_3, J_4 and J_5, we begin with the estimate of $F_i := \mathbb{P}^z(X_{\tau_n} \notin B_{n-i})$. If $X_{\tau_n} \notin B_{n-i}$, the process cannot have a jump larger than $2r_n$ before the time τ_n and $|X_{\tau_n} - X_{\tau_n -}| \ge r_{n-i} - r_n$. Note that $r_n/r_{n-1} \le 1/4$ and therefore $2r_n < r_{n-i} - r_n$. Then, we can apply Lemma 4.2 and obtain

$$
F_i \le \kappa_1\Big(\frac{2r_n}{r_{n-i} - r_n}\Big)^{\sigma} \qquad (1 \le i \le n).
$$

Thus, noting that $0 < \beta_1 \le 1 \le \beta_2$, we have

$$\begin{aligned} J_5 &= \frac{2K}{s_{n+1}} \cdot F_{n-1} \le (\log(n+2))^{\beta_1} \cdot \kappa_1 \Big(\frac{2r_n}{r_1 - r_n} \Big)^{\sigma} \\ &= (\log(n+2))^{\beta_1} \cdot \kappa_1 \Big(\frac{2}{n^{\beta_2 n} - 1} \Big)^{\sigma} \le \log(n+2) \cdot \kappa_1 \Big(\frac{4}{n^n} \Big)^{\sigma} \\ &\le C_1 e^{-n\sigma} \log n \le C_2 \exp\Big(-\frac{n\sigma}{2} \Big). \end{aligned} \tag{4.36}$$

The estimates of J_3 and J_4 are a little complicated.

$$\begin{aligned} J_3 &= \frac{s_{n-2}}{s_{n+1}} \cdot F_1 \le \Big(\frac{\log(n+2)}{\log(n-1)} \Big)^{\beta_1} \cdot \kappa_1 \Big(\frac{2r_n}{r_{n-1} - r_n} \Big)^{\sigma} \\ &\le \Big(\frac{\log(n+2)}{\log(n-1)} \Big)^{\beta_1} \cdot \kappa_1 \Big(\frac{2}{n^{\beta_2} - 1} \Big)^{\sigma} \le \Big(\frac{\log(n+2)}{\log(n-1)} \Big)^{\beta_1} \cdot \kappa_1 \Big(\frac{4}{n^{\beta_2}} \Big)^{\sigma} \\ &\le C_3 \Big(1 + \frac{3}{(n-1)\log(n-1)} \Big)^{\beta_1} \cdot \frac{1}{n^{\beta_2 \sigma}} \le \frac{C_3}{n^{\beta_2 \sigma}} + \frac{C_4}{n^{\beta_2 \sigma + 1} \log n}. \end{aligned} \tag{4.37}$$

In order to estimate J_4, we begin with the following decomposition.

$$\begin{aligned} J_4 &= \sum_{i=2}^{n-2} \frac{s_{n-i-1} - s_{n-i}}{s_{n+1}} \cdot F_i \\ &= \sum_{i=2}^{n_0 - 1} \frac{s_{n-i-1} - s_{n-i}}{s_{n+1}} \cdot F_i + \sum_{i=n_0}^{n-2} \frac{s_{n-i-1} - s_{n-i}}{s_{n+1}} \cdot F_i =: K_1 + K_2, \end{aligned} \tag{4.38}$$

where n_0 is an integer that satisfies $n_0 - 1 < \sqrt{n} \le n_0$. Note that $r_{n+1}/r_n \le 1/4$ and therefore

$$F_i \le \kappa_1 \Big(\frac{2r_n}{r_{n-i} - r_n} \Big)^{\sigma} \le \kappa_1 \Big(\frac{2}{4^i - 1} \Big)^{\sigma} \le \frac{C_5}{4^{(i-1)\sigma}}. \tag{4.39}$$

Noting that $(\log x)^{-\beta_1}$ is a convex monotone decreasing function on $x > 1$, we can estimate K_1 as follows;

$$\begin{aligned} K_1 &\le (\log(n+2))^{\beta_1} \Big(\frac{1}{(\log(n - \sqrt{n}))^{\beta_1}} - \frac{1}{(\log(n - \sqrt{n} + 1))^{\beta_1}} \Big) \cdot \sum_{i=2}^{n_0 - 1} \frac{C_5}{4^{(i-1)\sigma}} \\ &\le C_6 (\log(n+2))^{\beta_1} \cdot \beta_1 \frac{1}{(n - \sqrt{n})(\log(n - \sqrt{n}))^{\beta_1 + 1}} \\ &\le C_6 (2 \log n)^{\beta_1} \cdot \beta_1 \frac{2^{\beta_1 + 2}}{n (\log n)^{\beta_1 + 1}} \le \frac{C_7 \beta_1}{n \log n}. \end{aligned} \tag{4.40}$$

The estimate of K_2 is much easier;

$$K_2 \le \Big(\frac{\log(n+2)}{\log 2} \Big)^{\beta_1} \sum_{i=n_0}^{\infty} \frac{C_5}{4^{(i-1)\sigma}} \le C_8 \log(n+2) \cdot \frac{1}{4^{\sqrt{n}\sigma}} \le \frac{C_9}{2^{\sqrt{n}\sigma}}. \tag{4.41}$$

From (4.40) and (4.41), we obtain

$$J_4 \le \frac{C_7\beta_1}{n\log n} + \frac{C_9}{2^{\sqrt{n\sigma}}}. \tag{4.42}$$

Combining all the estimates of J_i's, we conclude that

$$\begin{aligned}
&J_1+J_2+J_3+J_4+J_5 \\
&\quad \le 1 + \frac{2\beta_1}{n\log n} - \frac{\kappa_7}{\beta_2 n\log n} + \frac{C_3}{n^{\beta_2\sigma}} + \frac{C_4}{n^{\beta_2\sigma+1}\log n} + \frac{C_7\beta_1}{n\log n} + \frac{C_9}{2^{\sqrt{n\sigma}}} + C_2\exp\Big(-\frac{n\sigma}{2}\Big) \\
&\quad \le 1 + \Big(C_{10}\beta_1 - \frac{\kappa_7}{\beta_2}\Big)\frac{1}{n\log n} + \frac{C_3}{n^{\beta_2\sigma}} + \frac{C_{11}}{n^{\beta_2\sigma+1}\log n}.
\end{aligned} \tag{4.43}$$

Now, choose $\beta_2 \ge 1$ so that $\beta_2\sigma > 1$, and then take β_1 small enough so that $C_{10}\beta_1 - \kappa_7/\beta_2 < -\kappa_7/2\beta_2$. Then, we conclude that there exist κ_8 and $n_1 \in \mathbb{N}$ such that the right hand side of (4.43) is bounded from above by $1-\kappa_8/(n\log n)$ for $n \ge n_1$. If we choose θ_1 sufficiently large so that

$$M_n - m_n \le 2K \le s_n \qquad (n \le n_1),$$

the induction hypothesis also holds for all $n \le n_1$. Now we conclude $|u(z)-u(y)| \le s_n$ for z, $y \in B(z_0, R/2)$ such that $|z-y| \le r_n$. Noting that $s_1 \ge 2K$, we can obtain the desired continuity. $\square$

Remark 4.3. *In the proof, we should choose s_n and r_n very carefully. In Theorem 1.2 of [14], they set*

$$s_n = \frac{\theta_1}{n^\zeta}, \qquad r_n = \frac{\theta_2}{4^n},$$

where $\zeta > 0$. However, this does not work well. The trouble is that J_2 may go to 0 as $n \to \infty$, while J_3 does not depend on n. To avoid this trouble, it is essential to choose r_n so that F_1 also depends on n.

Recently Kassmann and Mimica prove Hölder continuity of harmonic functions for more general non-local operators in [18].

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