Infinite Games, Inductive Definitions and Transfinite Recursion

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博士論文

Infinite Games, Inductive Definitions and Transfinite Recursion

(無限ゲーム, 帰納的定義, 超限的再帰)

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Contents

Motivation of this research

The purpose of this research is to investigate the logical strength of determinacy of Gale-Stewart games from the standpoint of reverse mathematics. More precisely, we observe the determinacy of infinite games on the hierarchy between Σ^0_2 and Δ^0_3 by using variations of inductive definitions. The inductive definition is first formalized as a subsystem of second order arithmetic by K. Tanaka in [17] in order to characterize the determinacy of Σ^0_2-games. The determinacy of Δ^0_3-games are pinned down with the inductive definition with transfinitely many operators ([9]). In this research, we prove that the determinacy on finer classes, so-called difference classes, can be characterized by inductive definitions with multiple operators and their transfinite recursion.

This thesis consists of six chapters. In chapter 1, we give an overview of backgrounds on most important concepts in this thesis. In chapter 2 and 4, we explain the basic knowledge and fundamental results for this research. In chapter 5, the proofs of the main theorem begins, theorem 5.2.2. We also explain some proofs which had been obtained by previous researches especially in [17] and [9] because theorem 5.2.2 can be viewed as a general version of them. Thus, we, in some parts, modify their proofs in order to be used for the proof of the main theorem.

We explain the some key topics in thesis thesis such as *reverse mathematics, determinacy of Gale-Stewart games, inductive definitions* and so forth below.

【Reverse Mathematics Program】

This research is a part of the *reverse mathematics program*, founded by Harvey Friedman in 1970's.

In the study of reverse mathematics, we formalize ordinary mathematics by using *an language of second order arithmetic* $\mathcal{L}_2$. This is a two-sorted language, whose variables are ranging over natural numbers and subsets of natural numbers. An arithmetic with two-sorted language is called *second order arithmetic*, denoted by Z_2. Z_2 consists of infinitely many axiom systems with different strengths. We explain the major subsystems of Z_2 in section 3.1 of chapter 3.

The main theme of the reverse mathematics program is the following:

Find out necessary and sufficient axiom systems to prove theorems of ordinary mathematics.

By the decades of studies, it is proved that most of classical mathematical theorems are equivalent to one of five subsystems of Z_2. These systems are called a big five and extensively studied by many researchers. A book titled "Subsystems of

Second order arithmetics", by Stephen G. Simpson, is the standard text book of this area [13].

【Determinacy of Gale-Stewart games】

In this research, we investigate logical strength of determinacy of Gale-Stewart games in second order arithmetic Z_2. This game is named after D. Gale and F. M. Stewart. This is a very simple game as follows: Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a set of infinite sequences of natural numbers. Two players, player I and player II, choose natural numbers in turn, and eventually an infinite sequence of natural numbers $n_0, n_1, n_2 \ldots$ will be constructed. Then, Player I wins if $n_0, n_1, n_2 \cdots \in A$, and player II wins if $n_0, n_1, n_2 \cdots \notin A$.

In such a game, it may be natural to think that computing a winning strategy becomes harder if a set A becomes more complicated, such as, a clopen, open, Borel, and so on. If one of the players has a winning strategy in a game G_A for any open set A, then we call the open game G_A is determinate, or simply *open determinacy*. (So, if the same thing holds for any Borel sets A, we call Borel game is determinate, or Borel determinacy.)

Indeed, Borel determinacy is too strong for second order arithmetic Z_2 to prove it. In order to prove Borel determinacy, we need stronger axiom systems, and actuary D. Martin in 1975 showed that Zermelo-Fraenkel set theory plus axiom of choice, denoted by ZFC, can prove Borel determinacy. However, it is known that ZFC can not prove the determinacy of all projective sets. Determinacy of games can be a quite strong statement, and it easily goes beyond axiom system Z_2 or even ZFC. Assuming the determinacy of games, we can get many interesting results, but some question may arise: "what does the determinacy assert?"

Indeed, determinacy of games can be regarded as statements asserting existences of sets with certain complexities. In this research, we give a characterization to relatively weak determinacy of games by subsystems of second order arithmetic Z_2, called *inductive definitions*.

【Inductive definitions and their transfinite recursion】

Inductive definition as a subsystem of second order arithmetic is first introduced by K.Tanaka in [17]. That is,

Definition 1 (K.Tanaka, 1991)**.** Let $\mathcal{C}$ be a class of $\mathcal{L}_2$-formulas. Γ-ID asserts that for any operator $\Gamma \in \mathcal{C}$, there exists a set $W \subseteq \mathbb{N} \times \mathbb{N}$ such that:

1. W is a pre-well-ordering on its field F,
2. $\forall x \in F \quad W_x = \Gamma(W_{<x}) \cup W_{<x}$,
3. $\Gamma(F) \subset F$.

where, $W_x = \{y \in F : (y,x) \in W\}$, $W_{<x} = \{y \in F : (y,x) \in W$, and $(x,y) \notin W\}$.

Inductive definitions are quite natural ways to define sets. We let Γ be an operator from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$. Then, applying it to an empty set $\emptyset$, we obtain a set $\Gamma(\emptyset)$. After that, again, apply Γ to $\Gamma(\emptyset)$ and take a union of them, we have $\Gamma(\emptyset) \cup \Gamma(\Gamma(\emptyset))$. If this procedure is continued and taking unions of them such as $\Gamma(\emptyset) \cup \Gamma(\Gamma(\emptyset)) \cup \Gamma(\Gamma(\emptyset) \cup \Gamma(\Gamma(\emptyset))) \cup \ldots$, the axiom scheme of inductive definition asserts that there exists a fixed points F such that $\Gamma(F) \subset F$. In this research, we basically consider Σ^1_1-operators. An operator $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is Σ^1_1 if its graph is expressed by a Σ^1_1 formula. Then, an axiom scheme, inductive definition with a Σ^1_1-operator, is denoted by Σ^1_1-ID_0.

In 1991, K. Tanaka formalized the inductive definitions in second order arithmetic and showed that Σ^1_1-ID_0 is equivalent to the determinacy of Σ^0_2 games. This is one of the most important results, and this research is based on it. One of the importances can be that Σ^1_1-ID_0 is introduced as a subsystem of Z_2. Sets defined by Σ^1_1-ID_0 are different from those defined by ordinary comprehension axioms. This difference makes us possible to investigate the structure of subsystems of Z_2 from different aspects.

In the sense of determinacy, it is not possible to characterize Σ^0_2-determinacy by comprehension axioms. The strongest determinacy which is pinned down by a subsystem of Z_2 is Δ^0_3-determinacy, and it is known to be equivalent to $[\Sigma^1_1]^{\mathsf{TR}}$-$\mathsf{ID}_0$ [9]. ($[\Sigma^1_1]^{\mathsf{TR}}$-$\mathsf{ID}_0$ asserts the existance of sets defined by Σ^1_1-ID_0 with transfinitly many operators.) In this research, in order to investigate the logical strength of determinacy between classes of Σ^0_2 and Δ^0_3, known as Wedge classes, we introduced the following axiom system. For easiness, we just see the case where the number of Σ^1_1-operators is two.

Definition 2. The formal definition of $[S_0, S_1]$-IDTR_0 consists of ACA_0 and the following axiom scheme: Let S_0 and S_1 are collections of operators. The axiom scheme $[S_0, S_1]$-IDTR_0 asserts the following. For any well-ordering $\preceq$ and any $\Gamma_0 \in S_0, \Gamma_1 \in S_1$, there exist $\langle W^r : r \in \text{field}(\preceq)\rangle$, $\langle V^{r,x} : r \in \text{field}(\preceq), x \in F^r_1\rangle$ and $\langle V^{r,\infty} : r \in \text{field}(\preceq)\rangle$ such that the following are all satisfied.

1. W^r is pre-well-ordering on its field F^r_1.
2. $\forall x \in F^r_1 \cup \{\infty\}$
   - $V^{r,x}$ is pre-well-ordering on its field $F^{r,x}_0$.
   - $V^{r,x}_y = \Gamma_0^{F_1^{\prec r} \oplus W^r_{<x}}(V^{r,x}_{<y}) \cup V^{r,x}_{<y}$ for all y $\in F^{r,x}_0$.
   - $W^r_x = \Gamma_1^{F_1^{\prec r}}(F^{r,x}_0) \cup W^r_{<x}$.

- $\Gamma_0^{F_1^{\prec r} \oplus W_{<x}^r}(F_0^{r,x}) \subset F_0^{r,x}$.

3. $W_\infty^r = W_{<\infty}^r = F_1^r$.

where $F_1^{\prec r} = \oplus\{F_1^{r_i} : r_i \prec r\}$. Note also that $X \oplus Y = \{2x : x \in X\} \cup \{2y+1 : y \in Y\}$.

This axiom system asserts the existence of sets defined by transfinite recursion of Σ_1^1-ID_0 with multiple operators. Then, by using this axiom system, we characterize the determinacy of classes between Σ_2^0 and Δ_3^0.

Theorem 3 (Main Theorem)**.** Over RCA_0, the following are equivalent. For any $k > 0$,

(1) $\Delta((\Sigma_2^0)_{k+1})$-$\mathsf{Det}$.

(2) $\mathsf{Sep}(\Delta_2^0, (\Sigma_2^0)_k)$-$\mathsf{Det}$.

(3) $[\Sigma_1^1]^k$-IDTR_0.

Conclusion

The following diagram shows the results on determinacy strength of Δ_3^0 games in second order arithmetic. The left column contains subsystems of second order arithmetic from weaker to stronger. The right column contains classes of the games in the Baire space. Each row represents that a certain axiom is equivalent to the determinacy of the corresponding games over appropriate systems (RCA_0, but with Π_3^1-TI for the last row).

Subsystem of SOA	Determinacy in Baire space
ATR_0	Δ_1^0 Σ_1^0
Π_1^1-CA_0	$\Delta((\Sigma_1^0)_2) = \mathsf{Sep}(\Delta_1^0, \Sigma_1^0)$ $(\Sigma_1^0)_2$
Π_1^1-TR_0	Δ_2^0
Σ_1^1-ID_0	Σ_2^0 $\mathsf{Sep}(\Delta_1^0, \Sigma_2^0)$ $\mathsf{Sep}(\Sigma_1^0, \Sigma_2^0)$
Σ_1^1-IDTR_0	$\Delta((\Sigma_2^0)_2) = \mathsf{Sep}(\Delta_2^0, \Sigma_2^0)$
$\vdots$	$\vdots$
$[\Sigma_1^1]^k$-ID_0	$(\Sigma_2^0)_k$
$[\Sigma_1^1]^k$-IDTR_0	$\Delta((\Sigma_2^0)_{k+1}) = \mathsf{Sep}(\Delta_2^0, (\Sigma_2^0)_k)$
$\vdots$	$\vdots$
$[\Sigma_1^1]^{\mathrm{TR}}$-$\mathsf{ID}_0$	Δ_3^0

In this thesis, we introduced the axiom of transfinite recursion of Σ^1_1 inductive definitions with k operators, denote $[\Sigma^1_1]^k$-IDTR_0, and showed that it is equivalent to the determinacy of $\Delta((\Sigma^0_2)_{k+1})$ sets. A key fact used in the proof is that a $\Delta((\Sigma^0_2)_{k+1})$ set is expressed as a $\mathsf{Sep}(\Delta^0_2, (\Sigma^0_2)_k)$ set, namely a Δ^0_2-separated union of a $(\Sigma^0_2)_k$ set and $(\Pi^0_2)_k$ set. By virtue of this fact, we can utilize a difference hierarchy for a Δ^0_2 set (cf. [16], [9]) to construct a winning strategy for a $\Delta((\Sigma^0_2)_{k+1})$ game.

In [9], the exact determinacy strength of Δ^0_3 sets has been pinned down in terms of transfinte combinations of Σ^1_1 inductive definitions. We should notice that their axiom for transfinte combinations of Σ^1_1 inductive definitions is much stronger than $[\Sigma^1_1]^k$-IDTR_0. However, it is worth studying such an axiom as $[\Sigma^1_1]^\alpha$-IDTR_0, where α is an ordinal, to refine their result on Δ^0_3-games.

Montalbán ans Shore [11] show that for any $m \geq 1$, Π^1_{m+2}-CA_0 proves the determinacy of $(\Sigma^0_3)_m$ sets, but Δ^1_{m+2}-CA_0 does not. Thus, $(\Sigma^0_3)_\omega$-determinacy is not provable over Z_2. Then, Montalbán [10] raises Question 28 to classify the precise strength of $(\Sigma^0_3)_m$-determinacy.

In [1], Bradfield has shown that the sets of Player I's winning positions of a $(\Sigma^0_2)_k$-game are exactly the same as the $(k+1)$-level of μ-calculus alternation hierarchy Σ^μ_{k+1}. Then, Bradfield [2] claims that the hierarchy $\langle \Sigma^\mu_n, n \in \omega \rangle$ is strict, that is, for any k in ω, we have $\Sigma^\mu_k \subsetneqq \Sigma^\mu_{k+1}$. This result easily follows from the previous result on multiple inductive definitions ([9]) together with observation that for any k in ω, Π^1_2-CA_0 proves the consistency of Δ^1_2-$\mathsf{CA}_0 + (\Sigma^0_2)_k$-determinacy, while it does not prove the consistency of $(\Sigma^0_2)_{<\omega}$-determinacy. (cf. Heinatsch and Möllerfeld [4])

From the main result of this paper, we will also obtain the following refinement. First of all, the hierarchy $\langle \Pi^\mu_n, n \in \omega \rangle$ is naturally defined and so is $\langle \Delta^\mu_n, n \in \omega \rangle$. Then, by the argument of this paper, we can associate a Δ^μ_{n+1} formula with transfinite recursion of a Σ^μ_k formula. Moreover, for any k in ω, we have $\Sigma^\mu_k \subsetneqq \Delta^\mu_{k+1} \subsetneqq \Sigma^\mu_{k+1}$ by a similar observation as above.

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