Behavior of solutions to impulsive differential equations

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博士論文

Behavior of solutions to impulsive differential equations

（衝撃微分方程式の解の挙動）

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Behavior of solutions to impulsive differential equations

A thesis presented

by

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Contents

Summary

Many real processes and phenomena in nature, science and technology are characterized by the fact that they possess contiguous time intervals of slow and fast development. An adequate apparatus for mathematical simulation of such processes and phenomena are the impulsive differential equations. The start of the theory of impulsive ordinary differential equations was made by Mil'man and Myshkis in the paper *On stability of motion in the presence of impulses*, Sib. Math. J. **1** (2) (1960), 233-237. (in Russian). Around 1991, the study of the impulsive partial differential equations started at several places in the world. Erbe, Freedman, Liu and Wu provided a natural framework for many evolutional processes in population dynamics in the paper *Comparison principles for impulsive parabolic equations with applications to models of single species growth*, J. Austral. Math. Soc., Ser. B **32** (1991), 382-400, one of the first several papers in this field. Since then many results have been published.

In this thesis, we consider three kinds of problems with impulsive conditions. The first concerns the existence of a global solution to an impulsive differential equation and its asymptotic behavior. It is well known that the growth of the solution to the semi-linear parabolic equation is related to that of the reaction function. We discuss the infinite recurrence of impulsive control which continues the solution globally in time.

The second concerns blow-up solutions to impulsive differential equations. Such problems arise from some discrete models of processes and phenomena which occur in discrete technologies, chemical reactor dynamics, combustion theory, thermal explosions, population dynamics etc. We are interested in studying the behavior of the solution which is influenced by the reaction function and the impulsive source. In the smooth case (without impulses) it is well known that if the reaction function is bounded from above by a certain linear growth condition, then the solution of the problem under consideration converges to a steady-state solution. In the case when the reaction function is bounded from below by either linear or nonlinear growth condition then the solution may grow unbounded as $t \to T^*$, where T^* is either a finite time or infinity. The blow-up time T^* depends on the initial data. If the initial data is small enough, then T^* becomes large. In our case (with impulses), we fix the initial data which is not too small. We will investigate how to control the impulsive source to delay the blow-up time T^* and to prevent the solution from growing unbounded in the desired time interval.

Finally, we study a problem derived from the nuclear reactor dynamics. This leads to a system of semilinear parabolic equations with an initial condition, Neumann boundary conditions and impulsive conditions. We are interested in studying the behavior of the solution which is influenced by the reaction function and the impulsive source. In our case when the reaction function is bounded from below by nonlinear growth condition then the solution may grow unbounded as $t \to T$, where T is either a finite time or infinity. This blow-up time T depends on the initial data and the control. We investigate the influence of the control on the solution. There are two cases, i.e., effective and ineffective controls, the precise meaning of which will be given later. We prove the existence of ineffective control, which implies the failure of control, resulting in the explosion. We restrict ourselves in the problem to the periodic solution to the semilimear parabolic equation. Although we deal in this thesis with a toy model, we believe that our results will shed some light on the real problem besides being of much interest from a theoretical point of view.

In what follows, we state the problems, the principal results and related facts.

I. We discuss the asymptotic behavior of global solutions to impulsive differential equations under impulsive control.

Consider the impulsive semilinear evolution equation.

$$\frac{du(t)}{dt} + A(t)u(t) = f(u(t)) \quad \text{for } t \in \bigcup_{i=0}^{\infty}(t_i, t_{i+1}), \tag{1}$$

$$u(0) = u_0, \tag{2}$$

$$u(t_k) = g_k(t_k, u(t_k-)) \qquad (k = 1, 2, \ldots), \tag{3}$$

where $g_k(t_k, \cdot) : X \to X \ \ (k = 1, 2, \ldots)$ are given continuous maps, $u(t_k-) = \lim_{s \to t_k - 0} u(s)$ and X is a Banach space. We suppose that $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are given numbers and $\lim_{n\to\infty} t_n = \infty$.

We give here some assumptions.

H01. The operator $A(t)$ is a closed operator defined densely in X for each $t \in [0, \infty)$. The resolvent set $\rho(A(t))$ of $A(t)$ contains the half-plane $Re\lambda \le 0$, and $(1 + |\lambda|)(A(t) - \lambda)^{-1}$ is uniformly bounded in $0 \le t < \infty$ and $Re\lambda \le 0$.

H02. The domain $D(A(t)) \equiv D$ of $A(t)$ is independent of t and, accordingly, $A(t)A(0)^{-1}$, being a bounded operator, is a Hölder continuous function of t in the norm of $B(X)$. In other words, there exist positive constants $\alpha \le 1$ and L such that

$$\| A(t)A(0)^{-1} - A(s)A(0)^{-1} \| \le L|t - s|^{\alpha}$$

is satisfied for $0 \le s < \infty$ and $0 \le t < \infty$.

Here $B(X)$ denotes the set of all bounded linear operator from X into X.

H03. $A(t)A(s)^{-1}$ is uniformly bounded, i.e., $\sup_{0 \le t,s < \infty} \| A(t)A(s)^{-1} \| < \infty$.

H04. f is a nonlinear mapping from the whole of X into X. For every $C > 0$ there exists a constant $k_C > 0$ such that

$$\| f(u) \| \le k_C, \ \ \| f(u) - f(v) \| \le k_C \| u - v \|$$

hold for $\| u \| \le C$ and $\| v \| \le C$. The constant k_C may be an increasing function of C.

Let δ_0 be an arbitrary element such that

$$\| U(t,s) \| \le Ce^{-\delta_0(t-s)}. \tag{4}$$

Here $U(t,s)$ is the fundamental solution of $d/dt + A(t)$. We assume

H05. $\| f(u(t)) \| \le f_0(t)$, where $f_0 : R_+ \to R_+$ is a bounded function and $R_+ = [0, \infty)$.

H06.

$$\| g_k(t_k, u(t_k-)) \|_X \le M_k \| u(t_k-) \|_X,$$

where M_k $(k = 1, 2, \ldots)$ are constants and

$$C \prod_{s<t_k<t} (CM_k) \le L_0 e^{\gamma(t-s)},$$

where $L_0 \ge 0$, γ are constants and the constant C was given in (4).

We estimate the growth of the solution u.

Theorem 1 *We suppose H01-H06 are satisfied. Then the following estimate holds.*

$$\| u(t) \|_X \le L_0 \| u_0 \|_X \, e^{(\gamma-\delta_0)t} + L_0 e^{(\gamma-\delta_0)t} \int_0^t e^{-(\gamma-\delta_0)s} f_0(s) ds,$$

where δ_0 is defined in (4).

Further on, suppose that $\gamma < \delta_0$ and let there exits $\limsup_{t\to\infty} f_0(t) < +\infty$. Then we have that

$$\lim_{t\to\infty} \| u(t) \|_X \le \frac{L_0}{\delta_0 - \gamma} \limsup_{t\to\infty} f_0(t).$$

Suppose the following condition in the place of H05.

H05′. The function $f : X \to X$ is continuous and satisfies that $\| f(u(t)) \|_X \le M \| u(t) \|_X$, where $M \ge 0$ is a constant.

Now we investigate the estimate of the growth of the solution u.

Theorem 2 *Suppose H01-H04,* H05′ *and H06 to be satisfied. Then the following estimate holds.*

$$\| u(t) \|_X \le L_0 \| u_0 \|_X \, e^{(\gamma-\delta_0+CM)t},$$

where δ_0 is defined as above.

1. *If we suppose that $\gamma + CM = \delta_0$, then $\| u(t) \|_X \le L_0 \| u_0 \|_X$.*
2. *If we suppose that $\gamma + CM < \delta_0$, then*

$$\lim_{t\to\infty} \| u(t) \|_X = 0.$$

We consider the following impulsive semilinear parabolic Cauchy problem.

$$\frac{\partial u}{\partial t} = \Delta u + u^p \qquad \text{in } \bigcup_{i=0}^{\infty} \{(t_k, t_{k+1}) \times R^n\}, \tag{5}$$

$$u(0,x) = \varphi(x) \qquad \text{in } R^n, \tag{6}$$

$$u(t_k, x) = g_k(u(t_k-, x)) \qquad \text{for } x \in R^n \qquad (k = 1, 2, \ldots), \tag{7}$$

where Δ is the Laplace operator, $p > 1$ is a constant and φ is a nonnegative bounded continuous function in R^n. We suppose that $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are given numbers, $\lim_{n\to\infty} t_n = \infty$, $g_k : R^1 \to R^1 \quad (k = 1,2,\ldots)$ are continuous and each $g_k(v)$ is increasing for $v \in R^1$.

Due to the possible nonuniqueness of solutions to the Cauchy problem (5) - (7), we shall restrict our attention to a certain class of solutions $u(t,x)$, namely, those with the following properties.

(i). $u(t,x) \geq 0 \qquad \text{in } \bigcup_{k=0}^{\infty} \{(t_k,t_{k+1}) \times R^n\}.$

(ii). $u(t,x)$ satisfies the integral equation in $[t_k,t_{k+1}) \times R^n \quad (k = 0,1,2,\ldots)$.

$$\begin{aligned} u(t,x) &= (4\pi(t-t_k))^{-n/2} \int_{R^n} \exp\left[-|x-y|^2/(4(t-t_k))\right] u(t_k,y)dy \\ &\quad + \int_{t_k}^{t} \int_{R^n} (4\pi(t-s))^{-n/2} \exp\left[-|x-y|^2/(4(t-s))\right] u^p(s,y)dyds, \end{aligned}$$

(iii). $u(t_k,x) = g_k(u(t_k-,x)) \quad (k = 1,2,\ldots)$.

On the other hand, it is proved that if u satisfies the integral equation and is bounded in $[t_k,t_{k+1}) \times R^n \quad (k = 0,1,2,\ldots)$ then u is unique and is a classical solution to the differential equation, i.e., u is in $C^{1,2}((t_k,t_{k+1}) \times R^n) \bigcap C([t_k,t_{k+1}) \times R^n), \ (k = 0,1,2,\ldots)$ and u satisfies the differential equation (5)-(7).

The behavior of the initial value near $x = \infty$ is essential to existence and non-existence of a global solution of the Cauchy problem. Instead of such a condition, we would like to make a sufficient impulsive condition to construct a global solution and investigate the behavior as t tends to infinity.

Theorem 3 *We put $M_0 = \| \psi \|_{L^\infty(R^n)}$. We assume that each $g_k : [0,\infty) \to [0,\infty)$ is a continuous function such that $g_k(v)$ is increasing and*

$$\| g_k(v) \|_{L^\infty(R^n)} \leq M_k \| v \|_{L^\infty(R^n)} \quad (k = 1,2,\ldots).$$

Let $M_1, M_2, \cdots, M_k, \cdots$ be constants and satisfy

$$M_0 M_1 \cdots M_k < [2(p-1)(t_{k+1}-t_k)]^{1/(1-p)} 2^{k(1-p)} \qquad (k = 0,1,2,\ldots).$$

Then there exists a unique solution $u(t,x)$ to (5) - (7) and we have

$$\| u(t,\cdot) \|_{L^\infty(R^n)} \leq 2^{p-1} [2(p-1)(t_{k+1}-t_k)]^{1/(1-p)} \quad \text{for } t \in [t_k,t_{k+1}) \quad (k = 0,1,2,\ldots). \tag{8}$$

We consider the following initial-boundary value problem with an impulsive condition.

$$\frac{\partial u}{\partial t} = \Delta u + u^p \qquad \text{in } \bigcup_{k=0}^{\infty} \{(t_k,t_{k+1}) \times \Omega\}, \tag{9}$$

$$u(t,x) = 0 \qquad \text{on } (0,\infty) \times \partial\Omega, \tag{10}$$

$$u(0,x) = \varphi(x) \qquad \text{in } \Omega, \tag{11}$$

$$u(t_k,x) = g_k(u(t_k-,x)) \qquad \text{in } \Omega \quad (k = 1,2,3,\ldots). \tag{12}$$

Here $p > 1$ is a constant, Ω is a bounded open set in R^n with a smooth boundary $\partial\Omega$, φ is a nonnegative bounded continuous function in Ω and $\varphi(x) = 0$ on $\partial\Omega$. Suppose that $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are given numbers, $\lim_{k\to\infty} t_k = \infty$ and $g_k : R^1 \to R^1 \quad (k = 1,2,\ldots)$ are continuous and each $g_k(v)$ is increasing for $v \in R^1$. We define $u(t_k-,x) = \lim_{s\to t_k-0} u(s,x)$.

The global existence and nonexistence of a solution to the above equation without an impulsive condition is well known. We here study the relation among the asymptotic behavior, the nonlinear term u^p and the impulsive condition. We will also investigate a sufficient impulsive condition for the global existence even if there is no global solution of the equation without an impulsive condition.

Due to the possible nonuniqueness of solutions to the problem (9)-(12), we shall restrict our attention to a certain class of solutions $u(t,x)$, namely, those with the following properties.

(i). $u(t,x) \geq 0 \qquad \text{in } \bigcup_{i=0}^{\infty}\{(t_k,t_{k+1}) \times \overline{\Omega}\}.$

(ii). u satisfies the integral equation in $(t_k,t_{k+1}) \times \overline{\Omega} \quad (k = 0,1,2,\ldots)$

$$u(t,x) = \int_\Omega U(t,t_k;x,y)u(t_k,y)dy + \int_{t_k}^t \int_\Omega U(t,s;x,y)u^p(s,y)dyds.$$

(iii). $u(t_k,x) = g_k(u(t_k-,x)) \qquad (k = 1,2,\ldots).$

On the other hand, it is proved that if u satisfies the integral equation and is bounded in $[t_k,t_{k+1}) \times \overline{\Omega} \quad (k = 0,1,2,\ldots)$ then u is unique and is a classical solution of the differential equation, i.e., u is in $C^{1,2}((t_k,t_{k+1}) \times \Omega) \bigcap C([t_k,t_{k+1}] \times \overline{\Omega}) \quad (k = 0,1,2,\ldots)$ and u satisfies the differential equation (9)-(12).

We make an assumption.

H. Each $g_k : [0,\infty) \to [0,\infty)$ is a continuous function such that $g_k(v)$ is nondecreasing and

$$\| g_k(v) \|_{L^\infty(\Omega)} \leq M_k \| v \|_{L^\infty(\Omega)} \quad (k = 1,2,\ldots).$$

Moreover, if $v(x)$ is continuous in Ω with zero boundary condition, then each $g_k(v(x))$ satisfies also zero boundary condition.

We present the following theorem.

Theorem 4 *Put $M_0 = \| \psi \|_{L^\infty(\Omega)}$. We assume that H is satisfied. Let $M_1, M_2, \ldots, M_k, \ldots$ be constants and satisfy*

$$M_0 M_1 \cdots M_k \leq L^{-(k+1)} e^{-L(t_{k+1}-t_0)}[2(p-1)(t_{k+1}-t_k)]^{1/(1-p)} 2^{k(1-p)} \quad (k = 0,1,2,\ldots). \tag{13}$$

Then there exists a unique solution $u(t,x)$ to (9) - (12) and it satisfies

$$\| u(t,\cdot) \|_{L^\infty(\Omega)} \leq 2^{p-1}[2(p-1)(t_{k+1}-t_k)]^{1/(1-p)} \quad for\, t \in [t_k,t_{k+1}) \quad (k = 0,1,2,\ldots). \tag{14}$$

II. We consider an impulsive initial-boundary value problem under Robin boundary condition. We give suitable impulsive conditions and control the blow-up times.

We use the following notation. Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. Supose that $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$ are given real numbers.

Define $J \equiv [0,T)$, $J_{imp} \equiv \{t_1,\ldots,t_p\}$, $Q_T \equiv (0,T) \times \Omega$, $\Gamma_T \equiv (0,T) \times \partial\Omega$, $P_k \equiv \{(t_k,x) \mid x \in \Omega\}$, $P \equiv \cup_{k=1}^{p} P_k$, $\Lambda_k \equiv \{(t_k,x) \mid x \in \partial\Omega\}$, $\Lambda \equiv \cup_{k=1}^{p} \Lambda_k$, $u_t = \partial u/\partial t$, $u_{x_i} = \partial u/\partial x_{x_i}$, $u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$.

Let $C^{1,2}(Q_T,P)$ be the class of all functions $u : [0,T] \times \bar{\Omega} \to R$ as follows.

(i) $u(t,x)$ is continuously differentiable in $\bar{Q}_T \setminus (P \cup \Lambda)$.

(ii) There exist $u_{x_i x_j}(t,x)$ $(i,j = 1,2,\ldots,n)$ which are continuous in $(t,x) \in Q_T \setminus P$.

(iii)

$$\begin{aligned}
\lim_{(s,y)\to(t,x)} u(s,y) &= u(t-,x) && \text{for } s<t, \\
\lim_{(s,y)\to(t,x)} u(s,y) &= u(t+,x) && \text{for } s>t,\ (t,x) \in \bar{Q}_T \\
u(t_k,x) &= u(t_k+,x) && \text{for } (t_k,x) \in J_{imp} \times \bar{\Omega}.
\end{aligned}$$

The boundary operator B is defined by $Bu \equiv \partial u/\partial \nu + \beta_0(x)u$, where $\beta_0 \in C(\Gamma_T \setminus \Lambda)$, $\beta_0(x) \geq 0$ and ν is the outward normal vector defined on $\Gamma_T \setminus \Lambda$. The orerator L is defined by

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}$$

and is an uniformly elliptic operator. The coefficients of L belong to $C^{1+\theta}(\bar{\Omega})$ $(0 < \theta < 1)$. The impulsive source is presented by the mappings $g_k :\ R^1 \to\ R^1$ $(k = 1,2,\ldots,p)$.

Consider the following impulsive initial-boundary value problem.

$$\begin{aligned}
u_t - Lu &= f(t,x,u) && \text{in } Q_T \setminus P, && (15)\\
Bu &= 0 && \text{on } \Gamma_T \setminus \Lambda, && (16)\\
u(0,x) &= u_0(x) && \text{in } \bar{\Omega}, && (17)\\
u(t_k,x) &= u(t_k-,x) + g_k(u(t_k-,x)) && \text{in } \bar{\Omega} \quad (k = 1,2,\ldots,p). && (18)
\end{aligned}$$

By $L_{loc}(R_+)$ we denote the set of all functions $f(t,x,u)$ which are locally Lipschitz continuous in $u \in\ R_+ \equiv [0,\infty)$. Further, introduce the following assumptions.

H11. $f \in L_{loc}(R_+)$ and is Hölder continuous for $(t,x) \in [0,T] \times \bar{\Omega}$. And there exists a constant $\alpha > 0$ such that

$$f(t,x,\eta) \geq \lambda_0 \eta + \alpha \eta^{1+\gamma} \quad \text{for } \eta \geq 0,$$

where γ is defined as follows. $\gamma = 0$ for $t \in [0,t_p)$ and $\gamma = \gamma_1$ ($\gamma_1 = const > 0$) for $t \in [t_p,T]$. The constant λ_0 stands for the principal eigenvalue of $(-L)$ and Φ_0 is the corresponding eigenfunction. In what follows we assume $\Phi_0(x)$ is normalized by $\max\{\Phi_0(x) :\ x \in \bar{\Omega}\} = 1$.

H12. The mapping $g_k :\ R^1 \to\ R^1$ $(k = 1,2,\ldots,p)$ possesses the following properties.

(i) $z+g_k(z)$ is nondecreasing smooth function in $z\in R_+$ for each k.

(ii) There exists positive numbers $E_k>0$, $e_k>0$ such that, for $\eta\ge 0$, $E_k\eta\ge g_k(\eta)\ge e_k\eta$ $(k=1,2,\dots,p)$.

H13. For $\hat{\delta},\tilde{\delta}>0$

$$\tilde{\delta}\Phi_0(x)\ge u_0(x)\ge\hat{\delta}\Phi_0(x),$$

where $\Phi_0(x)$ is the eigenfunction corresponding to the principal eigenvalue λ_0 of the operator $(-L)$ defined in H11.

Theorem 5 *Assume that H11-H12 hold and let u be the nonnegative solution to (15)-(18). Then for each constant* $\hat{\delta}>0$ *and each* u_0 $(u_0\ge\hat{\delta}\Phi_0)$*,*

$$u(t,x)\ge A(k)\delta\Phi_0(x)\exp[\alpha t]\quad\text{for}\quad (t,x)\in[t_k,t_{k+1})\times\bar{\Omega}\quad(k=0,1,\dots,p),$$

where

$$A(0)\equiv 1\text{ and }A(k)\equiv\prod_{i=1}^{k}(1+e_i)\qquad(k=1,2,\dots,p).$$

Theorem 6 *Assume that H11-H13 hold. Then there exists* $T^*\in(t_p,T)$ *such that a unique solution* $u(t,x)$ *of the impulsive initial-boundary value problem (15)-(18) in* $[0,T^*)\times\bar{\Omega}$ *has the folloeing property.*

$$B(k)\hat{\delta}\Phi_0(x)\exp[\beta t]\ge u(t,x)\ge A(k)\tilde{\delta}\Phi_0(x)\exp[\alpha t]$$
$$\text{for}\quad (t,x)\in(t_k,t_{k+1})\times\bar{\Omega}\quad(k=1,2,\dots,p-1),$$

where

$$B(0)\equiv 1\text{ and }B(k)\equiv\prod_{i=1}^{k}(1+E_i)\qquad(k=1,2,\dots,p).$$

Moreover, we have $\lim_{t\to T^*}[\max_{x\in\bar{\Omega}}u(t,x)]=+\infty$*, where* T^* *satisfies* $T^*\le t_p+1/[\alpha\gamma_1(\delta_p\Phi_m)^{\gamma_1}]$, $\delta_p\equiv A(p)\hat{\delta}\exp[\alpha t_p]$ *and* $\Phi_m>0$ *is the minimum value in* $\bar{\Omega}$ *of the eigenfunction* $\Phi_0(x)$*.*

Consider the following impulsive parabolic initial-boundary value problem.

$$\begin{aligned}
u_t-\Delta u &= f(t,x,u) && \text{in } Q_T\setminus P, && (19)\\
Bu &= 0 && \text{on } \Gamma_T\setminus\Lambda, && (20)\\
u(0,x) &= u_0(x) && \text{on } \bar{\Omega}, && (21)\\
u(t_k,x) &= g_k(u(t_k-,x))\quad(k=1,2,\dots,m) && \text{on } \bar{\Omega}, && (22)
\end{aligned}$$

where $u_0(x)$ is nonnegative and is in $C^2(\bar{\Omega})$.

We introduce the following assumptions.

H21. Let $f\in L_{loc}(R_+)$ and there exist positive constants γ, μ, σ_0, σ_1 such that for each $\eta\ge 0$ and $x\in\bar{\Omega}$,

$$f(t,x,\eta)\ge\lambda_0\eta+\sigma_0 t^{\mu-1}\eta^{\gamma+1}\quad\text{for}\quad t\in[t_k,t_{k+1})$$

and

$$f(t,x,\eta) \le \lambda_0\eta + \sigma_1 t^{\mu-1}\eta^{\gamma+1} \ \text{ for } \ t \in [t_k, t_{k+1}) \quad (k = 0,1,\ldots,m),$$

where λ_0 stands for the principal eigenvalue of the eigenvalue problem

$$-\Delta\Phi = \lambda\Phi \ \text{ in } \ \Omega, \ \ B\Phi = 0 \ \text{ on } \ \partial\Omega.$$

H22. The mapping $g_k : R^1 \to R^1$ $(k = 1,2,\ldots,m)$ possess the following properties.

(i) $g_k(z)$ are nondecreasing smooth functions in R_+.

(ii) For $k = 1,2,\ldots,m,$ there exist positive numbers $E_k,\ e_k$ such that for $\eta \ge 0$,

$$E_k\eta \ge g_k(\eta) \ge e_k\eta.$$

The constants E_k and e_k actually control the impulsive source g_k.

H23. There exist $\hat{\delta}_0 > 0$ and $\bar{\delta}_0 > 0$ such that

$$\bar{\delta}_0\Phi_0(x) \ge u_0(x) \ge \hat{\delta}_0\Phi_0(x) \ \text{ for } \ x \in \bar{\Omega}.$$

They satisfy

$$0 < \hat{\delta}_0 < \mu^{1/\gamma}\Psi_0^{-1}[\gamma\sigma_0(t_1^\mu - t_0^\mu)]^{-1/\gamma}$$

and

$$0 < \bar{\delta}_0 < \mu^{1/\gamma}[\gamma\sigma_1(t_1^\mu - t_0^\mu)]^{-1/\gamma},$$

where $\Psi_0 = \min\limits_{x\in\bar{\Omega}}\Phi_0(x)$.

We first choose $\hat{\delta}_0$ and $\bar{\delta}_0$ suitablly and fix them. Let us introduce the following notation.

$$\begin{aligned}
N_k &\equiv 1 + \gamma\sigma_0\mu^{-1}\,(\hat{\delta}_k\Psi_0)^\gamma\, t_k^\mu,\\
\bar{N}_k &\equiv 1 + \gamma\sigma_1\mu^{-1}\bar{\delta}_k^\gamma\, t_k^\mu,\\
M_k &\equiv \gamma\sigma_0\mu^{-1}(\hat{\delta}_k\Psi_0)^\gamma \Big/ N_k,\\
\bar{M}_k &\equiv \gamma\sigma_1\mu^{-1}\bar{\delta}_k^\gamma \Big/ \bar{N}_k \qquad\qquad (k = 0,1,\ldots,m).
\end{aligned}$$

We define $\hat{\delta}_k$ and $\bar{\delta}_k$ so that $\hat{\delta}_k \le \bar{\delta}_k$ inductively.

$$\begin{aligned}
\hat{\delta}_{k+1} &= e_{k+1}\frac{\hat{\delta}_k}{(N_k(1 - M_k t_{k+1}^\mu))^{1/\gamma}},\\
\bar{\delta}_{k+1} &= E_{k+1}\frac{\bar{\delta}_k}{(\bar{N}_k(1 - \bar{M}_k t_{k+1}^\mu))^{1/\gamma}} \qquad (k = 0,1,\ldots,m-1).
\end{aligned}$$

H24. The constants $e_1, e_2, \ldots, e_{m-1}$ satisfy the inequalities

$$0 < e_k < \left\{ \frac{\mu N_{k-1}(1 - M_{k-1} t_k^{\mu})}{\gamma \sigma_0 (t_{k+1}^{\mu} - t_k^{\mu})} \right\}^{1/\gamma} (\hat{\delta}_{k-1} \Psi_0)^{-1} \quad (k = 1, 2, \ldots, m-1).$$

We choose $e_1, e_2, \ldots, e_{m-1}$ so that they satisfy H24 and fix them. Then we have

$$0 < 1 - M_k t^{\mu} \ \text{ for } \ t \in [t_k, t_{k+1}] \quad (k = 0, 1, 2, \ldots, m-1).$$

The constant e_m will be chosen in a different manner.

H25. The constants $E_1, E_2, \cdots, E_{m-1}$ satisfy the inequalities

$$0 < E_k < \left\{ \frac{\mu \bar{N}_{k-1}(1 - \bar{M}_{k-1} t_k^{\mu})}{\gamma \sigma_1 (t_{k+1}^{\mu} - t_k^{\mu})} \right\}^{1/\gamma} \bar{\delta}_{k-1}^{-1} \quad (k = 1, 2, \ldots, m-1).$$

We choose also $E_1, E_2, \ldots, E_{m-1}$ so that they satisfy H25 and fix them. Then we have

$$0 < 1 - \bar{M}_k t^{\mu} \quad \text{for } \ t \in [t_k, t_{k+1}] \quad (k = 0, 1, 2, \ldots, m-1).$$

The constants E_m will be chosen in a different manner.

Theorem 7 *Assume that conditions H21-H25 hold and*

$$t_m < T_1 = \bar{M}_m^{-1/\mu} < T_2 = M_m^{-1/\mu} < T.$$

Then there exist $T^* \in [T_1, T_2]$ and a unique solution u to (19)-(22) such that

$$\lim_{t \to T^*} [\max_{x \in \bar{\Omega}} u(t,x)] = +\infty.$$

Consider the initial-boundary value problem for impulsive parabolic equations with an impulsive condition.

$$\begin{aligned}
u_t - Lu &= f(t,x,u) && \text{in } Q_T \setminus P, && (23)\\
Bu &= h(t,x) && \text{on } \Gamma_T \setminus \Lambda, && (24)\\
u(0,x) &= u_0(x) && \text{in } \bar{\Omega}, && (25)\\
u(t_k,x) &= u(t_k-,x) + g_k(u(t_k-,x)) && \text{in } \bar{\Omega} && (26)\\
& && (k = 1, 2, \ldots, p). &&
\end{aligned}$$

We give here the following assumptions.
H31.

(i) $f \in L_{loc}(R_+)$ and f is Hölder continuous for $(t,x) \in [0,T] \times \bar{\Omega}$.

(ii) There exist real numbers $\alpha > 0$ and $\beta > 0$ such that for $\eta \geq 0$,

$$[\alpha(1+t-t_k)^{-1}+\lambda_0]\eta \leq f(t,x,\eta) \leq \beta(1+t-t_k)^{-1}\eta \quad \text{in} \quad [t_k,t_{k+1})\times\Omega \;\; (k=1,2,\ldots,p-1)$$

and

$$[\alpha(1+t-t_p)^{-1}+\lambda_0]\eta \leq f(t,x,\eta) \quad \text{in} \;\; [t_p,t_{p+1})\times\Omega.$$

Here λ_0 stands for the principal eigenvalue of $(-L)$ defined by

$$-L\Phi = \lambda\Phi \;\text{ in } \Omega, \quad B\Phi = 0 \;\text{ on } \partial\Omega.$$

H32. The mappings $g_k : R^1 \to R^1 \quad (k=1,2,\ldots,p)$ possess the following properties.

(i) $z+g_k(z)$ are nondecreasing smooth functions in $z \in R_+$ for each k.

(ii) There exist positive numbers $E_k > 0$, $e_k > 0$ such that, for $\eta \geq 0$, $E_k\eta \geq g_k(\eta) \geq e_k\eta \quad (k=1,2,\ldots,p)$.

H33. For $\hat{\delta} > 0$, we have $u_0 \geq \hat{\delta}\Phi_0(x)$ for $x \in \bar{\Omega}$ and $u_0 \in C^{2+\theta}(\bar{\Omega})$, where $\Phi_0(x)$ is the eigenfunction corresponding to the principal eigenvalue λ_0 of $(-L)$ defined in H31.

H34. The function $h \in C^{1+\theta}([0,T]\times\partial\Omega)$ and is nonnegative.

We put $\rho = \max_{(t,x)\in[0,T]\times\bar{\Omega}} h(t,x)$. Let w_ρ be the solution to the problem

$$\begin{aligned} -Lw_\rho &= 0 \quad \text{in } \Omega, \\ Bw_\rho &= \rho \quad \text{on } \partial\Omega. \end{aligned}$$

There exists a constant $\gamma \geq 1$ such that $\gamma w_\rho \geq 1$ on $\bar{\Omega}$. We put $\Psi_0 \equiv \gamma w_\rho$. Then Ψ_0 satisfies

$$\begin{aligned} -L\Psi_0 &= 0 \quad \text{in } \Omega, \\ B\Psi_0 &= \gamma\rho \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover $\Psi_0 \geq \Phi_0$ on $\bar{\Omega}$ since $\max_{x\in\bar{\Omega}} \Phi_0(x) = 1$.

Having in maind the above stated function $\Psi_0(x)$, we introduce the following assumption.

H35. For $\tilde{\delta} > 1$, we have $\tilde{\delta}\Psi_0(x) \geq u_0(x)$ on $\bar{\Omega}$.

Now we have the following theorem.

Theorem 8 *Let the hypotheses H31-H35 be satisfied. Then there exists $T^* \in (t_p,T)$ such that the impulsive initial-boundary value problem (23)-(26) has a unique solution $u=u(t,x)$ in $[0,T^*)\times\bar{\Omega}$ possessing the following properties.*

$$u(t,x) \geq A(k)\hat{\delta}\Phi_0(x)[1+t-t_k]^{\alpha} \quad \text{in } [t_k,t_{k+1})\times\bar{\Omega} \;\; (k=1,2,\ldots,p)$$

and

$$u(t,x) \leq B(k)\tilde{\delta}\Psi_0(x)[1+t-t_k]^{\beta} \quad \text{in } [t_k,t_{k+1})\times\bar{\Omega} \quad (k=1,2,\ldots,p-1).$$

Here

$$A(0) \equiv 1 \text{ and } A(k) \equiv \prod_{i=1}^{k}(1+e_i)[1+t_i-t_{i-1}]^{\alpha}$$
$$(k,1,2,\ldots,p)$$

and

$$B(0) \equiv 1 \text{ and } B(k) \equiv \prod_{i=1}^{k}(1+E_i)[1+t_i-t_{i-1}]^{\beta} \quad (k=1,2,\ldots,p-1)$$

Consider the following impulsive parabolic initial-boundary value problem.

$$\begin{aligned} u_t - \Delta u &= f(t,x,u) && \text{in } Q_T \setminus P, && (27)\\ Bu &= h(x) && \text{on } \Gamma_T \setminus \Lambda, && (28)\\ u(0,x) &= u_0(x) && \text{in } \bar{\Omega}, && (29)\\ u(t_k,x) &= g_k(u(t_k-,x)) && \text{in } \bar{\Omega} \quad (k=1,2,\ldots,m), && (30) \end{aligned}$$

where $h(x)$ is nonnegative and in $C^{1+\theta}(\partial\Omega)$.

Now we consider the linear problem

$$\begin{aligned} \Delta w &= 0 && \text{in } \Omega,\\ Bw &= h(x) && \text{on } \partial\Omega. \end{aligned}$$

For $h \in C^{1+\theta}(\partial\Omega)$ $(0<\theta<1)$, there exists a unique solution $w \in C^{2+\theta}(\bar{\Omega})$ to the above linear problem. Moreover, if $h(x)$ is nonnegative, then $w \geq 0$ on $\bar{\Omega}$ and $w > 0$ in Ω.

Let $v(t,x) = u(t,x) - w(x)$. Then $v(t,x)$ satisfies the following equation.

$$\begin{aligned} v_t - \Delta v &= f(t,x,v+w) && \text{in } Q_T \setminus P, && (31)\\ Bv &= 0 && \text{on } \Gamma_T \setminus \Lambda, && (32)\\ v(0,x) &= u_0(x) - w(x) && \text{in } \bar{\Omega}, && (33)\\ v(t_k,x) &= g_k(v(t_k-,x)+w(x)) - w(x) && \text{in } \bar{\Omega} && (34)\\ & && (k=1,2,\ldots,m). \end{aligned}$$

Let us introduce the following assumptions.

H41. Let $f \in L_{loc}(\mathbf{R}_+)$ and there exist positive constants γ, σ_0, σ_1 such that for any $\eta \geq 0$ and $(t,x) \in [0,T) \times \bar{\Omega}$,

$$\sigma_0 \eta^{\gamma+1} \leq f(t,x,\eta) \leq \sigma_1 \eta^{\gamma+1}.$$

Let λ_0 be the principal eigenvalue of the eigenvalue problem

$$-\Delta\Phi = \lambda\Phi \text{ in } \Omega, \; B\Phi = 0 \text{ on } \partial\Omega.$$

In what follows we assume that $\Phi_0(x)$ is the eigenfunction corresponding to λ_0 and is normalized by $\max\{\Phi_0(x) : x \in \bar{\Omega}\} = 1$.

H42. The real numbers $\{t_i\}$ $(i=1,2,\ldots,m)$ are chosen so that

$$\begin{aligned} t_1 &< [\gamma\sigma_1(v_M+w_M)^{\gamma}]^{-1},\\ t_{k+1}-t_k &< [C_0\gamma\sigma_1(\hat{\delta}_0+w_M)^{\gamma}]^{-1} \quad (k=1,2,\cdots,m-1). \end{aligned}$$

Here $v_M = \max_{x\in\bar\Omega}[u_0(x) - w(x)]$, $w_M = \max_{x\in\bar\Omega} w(x)$ and $C_0 > 1$ is a constant.

H43. The initial function $u_0(x)$ is subjected to the inequality

$$0 < \hat\delta_0 \Phi_0(x) \le u_0(x) - w(x) \le \bar\delta_0 \text{ in } \bar\Omega$$

for $\hat\delta_0 > 0$ and $\bar\delta_0 > 0$. The constant $\hat\delta_0$ and $w_m = \min_{x\in\bar\Omega} w(x)$ satisfy

$$\hat\delta_0 + w_m > \left(\frac{\lambda_0}{\sigma_0 \phi_m^\gamma}\right)^{1/\gamma}.$$

Here $\phi_m = \min_{x\in\bar\Omega}\Phi(x)$. And the constant $\bar\delta_0$ satisfies

$$t_1 < [\gamma\sigma_1(\bar\delta_0 + w_M)^\gamma]^{-1} < [\gamma\sigma_1(v_M + w_M)^\gamma]^{-1}.$$

Then $1 - \bar M_0 t > 0$ for $t \in [t_0, t_1]$. We first choose $\hat\delta_0$ and $\bar\delta_0$ suitablly and fix them.

H44. Let the mappings $g_k : R^1 \to R^1$ $(k = 1, 2, \dots, m)$ be defined as follows.

(i) Let $\tilde g_k(\zeta(x)) = g_k(\zeta(x) + w(x)) - w(x)$ for $\zeta \in C(\bar\Omega)$. And $\tilde g_k(\eta)$ is nondecreasing smooth function in $\eta \in R_+$.

(ii) There exist positive numbers E_k, e_k which satisfy

$$\begin{aligned} E_k\zeta(x) + (1-E_k)\hat\delta_0\Phi_0(x) &\ge \tilde g_k(\zeta(x)) \ge e_k\zeta(x) + (1-e_k)\hat\delta_0\Phi_0(x) \\ &\quad (k = 1, 2, \dots, m) \end{aligned}$$

for $\zeta(x)$ such that $\zeta(x) \ge \hat\delta_0\Phi_0(x)$ on $\bar\Omega$ and is in $C(\bar\Omega)$. The constants E_k and e_k actually control the impulsive source g_k.

Let us introduce the following notation.

$$\begin{aligned} \bar N_k &= 1 + \gamma\sigma_1(\bar\delta_k + w_M)^\gamma t_k, \\ \bar M_k &= \gamma\sigma_1(\bar\delta_k + w_M)^\gamma / \bar N_k \qquad (k = 0, 1, \dots, m), \\ \rho &= \sigma_0\phi_m^\gamma/\lambda_0. \end{aligned}$$

We define $\hat\delta_k$ and $\bar\delta_k$ so that $\hat\delta_k \le \bar\delta_k$ inductively.

$$\begin{aligned} \bar\delta_{k+1} &= E_{k+1}(\bar\delta_k + w_M)[\bar N_k(1 - \bar M_k t_{k+1})]^{-1/\gamma} - E_{k+1}w_M + (1 - E_{k+1})\hat\delta_0, \\ \hat\delta_{k+1} &= e_{k+1}e^{-\lambda_0(t_{k+1}-t_k)}[(\hat\delta_k + w_m)^{-\gamma} - \rho(1 - e^{-\lambda_0\gamma(t_{k+1}-t_k)})]^{-1/\gamma} \\ &\qquad - e_{k+1}w_m + (1 - e_{k+1})\hat\delta_0 \\ &\qquad\qquad (k = 0, 1, \dots, m-1). \end{aligned}$$

H45. Each E_k $(k = 1, 2, \dots, m-1)$ satisfies

$$\begin{aligned} &E_k\{(\bar\delta_{k-1} + w_M)[\bar N_{k-1}(1 - \bar M_{k-1}t_k)]^{-1/\gamma} - w_M - \hat\delta_0\} \\ &\quad < (C_0^{1/\gamma} - 1)(\hat\delta_0 + w_M). \end{aligned}$$

Under the hypothesis H45, we have $0 < 1 - \bar M_k t$ for $t \in [t_k, t_{k+1}]$ $(k = 0, 1, 2, \dots, m-1)$.

Put $T_1 = t_m + 1/\sigma_1\gamma(\bar\delta_m + w_M)^\gamma = \bar M_m^{-1}$ and $T_2 = t_m + (\lambda_0\gamma)^{-1}\log[\rho(\hat\delta_m + w_m)^\gamma/\{\rho(\hat\delta_m + w_m)^\gamma - 1\}]$. Then we have the following.

Theorem 9 *Assume that conditions H41-H45 hold and*

$$t_m < T_1 < T_2 < T.$$

Then there exist $T^* \in [T_1, T_2]$ and a unique solution u to (27)-(30) such that

$$\lim_{t\to T^*}[\max_{x\in\Omega} u(t,x)] = +\infty.$$

III. We consider effective and ineffective impulsive controls. We treat the problem which arises from the nuclear reactor dynamics. The general formulation of this problem is too complicated. So we restrict ourselves in the initial and periodic boundary value problem. We explain the meaning of effective and ineffective impulsive controls and show the existence of ineffective control.

Namely, we consider the following initial and periodic boundary value problem.

$$u_t - u_{xx} = u^2 \quad \text{for } t > 0,\ t \neq t_i,\ x \in R^1 \quad (i = 1,2,\ldots), \tag{35}$$

$$u(t, x+2\pi) = u(t,x) \quad \text{for } t > 0,\ x \in R^1, \tag{36}$$

$$u(0,x) \equiv \lim_{t\to+0} u(t,x) = \frac{1}{2}\{\lim_{x'\to x-0} u_0(x') + \lim_{x'\to x+0} u_0(x')\}, \tag{37}$$

where $u_t = \partial u/\partial t, u_{xx} = \partial^2 u/\partial x^2$, $u_0(x+2\pi) = u_0(x)$ $x \in R^1$ and u_0 is a nonnegative function of bounded variation.

We add the first type of impulsive control.

$$\max_{x\in R^1} u(t,x) < S \quad \text{for } t \in [t_{i-1}, t_i), \tag{38}$$

$$\max_{x\in R^1} u(t_i-,x) = S, \tag{39}$$

$$u(t_i,x) = \lim_{t\to t_i+0} u(t,x) = \alpha u(t_i-,x) \tag{40}$$

$$\text{for } x \in R^1 \quad (i = 1,2,\ldots),$$

$$0 = t_0 < t_1 < \cdots < t_i < \cdots,$$

where $\max_{x\in R^1} u(t_i-,x) = \lim_{t\to t_i-0}[\max_{x\in R^1} u(t,x)]$. Let S and α $(0 < \alpha < 1)$ be given constants. Let u_0 satisfy $\max_{x\in R^1} u_0(x) < S$.

Then the solution $u(t,x)$ will be continued globally in time. And the length of each time interval $(t_{i+1} - t_i)$ $(i = 1,\ 2,\ \ldots)$ is greater than $(1-\alpha)/\alpha S$.

Secondarily we replace the first type of impulsive control by the following one. Let a sequence $\{t_i\}$ be given and satisfy the following. $0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots$ and $t_i - t_{i-1} = s^0$ $(i = 1,2,\ldots)$. Here s^0 is a positive constant and satisfies $s_0 < 2\pi U_0^{-1}$. We put the following impulsive control:

$$u(t_i,x) = \lim_{t\to t_i+0} u(t,x) = \alpha u(t_i-,x) \quad \text{for } x \in R^1 \quad (i = 1,2,\ldots).$$

We put $U_i = \int_{-\pi}^{\pi} u(t_i,y)dy$ $(i = 0,1,\ldots)$. It is easy to see that

$$U_i \geq \frac{2\pi\alpha}{2\pi - s^0 U_{i-1}} U_{i-1} \quad i = 1,2,\ldots.$$

If we assume that a constant α satisfies $2\pi\alpha/(2\pi - s^0U_0) > 1$, then we have $U_1 > U_0$. We generally have $U_i > (2\pi\alpha/(2\pi - s^0U_0))^i U_0$. Then there exists the maximum integer n such that $2\pi - s^0U_n > 0$. This means that there exists T $(0 < T < +\infty)$ and we have $\lim_{t\to T-0}[\max_{x\in R^1} u(t,x)] = +\infty$.

Finally we replace the first type of impulsive control by

$$\begin{aligned}
\min_{x\in R^1} u(t,x) &< S \quad \text{for } t \in [t_{i-1},t_i), && (41)\\
\min_{x\in R^1} u(t_i-,x) &= S, && (42)\\
u(t_i,x) &= \frac{1}{2}\{\lim_{x'\to x-0} \varphi(x')u(t_i-,x') + \lim_{x'\to x+0} \varphi(x')u(t_i-,x')\},\\
&\qquad \text{for } x \in R^1 \quad (i = 1,2,\ldots) && (43)\\
0 &= t_0 < t_1 < \cdots < t_i < \cdots,
\end{aligned}$$

where $\min_{x\in R^1} u(t_i-,x) = \lim_{t\to t_i-0}[\min_{x\in R^1} u(t,x)]$ and $\varphi(x) = 1$ for $-a+2n\pi \le x \le a+2n\pi$ $(0 < a < \pi,\ n:\text{integer})$ and $\varphi(x) = 0$ otherwise. Let S be a given constant. Let u_0 satisfy $\min_{x\in R^1} u_0(x) < S$. A function u and a sequence $\{t_i\}$ are unknown. We will consider whether the solution u will grow unbounded or not. Namely, we consider whether the total mass of $u(t,x)$ will grow unbouded or not. If it occurs, the length of each time interval $[t_i,t_{i-1})$ may decrease. It is not easy to compare the amount of increase of the total mass and the amount of loss which is caused by the impulsive control. In the above second case, the relation between U_i and U_{i-1} is clear. But in the last case, we have nothing about the relation. We will mainly consider the phenomenon of the last type.

We state the meanings of effective and ineffective impulsive controls. If there exists a global solution to an impulsive differential equation and the length of each time interval between contiguous controls is greater than some positive constant, then we say that the impulsive control is effective. If the control is not effective, then we say that it is ineffective. For example, the impulsive control (38)-(40) is effective. We discuss the existence of ineffective impulsive control.

We consider the system in each time interval $(t_{i-1},\ t_{i-1}+T_i)$. For $x \in R^1$,

$$\begin{aligned}
u_t^{(i)} - u_{xx}^{(i)} &= \{u^{(i)}\}^2 && \text{in } (t_{i-1},t_{i-1}+T_i), && (44)\\
u^{(i)}(t,x+2\pi) &= u^{(i)}(t,x) && \text{in } (t_{i-1},t_{i-1}+T_i), && (45)\\
u^{(i)}(t_{i-1},x) &= \frac{1}{2}\{\lim_{x'\to x-0} u(t_{i-1},x') + \lim_{x'\to x+0} u(t_{i-1},x')\}, &&&& (46)\\
\lim_{t\to t_{i-1}+T_i-0} \max_{x\in R^1} u^{(i)}(t,x) &= +\infty, && (i = 1,2,\ldots). && (47)
\end{aligned}$$

Here u is a solution to (35)-(37) with (41)-(43). It is easy to see that each $u^{(i)}(t,x)$ coincides with $u(t,x)$ in $[t_{i-1},t_i) \times R^1$. The constant T_i is called the blow-up time and represents the growth of $u^{(i)}$. It is well known that T_i is finite. There are two cases. One is that there exists t_i such that $t_i < t_{i-1}+T_i$ for each i. Another is that there does not exist t_i for some i, i.e., $u^{(i)}(t,x)$ blows up before the minimum value of $u^{(i)}(t,x)$ arrives at S. If the latter case occurs, then this process will be suspended suddenly. So, this control is ineffective. In what follows, we assume that the number of recurrences of control is infinite.

We define the set **IN** of initial functions as follows. The function u_0 is in the set **IN** if u_0 satisfies the following.

1. u_0 is defined in R^1 and $u_0(x+2\pi) = u_0(x)$ for $x \in R^1$.
2. u_0 is a nonnegative function and has at most a finite number of discontinuous pionts in $[-\pi, \pi]$.
3. $u_0(-x) = u_0(x)$ for $x \in [-\pi, \pi]$.
4. $u_0(x) = 0$ for $x \in [-\pi + 2n\pi,\ -a+2n\pi) \cup (a+2n\pi,\ \pi+2n\pi]$. Here a $(0 < a < \pi)$ is a positive constant and n is an integer.
5. $u_0(x)$ is non-increasing in $[0,\ \pi]$.

We assume that u_0 in (37) is in IN.

We have two theorems related to ineffective control.

Theorem 10 *We assume that there exists a positive constant κ (≥ 1) such that $T_i U_{i-1} \geq 1/\kappa$ for $i = 1, 2, \ldots$ and κ is independent of a and S. Here $U_i = \int_{-\pi}^{\pi} u(t_i, y)dy$. If we choose the constants a and S suitably, then we have*

$$U_i > U_{i-1} \qquad\qquad (i = 1, 2, \ldots).$$

Moreover there exists T^ω $(0 < T^\omega < \infty)$ such that $\lim_{t \to T^\omega - 0} [\max_{x \in R^1} u(t,x)] = +\infty$. Here $u(t,x)$ is the solution to (35)-(37) and (41)-(43).

Theorem 11 *We assume that there does not exist a positive constant κ (≥ 1) such that $T_i U_{i-1} \geq 1/\kappa$ for $i = 1, 2, \ldots$ and κ is independent of a and S. If we take a and S suitably for any $\varepsilon > 0$, then there exists i such that $T_i < \varepsilon$, i.e., $0 < t_i - t_{i-1} < \varepsilon$.*

Moreove, we give the theorem which states the growth of the solution. We consider the following.

$$\begin{aligned}
v_t - v_{xx} &= v^2 && \text{for } 0 < t < T,\ x \in R^1, && (48)\\
v(t, x+2\pi) &= v(t,x) && \text{for } 0 < t < T,\ x \in R^1, && (49)\\
v(0,x) &\equiv \lim_{t \to +0} v(t,x) = \frac{1}{2}\{\lim_{x' \to x-0} u_0(x') + \lim_{x' \to x+0} u_0(x')\}. && && (50)
\end{aligned}$$

We assume that

$$\begin{aligned}
\min_{x \in R^1} v(t,x) &< S && \text{for } t \in [0, t_1),\\
\min_{x \in R^1} v(t_1-, x) &= S,\\
v(t_1, x) &= \varphi(x) v(t_1-, x) && \text{for } x \in R^1,
\end{aligned}$$

where $\varphi(x)$ is defined in (43). Moreover, we assume that a constant a satisfies the inequalities $a^2 + C_0 C_1 a - \pi C_0 C_1 > 0$ and $\pi/2 < a < \pi$. We fix a.

Theorem 12 *Let u_0 be in IN. Under the above assumptions, there exists a positive constant S_a such that for each S $(S > S_a)$ we have*

$$V_1 > V_0, \tag{51}$$

where $V_0 = \int_{-\pi}^{\pi} v(0,y)dy$ and $V_1 = \int_{-\pi}^{\pi} v(t_1,y)dy$.