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ORTHOGONAL DRAWINGS OF SERIES-PARALLEL GRAPHS WITH MINIMUM BENDS*

XIAO ZHOU† AND TAKAO NISHIZEKI†

Abstract. In an orthogonal drawing of a planar graph G, each vertex is drawn as a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common end. A bend is a point where an edge changes its direction. A drawing of G is called an optimal orthogonal drawing if the number of bends is minimum among all orthogonal drawings of G. In this paper we give an algorithm to find an optimal orthogonal drawing of any given series-parallel graph of the maximum degree at most three. Our algorithm takes linear time, while the previously known best algorithm takes cubic time. Furthermore, our algorithm is much simpler than the previous one. We also obtain a best possible upper bound on the number of bends in an optimal drawing.

Key words. orthogonal drawing, bend, series-parallel graph, planar graph

AMS subject classifications. 05C85, 05C10

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1. Introduction.

Automatic graph drawings have numerous applications in VLSI circuit layouts, networks, computer architecture, circuit schematics, etc. [3, 11]. Many graph drawing styles have been introduced [1, 3, 9, 11, 15, 17]. Among them, an "orthogonal drawing" has attracted much attention due to its various applications, especially in circuit schematics, entity relationship diagrams, data flow diagrams, etc. [14, 16, 19, 20]. An *orthogonal drawing* of a planar graph G is a drawing of G such that each vertex is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common end. A point where an edge changes its direction in a drawing is called a *bend* of the drawing. Figure 1.1(a) depicts an orthogonal drawing of the planar graph in Figure 1.1(b); the drawing has exactly one bend on the edge joining vertices g and t. If a planar graph G has a vertex of degree five or more, then G has no orthogonal drawing. On the other hand, if G has no vertex of degree five or more, that is, the maximum degree Δ of G is at most four, then G has an orthogonal drawing, but may need bends. If a planar graph represents a VLSI routing, then one may be interested in an orthogonal drawing such that the number of bends is as small as possible, because bends increase the manufacturing cost of a VLSI chip. An orthogonal drawing of a planar graph G is called an *optimal* orthogonal drawing if it has the minimum number of bends among all possible orthogonal drawings of G.

The problem of finding an optimal orthogonal drawing is one of the most famous problems in the graph drawing literature [3, 11] and has been studied both in the fixed embedding setting [7, 14, 16, 18, 20] and in the variable embedding setting [5, 6, 8, 12, 13]. A planar graph with a fixed embedding is called a *plane graph*. As a result in the fixed embedding, Tamassia [20] presented an algorithm to find an optimal orthogonal drawing of a plane graph G in time $O(n^2 \log n)$, where n is the

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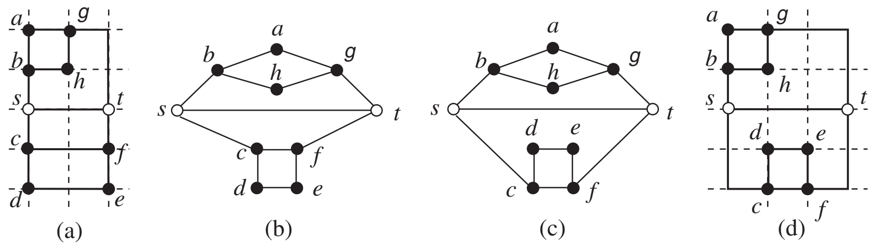


FIG. 1.1. (a) *An optimal orthogonal drawing with one bend,* (b), (c) *two embeddings of the same planar graph, and* (d) *an orthogonal drawing with three bends.*

number of vertices in G; he reduced the optimal drawing problem to a min-cost flow problem. Then Garg and Tamassia improved the complexity to $O(n^{7/4}\sqrt{\log n})$ [7]. As a result in the variable embedding setting, Garg and Tamassia showed that the problem is NP-complete for planar graphs of $\Delta \leq 4$ [8]. However, Di Battista, Liotta, and Vargiu [5] showed that the problem can be solved in polynomial time for a planar graph G of $\Delta \leq 3$. Their algorithm finds an optimal orthogonal drawing among all possible plane embeddings of G. They use the properties of "spirality," min-cost flow techniques, and a data structure, call a SPQ*R tree that implicitely represents all of the plane embeddings of G. The algorithm is complicated and takes time $O(n^5 \log n)$ for a planar graph of $\Delta \leq 3$. Using the algorithm, one can find more efficiently an optimal orthogonal drawing for a biconnected series-parallel simple graph; it takes time $O(n^4)$ if $\Delta \leq 4$ [5] and takes time $O(n^3)$ if $\Delta \leq 3$ [4, 5]. Note that every series-parallel graph is planar. Series-parallel graphs arise in a variety of problems such as scheduling, electrical networks, data-flow analysis, database logic programs, and circuit layout [21]. On the other hand, Garg and Liotta give an algorithm to find a nearly optimal orthogonal drawing for a biconnected planar graph of $\Delta \leq 3$ [6]. Their algorithm finds a drawing having at most three more bends than the optimal one in time $O(n^2)$. The complexities $O(n^5 \log n)$, $O(n^4)$, and $O(n^3)$ above for an exact algorithm in the variable embedding setting are very high, and it is expected to obtain an efficient algorithm for a particular class of planar graphs of $\Delta \leq 3$ [2].

In this paper we deal with the class of series-parallel (multi)graphs of $\Delta \leq 3$ and give a simple linear algorithm to find an optimal orthogonal drawing in the variable embedding setting. The graph G in Figure 1.1 is series-parallel and has various plane embeddings; two of them are illustrated in Figures 1.1(b) and (c); there is no plane embedding having an orthogonal drawing with no bend; however, the embedding in Figure 1.1(b) has an orthogonal drawing with one bend, as illustrated in Figure 1.1(a), and hence the drawing is optimal; the embedding in Figure 1.1(c) needs three bends, as illustrated in Figure 1.1(d); given G, our algorithm finds an optimal drawing in Figure 1.1(a). Our algorithm works well even if G has multiple edges or is not biconnected and is much simpler and faster than the algorithms for biconnected series-parallel simple graphs in [4, 5]; we use neither the min-cost flow technique nor the SPQ*R tree, but uses some structural features of series-parallel graphs, which have not been exploited in [21]. We furthermore obtain a best possible upper bound on the minimum number of bends. An early version of the paper has been presented at [22].

The rest of the paper is organized as follows. In section 2 we present some definitions and our main idea. In section 3 we present an algorithm and an upper bound for biconnected series-parallel graphs. In section 4 we present an algorithm and an upper bound for nonbiconnected series-parallel graphs. Finally section 5 is a conclusion.

2. Preliminaries. In this section we present some definitions and our main idea.

Let $G = (V, E)$ be an undirected graph, with vertex set V and edge set E. We denote the number of vertices in G by $n(G)$ or simply by n. For a vertex $v \in V$, we denote by $G - v$ the graph obtained from G by deleting v. An edge joining vertices u and v is denoted by uv. We denote by $G - uv$ the graph obtained from G by deleting uv. We denote the degree of a vertex v in G by $d(v, G)$ or simply by $d(v)$. We denote the maximum degree of G by $\Delta(G)$ or simply by Δ. A connected graph is *biconnected* if there is no vertex whose removal results in a disconnected graph or a single-vertex graph K_1. A *plane graph* is a fixed embedding of a planar graph.

Let G be a planar graph with $\Delta \leq 3$. We denote by $bend(G)$ the number of bends of an optimal orthogonal drawing of G in the variable embedding setting. (Thus $bend(G) = 1$ for the graph G in Figure 1.1.) Let D be an orthogonal drawing of G. The number of bends in D is denoted by $bend(D)$. Of course, $bend(G) \leq bend(D)$. Let $G(D)$ be a plane graph obtained from a drawing D by replacing each bend in D with a new vertex. Figures 2.1(a) and (b) depict $G(D)$ for the drawings D in Figures 1.1(a) and (d), respectively. An angle formed by two edges e and e' incident to a vertex v in $G(D)$ is called an *angle of vertex* v if e and e' appear consecutively around v. An angle of a vertex in $G(D)$ is called an *angle of the plane graph* $G(D)$. In an orthogonal drawing, every angle is $\pi/2$, π, $3\pi/2$, or 2π. Consider a labeling l, which assigns a label 1, 0, -1, or -2 to every angle of $G(D)$. Labels 1, 0, -1, and -2 correspond to angles $\pi/2$, π, $3\pi/2$, and 2π, respectively. We call l a *regular labeling* of $G(D)$ if l satisfies the following three conditions (a)–(c) [11, 20]:

(a) for each vertex v of $G(D)$,
  (a-1) if $d(v) = 1$, then the label of the angle of v is -2;
  (a-2) if $d(v) = 2$, then the labels of the two angles of v total to 0; and
  (a-2) if $d(v) = 3$, then the labels of the three angles of v total to 2;
(b) the sum of the labels of each inner face is 4; and
(c) the sum of the labels of the outer face is -4.

Figures 2.1(a) and (b) illustrate regular labelings for the orthogonal drawings in Figures 1.1(a) and (d), respectively. If D is an orthogonal drawing of G, then clearly $G(D)$ has a regular labeling. Conversely, every regular labeling of $G(D)$ corresponds to an orthogonal drawing of G [20]. An orthogonal (geometric) drawing of G can be obtained from a regular labeling of $G(D)$ in linear time, that is, in time $O(n(G) + bend(D))$ [11, 20]. Therefore, from now on, we call a regular labeling of $G(D)$ an *orthogonal*

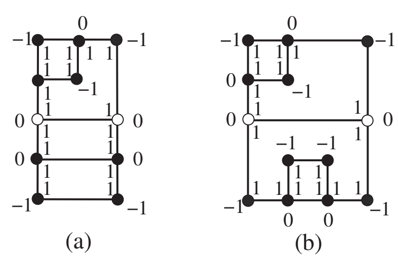


FIG. 2.1. *Regular labelings of* $G(D)$ *corresponding to the drawings* D *in Figures* 1.1(a) *and* (d), *respectively.*

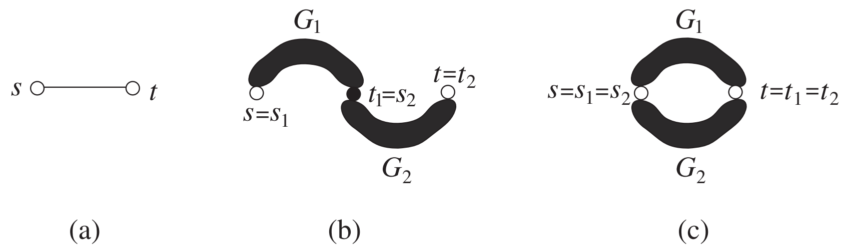


FIG. 2.2. (a) K_2, (b) *series, and* (c) *parallel connections.*

drawing of a planar graph G or simply, a *drawing of* G, and obtain a regular labeling of G in place of an orthogonal (geometric) drawing of G.

For a drawing D of a planar graph G and for a subgraph G' of G, we denote by $D|G'$ the drawing of G' in D. Clearly

$$bend(G') \leq bend(D|G') \leq bend(D). \tag{2.1}$$

Let $\overline{G'}$ be the complement of G', that is, the subgraph of G induced by all of the edges that are not contained in G'. Then

$$bend(D) = bend(D|G') + bend(D|\overline{G'}). \tag{2.2}$$

A *series-parallel graph (with terminals s and t)* is recursively defined as follows:

(a) A graph G of a single edge is a series-parallel graph. The ends s and t of the edge are called the *terminals* of G. (See Figure 2.2(a).)

(b) Let G_1 be a series-parallel graph with terminals s_1 and t_1, and let G_2 be a series-parallel graph with terminals s_2 and t_2.
   - (i) A graph G obtained from G_1 and G_2 by indentifying vertex t_1 with vertex s_2 is a series-parallel graph, whose terminals are $s = s_1$ and $t = t_2$. Such a connection is called a *series connection.* (See Figure 2.2(b).)
   - (ii) A graph G obtained from G_1 and G_2 by identifying s_1 with s_2 and t_1 with t_2 is a series-parallel graph, whose terminals are $s = s_1 = s_2$ and $t = t_1 = t_2$. Such a connection is called a *parallel connection.* (See Figure 2.2(c).)

For example, the graph in Figure 1.1 is series-parallel.

Throughout the paper we assume that the maximum degree of a given series-parallel graph G is at most three, that is, $\Delta \leq 3$. We may assume without loss of generality that G is a simple graph, that is, G has no multiple edges, as follows. If a series-parallel multigraph G consists of exactly three multiple edges, then G has an optimal drawing of four bends; otherwise, insert a dummy vertex of degree two into an edge of each pair of multiple edges in G, and let G' be the resulting series-parallel simple graph, then an optimal drawing of the multigraph G can be immediately obtained from an optimal drawing of the simple graph G' by replacing each dummy vertex with a bend.

A drawing D of a series-parallel graph G is *outer* if the two terminals s and t of G are drawn on the outer face of D. A drawing D is called an *optimal outer drawing* of G if D is outer and $bend(D) = bend(G)$. The graph in Figure 1.1 has an optimal outer drawing, as illustrated in Figure 1.1(a). On the other hand, the graph in Figure 2.3(a) has no optimal outer drawing for the specified terminals s and t; the no-bend drawing

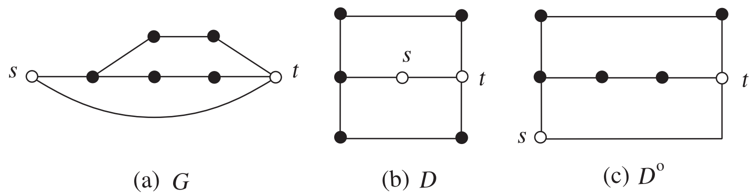


Fig. 2.3. (a) *A biconnected series-parallel graph* G, (b) *an optimal drawing* D, *and* (c) *an outer drawing* D^{o}.

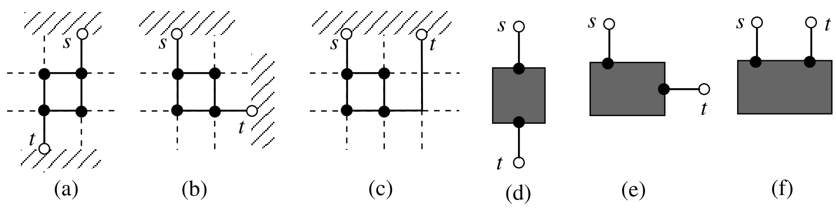


Fig. 2.4. (a)–(c) *I-, L-, and U-shaped drawings, and* (d)–(f) *their schematic representations.*

D in Figure 2.3(b) is optimal but is not outer, because s is not on the outer face; and the drawing D^{o} with one bend in Figure 2.3(c) is outer but is not optimal.

Our main idea is to notice that a series-parallel graph G has an optimal outer drawing if G is "2-legged." We say that G is 2-*legged* if $n(G) \geq 3$ and $d(s) = d(t) = 1$ for the terminals s and t of G. The edge incident to s or t is called a *leg* of G, and the neighbor of s or t is called a *leg-vertex.* For example, the series-parallel graphs in Figures 2.4(a)–(c) are 2-legged.

We will show in section 3 that every 2-legged series-parallel graph G has an optimal outer drawing, and the drawing has one of the three shapes: "I-shape," "L-shape," and "U-shape," defined as follows. An outer drawing D of G is *I-shaped* if D intersects neither the north side of terminal s nor the south side of terminal t after rotating the drawing and renaming the terminals if necessary, as illustrated in Figure 2.4(a). D is *L-shaped* if D intersects neither the north side of s nor the east side of t after rotating the drawing and renaming the terminals if necessary, as illustrated in Figure 2.4(b). D is *U-shaped* if D does not intersect the north sides of s and t after rotating the drawing and renaming the terminals if necessary, as illustrated in Figure 2.4(c). In Figures 2.4(a)–(c) each side is shaded. The north side and the south side of a terminal contain the horizontal line passing through the terminal, while the east side of a terminal contains the vertical line passing through the terminal. The schematic representations of I-, L-, and U-shaped drawings are depicted in Figures 2.4(d), (e), and (f), respectively. D is called an *optimal X-shaped drawing*, X=I, L and U, if D is X-shaped and $bend(D) = bend(G)$.

More precisely, we will show in section 3 that every 2-legged series-parallel graph G, with $n(G) \geq 3$ has both an optimal I-shaped drawing and an optimal L-shaped drawing and that G has an optimal U-shaped drawing, too, unless G is a "diamond graph," defined as follows. A *diamond graph* is either a graph in Figure 2.5(a) or obtained from two diamond graphs G' and G'' by connecting them in parallel and adding two legs, as illustrated in Figures 2.5(b) and (c).

For example, the 2-legged series-parallel graph in Figure 2.6(a) is a diamond graph and has both an optimal (no-bend) I-shaped drawing and an optimal (no-bend) L-

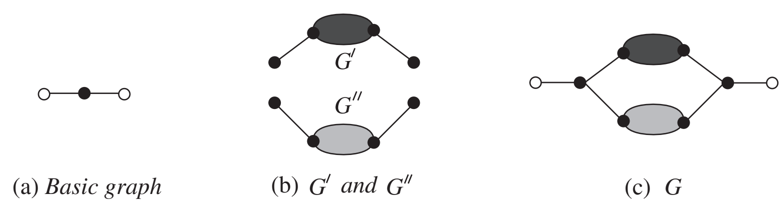


FIG. 2.5. *Recursive definition of a diamond graph.*

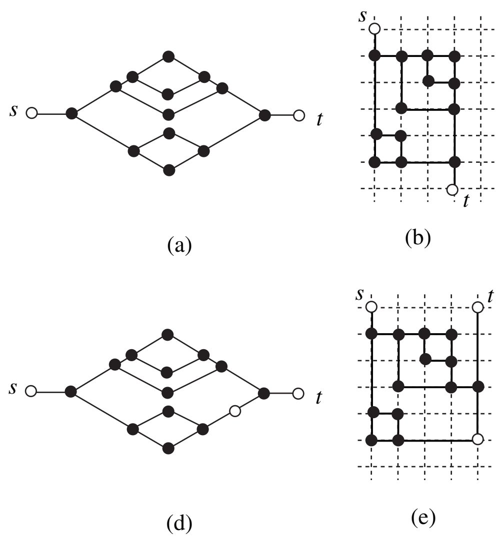


FIG. 2.6. (a) *Diamond graph,* (b) *I-shaped drawing,* (c) *L-shaped drawing,* (d) *nondiamond graph, and* (e) *U-shaped drawing.*

shaped drawing, as illustrated in Figures 2.6(b) and (c), but does not have an optimal (no-bend) U-shaped drawing. On the other hand, the 2-legged series-parallel graph in Figure 2.6(d) is obtained from the diamond graph in Figure 2.6(a) simply by inserting a new vertex of degree two in an edge and is not a diamond graph anymore. It has an optimal (no-bend) U-shaped drawing, too, as illustrated in Figure 2.6(e). Thus the diamond graph in Figure 2.6(a) has a U-shaped drawing with one bend.

3. Optimal drawing of biconnected series-parallel graph.

In this section we give a linear algorithm to find an optimal drawing of a biconnected series-parallel graph G of $\Delta \leq 3$. We first give an algorithm for 2-legged series-parallel graphs in subsection 3.1. Using the algorithm, we then give an algorithm for biconnected series-parallel graphs in subsection 3.2.

3.1. 2-legged series-parallel graph.

We first have the following lemma on a diamond graph.

LEMMA 3.1. *If G is a diamond graph, then*

(a) *G has both a no-bend* I*-shaped drawing D_{I} and a no-bend* L*-shaped drawing D_{L};*

(b) *D_{I} and D_{L} can be found in linear time; and*

(c) *every no-bend drawing of G is either* I*-shaped or* L*-shaped, and hence G does not have a no-bend* U*-shaped drawing.*

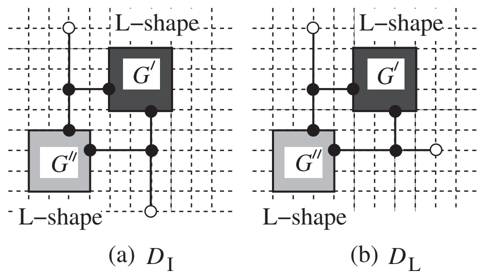


FIG. 3.1. *Illustration of the proof of Lemma* 3.1.

Proof. We prove the lemma by induction on the number $n(G)$ of vertices of G. Since G is a diamond graph, G has at least three vertices. If $n(G) = 3$, as illustrated in Figure 2.5(a), then (a), (b), and (c) trivially hold true. One may thus assume that $n(G) \geq 4$, and inductively assume that (a), (b), and (c) hold for every diamond graph of at most $n(G) - 1$ vertices.

We first prove that (a) holds for G. The definition of a diamond graph implies that one can obtain two diamond subgraphs G' and G'' from G by deleting the two terminals of G and splitting each of the two leg-vertices into two vertices, as illustrated in Figures 2.5(b) and (c). By the inductive hypothesis, (a) holds for G', and hence G' has a no-bend L-shaped drawing $D'_{\rm L}$. Similarly, G'' has a no-bend L-shaped drawing $D''_{\rm L}$. Combining a flipped drawing of $D'_{\rm L}$ with the drawing $D''_{\rm L}$ and drawing the two legs appropriately, one can construct a no-bend I-shaped drawing $D_{\rm I}$ and a no-bend L-shaped drawing $D_{\rm L}$ of G, as illustrated in Figures 3.1(a) and (b), respectively. Thus (a) holds true.

One can obtain (regular labelings of) $D_{\rm I}$ and $D_{\rm L}$ from (those of) $D'_{\rm L}$ and $D''_{\rm L}$ in constant time by deciding the labels of the angles of the terminals and leg vertices of G. Thus (b) holds true for G.

We now prove that (c) holds for G. Let D^* be an arbitrary no-bend drawing of G. Then both the drawing $D^*|G'$ of G' in D^* and the drawing $D^*|G''$ of G'' in D^* are no-bend drawings. By the inductive hypothesis, each of $D^*|G'$ and $D^*|G''$ is either I-shaped or L-shaped. Since D^* is a no-bend drawing, both $D^*|G'$ and $D^*|G''$ must be L-shaped and D^* must be either I-shaped or L-shaped. Thus (c) holds true. ◻

The proof of Lemma 3.1 immediately yields a linear algorithm **Diamond**(G**,** $D_{\rm I}$**,** $D_{\rm L}$), which recursively finds both a no-bend I-shaped drawing $D_{\rm I}$ and a no-bend L-shaped drawing $D_{\rm L}$ of a given diamond graph G.

The following lemma holds for a 2-legged series-parallel graph G which is not a diamond graph.

LEMMA 3.2. *The following* (a), (b), *and* (c) *hold for a* 2*-legged series-parallel graph G, with $n(G) \geq 3$ unless G is a diamond graph:*

(a) *G has three optimal* I-, L-, *and* U-*shaped drawings $D_{\rm I}$, $D_{\rm L}$, and $D_{\rm U}$;*

(b) *$D_{\rm I}$, $D_{\rm L}$, and $D_{\rm U}$ can be found in linear time; and*

(c) *$bend(G) \leq (n(G) - 2)/3$.*

Proof. We prove the lemma by induction on the number $n(G)$ of vertices of G. Assume that G is a 2-legged series-parallel graph, with $n(G) \geq 3$, and G is not a diamond graph. Then $n(G) \geq 4$. If $n(G) = 4$, that is, G is a path of four vertices, then the lemma trivially holds true, as illustrated in Figure 3.2. One may thus assume $n(G) \geq 5$, and inductively assume that the lemma holds for every 2-legged

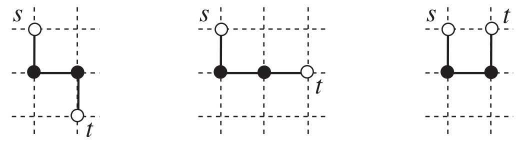


FIG. 3.2. *Optimal I-, L-, and U-shaped drawings of a path of four vertices.*

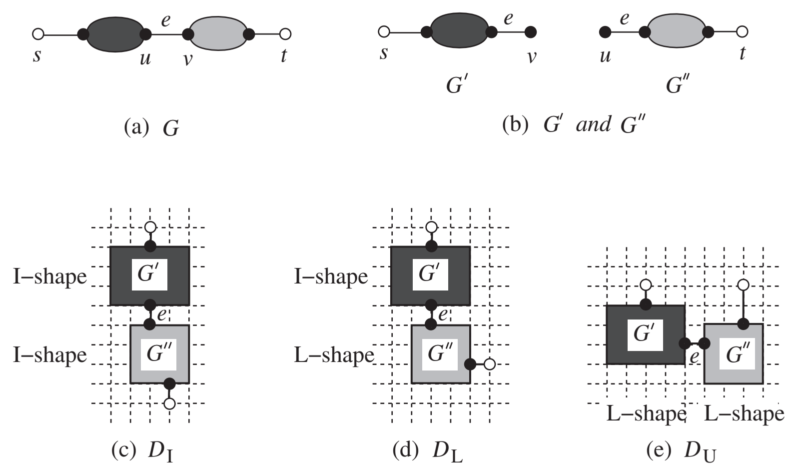


FIG. 3.3. *Illustration of the proof of Lemma* 3.2.

series-parallel graph of at most $n(G)-1$ vertices. We now prove that the lemma holds for G.

(a) We first prove that (a) holds for G. Let s and t be the terminals of G, then $d(s) = d(t) = 1$. Since G is a 2-legged series-parallel graph, the graph $G - s - t$ obtained from G by deleting vertices s and t is a series-parallel graph having the leg-vertices as the terminals, and hence $G - s - t$ is either a series connection or a parallel connection of two subgraphs.

Consider first the case where $G-s-t$ is a series connection of two subgraphs. In this case, since $\Delta(G) \leq 3$, G has a bridge $e = uv$ other than the two legs, that is, $G-e$ is disconnected, as illustrated in Figure 3.3(a). Then G contains two subgraphs G' and G''; G' is a 2-legged series-parallel graph with terminals s and v, and G'' is a 2-legged series-parallel graph with terminals u and t, as illustrated in Figure 3.3(b). If G' is not a diamond graph, then $4 \leq n(G') < n(G)$, and hence by the inductive hypothesis, G' has three optimal drawings of I-, L-, and U-shapes. If G' is a diamond graph, then by Lemma 3.1(a), G' has two optimal (no-bend) drawings of I- and L-shapes. Hence, in either case, G' has two optimal drawings of I- and L-shapes. Similarly, G'' has two optimal drawings of I- and L-shapes. Combining these drawings of G' and G'', one can easily construct three optimal drawings D_{I}, D_{L}, and D_{U} of G, as illustrated in Figures 3.3(c), (d), and (e). Thus (a) holds true.

Consider next the case where $G - s - t$ is a parallel connection of two subgraphs. Then G has exactly one biconnected component, and the degrees of the leg-vertices are three, as illustrated in Figure 3.4(a). Deleting the terminals of G and splitting the two leg-vertices of G, one can obtain two edge-disjoint 2-legged series-parallel

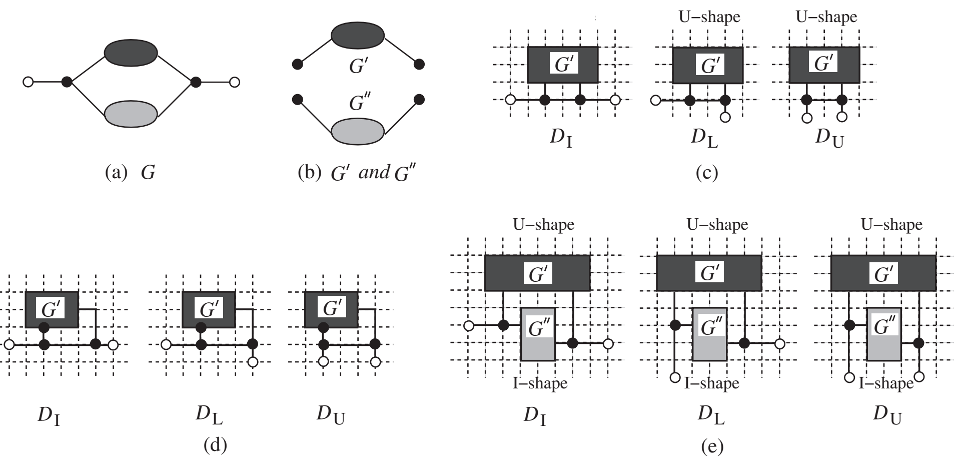


FIG. 3.4. *Illustration of the proof of Lemma* 3.2.

subgraphs G' and G'', as illustrated in Figure 3.4(b). Since $n(G'), n(G'') \geq 2$, there are the following two cases to consider.

Case 1: Either $n(G') = 2$ or $n(G'') = 2$.

One may assume that $n(G'') = 2$, and hence G'' consists of a single edge. Since G is a simple graph, we have $n(G') \geq 3$.

Consider first the case where G' is not a diamond graph. Then $4 \leq n(G') < n(G)$, and hence by the inductive hypothesis, G' has an optimal U-shaped drawing D'_{U}. From a flipped drawing of D'_{U}, one can easily construct three optimal drawings D_{I}, D_{L}, and D_{U} of G, as illustrated in Figure 3.4(c).

Consider next the case where G' is a diamond graph. Then by Lemma 3.1(a), G' has a no-bend L-shaped drawing D'_{L}. From a flipped drawing of D'_{L}, one can easily construct three drawings D_{I}, D_{L}, and D_{U} of G, with one bend, as illustrated in Figure 3.4(d). Since G' is a diamond graph, by Lemma 3.1(c), every no-bend drawing of G' is either I-shaped or L-shaped. Therefore, we have $bend(G) \geq 1$; if G had a no-bend drawing D, then $D|G'$ would be I- or L-shaped, and hence D would have one or more bends on the edge in G'', a contradiction. Thus the constructed drawings D_{I}, D_{L}, and D_{U} having exactly one bend are optimal.

Case 2: $n(G'), n(G'') \geq 3$.

Since G is not a diamond graph, either G' or G'' is not a diamond graph. One may assume without loss of generality that G' is not a diamond graph. Then $4 \leq n(G') < n(G)$, and hence by the inductive hypothesis, G' has an optimal U-shaped drawing D'_{U}. On the other hand, G'' has an optimal I-shaped drawing D''_{I}; if G'' is a diamond graph, then by Lemma 3.1(a), G'' has an optimal (no-bend) I-shaped drawing D''_{I}; otherwise, by the inductive hypothesis, G'' has an optimal I-shaped drawing D''_{I}. From the U-shaped drawing D'_{U} and the I-shaped drawing D''_{I}, one can easily construct I-, L-, and U-shaped drawings $D_{\mathrm{I}}, D_{\mathrm{L}}$, and D_{U} of G without introducing any new bends, as illustrated in Figure 3.4(e). Since D_{I} has no bend on the legs, by (2.1) and (2.2) we have

$$bend(D_{\mathrm{I}}) = bend(D'_{\mathrm{U}}) + bend(D''_{\mathrm{I}}).$$

Since D'_{U} and D''_{I} are optimal drawings of G' and G'', respectively,

$$bend(D'_{\mathrm{U}}) = bend(G')$$

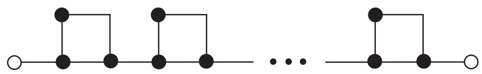


FIG. 3.5. *A graph attaining the bound in Lemma* 3.2(c).

and

$$bend(D''_{\mathrm{I}}) = bend(G'').$$

Let D^* be an arbitrary optimal drawing of G, then clearly

$$bend(G') \leq bend(D^*|G'),$$
$$bend(G'') \leq bend(D^*|G''),$$

and

$$bend(D^*|G') + bend(D^*|G'') \leq bend(D^*) = bend(G).$$

From these six equations we have $bend(D_{\mathrm{I}}) \leq bend(G)$, and hence D_{I} is an optimal drawing of G. Similarly, D_{L} and D_{U} are optimal drawings of G. Thus (a) holds true.

(b) We now prove (b). One can construct the (regular labeling of) drawings D_{I}, D_{L}, and D_{U} above from the (regular labeling of) drawings of G' and G'' simply by deciding the labels of the terminals and leg-vertices in G; this can be done in constant time. Thus (b) holds true for G.

(c) We finally prove (c). If a new bend is produced in a construction of G, as illustrated in Figures 3.3 and 3.4, then the construction is one in Figure 3.4(d), $n(G) \geq 5$, and $bend(G) = bend(G')+1 = 1 \leq (n(G)-2)/3$. In any other construction, no new bend is produced. Noting this fact, one can inductively prove that $n(G) \leq (n(G)-2)/3$. $\square$

We denote by K_n a complete graph of $n(\geq 1)$ vertices. Let G be a 2-legged series-parallel graph obtained from copies of K_2 and K_3 by connecting them alternately in series, as illustrated in Figure 3.5. Then $bend(G) = (n(G)-2)/3$. Thus the bound in Lemma 3.2(c) is best possible.

The proof of Lemma 3.2 immediately yields a linear algorithm **NonDiamond**(G, D_{I}, D_{L}, D_{U}), which recursively finds three optimal I-, L-, and U-shaped drawings D_{I}, D_{L}, and D_{U} of a given 2-legged series-parallel graph G unless G is a diamond graph. By algorithms **Diamond** and **NonDiamond**, one can find an optimal drawing of a 2-legged series-parallel graph G. Note that **NonDiamond** may call **Diamond**.

For a series-parallel graph G with $d(s)$, $d(t) \leq 2$, one can easily find an optimal outer drawing D of G, as follows. Add to G a dummy edge $s's$ for a new vertex s' if $d(s) = 2$, and add to G a dummy edge tt' for a new vertex t' if $d(t) = 2$. Then the resulting graph G' is a 2-legged series-parallel graph, and $\Delta(G') \leq 3$. Find an optimal drawing D' of G' by **Diamond** or **NonDiamond** and delete the dummy edges from D'. Then the resulting drawing D of G is clearly optimal and outer.

3.2. Biconnected series-parallel graphs. A biconnected series-parallel graph G can be defined (without specifying terminals) as a biconnected graph which has no K_4 as a minor. For every edge uv in G, G is a series-parallel graph with terminals u and v.

A cycle C of four vertices in a graph G is called a *diamond* if two nonconsecutive vertices of C have degree two in G and the other two vertices of C have degree three

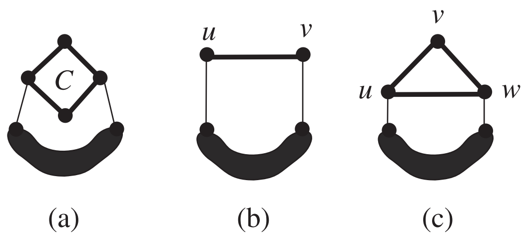


FIG. 3.6. *Substructures contained in a biconnected series-parallel graph of* $\Delta \leq 3$.

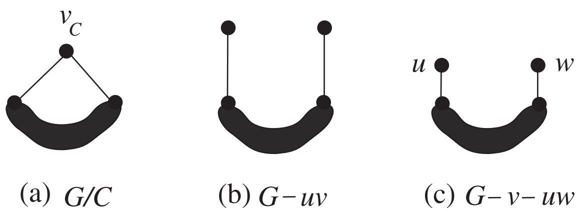


FIG. 3.7. *Smaller graphs* G'.

and are not adjacent in G, as illustrated in Figure 3.6(a). We denote by G/C the graph obtained from G by contracting C to a new single vertex v_C, as illustrated in Figure 3.7(a). (Note that $G_C = G/C$ is series-parallel if G is series-parallel. One can observe that, from every diamond graph, one can obtain a graph in Figure 2.5(a) by repeatedly contracting a diamond.)

Noting that every biconnected series-parallel graph G has a vertex of degree two, one can easily observe that the following Lemma 3.3 holds. (Lemma 3.3 is also an immediate conquence of Lemma 2.1 in [10] on a general series-parallel graph.)

LEMMA 3.3. *Every biconnected series-parallel graph* G *of* $\Delta \leq 3$ *has, as a subgraph, one of the following three substructures* (a)–(c) *illustrated in Figure* 3.6*:*

(a) *a diamond* C*;*
(b) *two adjacent vertices* u *and* v *such that* $d(u) = d(v) = 2$*; and*
(c) *a complete graph* K_3 *of three vertices* u, v, *and* w *such that* $d(v) = 2$.

Our idea is to reduce the optimal drawing problem for a biconnected series-parallel graph G to that for a smaller graph G' illustrated in Figure 3.7, as in the following Lemmas 3.4 and 3.5.

LEMMA 3.4. *If a biconnected series-parallel graph* G, *with* $n(G) \geq 6$ *has a diamond* C, *then* $bend(G) = bend(G')$ *for* $G' = G/C$.

Proof. Assume that a biconnected series-parallel graph G has a diamond C and that s' and t' are the two vertices of C having degree three in G, as illustrated in Figure 3.8(a). Let $G' = G/C$ be the graph obtained from G by contracting C to a new single vertex v_C, as illustrated in Figure 3.8(b). Since $n(G) \geq 6$ and G is biconnected, the neighbor of s' outside C is different from that of t'. Therefore, G' is a biconnected series-parallel (simple) graph. Let D'^* be an optimal drawing of G', then $bend(D'^*) = bend(G')$. (See Figure 3.8(e).) Replace the vertex v_C in D'^* with a rectangular drawing of C, and let D be the resulting drawing of G, as illustrated in Figure 3.8(d). We claim that D is an optimal drawing of G, and hence $bend(G) = bend(G')$.

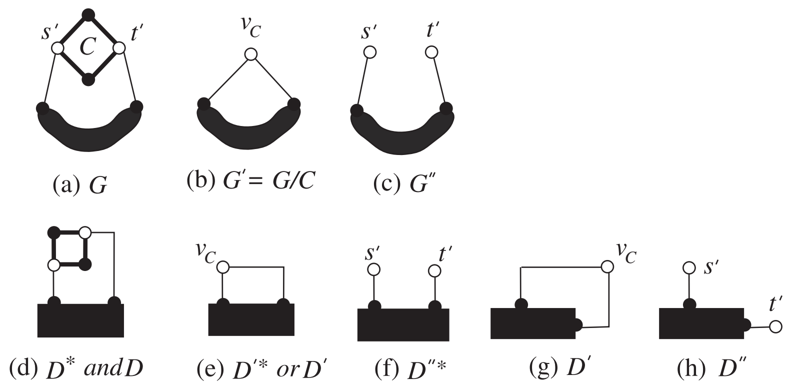


FIG. 3.8. *Illustration for the proof of Lemma* 3.4.

The (regular labeling of) drawing D is constructed from the (regular labeling of) optimal drawing $D^{'*}$ of G' without introducing any new bend, and hence we have $bend(D) = bend(D^{'*}) = bend(G')$. Let D^* be an arbitrary optimal drawing of G, then $bend(D^*) = bend(G)$. If

$$bend(G') \leq bend(D^*), \tag{3.1}$$

then $bend(D) = bend(G') \leq bend(D^*) = bend(G)$, and hence D is an optimal drawing of G. It thus suffices to verify (3.1). There are the following two cases to consider.

Case 1: $bend(D^*|C) = 0$.

In this case, $D^*|C$ is a rectangle, as illustrated in Figure 3.8(d). Contracting C in D^* to a single vertex v_C and appropriately deciding the two labels of v_C, one can obtain (a regular labeling of) a drawing D' of $G' = G/C$ from D^* without introducing any new bend, as illustrated in Figure 3.8(e). We thus have $bend(G') \leq bend(D') = bend(D^*)$.

Case 2: $bend(D^*|C) \geq 1$.

In this case, $D^*|C$ is a rectilinear polygon having five or more (geometric) vertices including at least one bend. Delete from G the two vertices of C having degree two, and let G'' be the resulting graph, as illustrated in Figure 3.8(c). Since G is biconnected, G'' is a 2-legged series-parallel graph with terminals s' and t'. (Note that every biconnected series-parallel graph is a series-parallel graph having the ends of an arbitrary edge as terminals, and that a graph obtained from a series-parallel graph by contracting an edge is series-parallel.) Since G'' is the complement of C in G, by (2.2) we have

$$bend(D^*|G'') + bend(D^*|C) = bend(D^*). \tag{3.2}$$

There are the following two subcases to consider.

Case 2.1: G'' is not a diamond graph.

Since $n(G) \geq 6$, $n(G'') = n(G) - 2 \geq 4$. Since G'' is not a diamond graph, by Lemma 3.2(a), G'' has an optimal U-shaped drawing $D^{''*}$, as illustrated in Figure 3.8(f). $D^{''*}$ can be modified to a drawing D' of G' by identifying s' with t' as a single vertex v_C and introducing a new bend, as illustrated in Figure 3.8(e), and

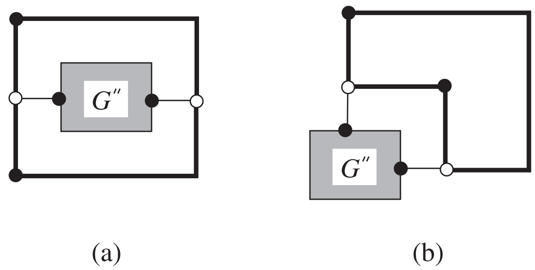


FIG. 3.9. *An optimal drawing D^* of G such that $D^*|G''$ is I-shaped in* (a) *or L-shaped in* (b).

hence by (3.2),

$$\begin{aligned} bend(G') &\leq bend(D') \\ &= bend(D^{''*}) + 1 \\ &= bend(G'') + 1 \\ &\leq bend(D^*|G'') + bend(D^*|C) \\ &= bend(D^*). \end{aligned}$$

Case 2.2: G'' is a diamond graph.

Since $bend(D^*|C) \geq 1$, by (2.1) we have $bend(D^*) \geq bend(D^*|C) \geq 1$. We now claim that

$$bend(D^*) \geq 2. \tag{3.3}$$

Suppose for a contradiction that $bend(D^*) = 1$. Since $bend(D^*|C) \geq 1$, by (3.2) we have $bend(D^*|C) = 1$ and $bend(D^*|G'') = 0$, and hence $D^*|G''$ is a no-bend drawing of G''. Then by Lemma 3.1(c), $D^*|G''$ is either I-shaped or L-shaped, as illustrated in Figures 3.9(a) and (b), respectively. In either case, the drawing of C needs two or more bends in D^*, that is, $bend(D^*|C) \geq 2$, as illustrated in Figure 3.9, where C is drawn by thick lines. This is contrary to $bend(D^*|C) = 1$.

Since G'' is a diamond graph, by Lemma 3.1(a) G'' has a no-bend L-shaped drawing D'', as illustrated in Figure 3.8(h). The drawing D'' can be modified to a drawing D' of $G' = G/C$ by identifying the two vertices s' and t' as a single vertex v_C and introducing two new bends, as illustrated in Figure 3.8(g), and hence we have

$$bend(G') \leq bend(D') = bend(D'') + 2 = 2. \tag{3.4}$$

By (3.3) and (3.4) we have $bend(G') \leq 2 \leq bend(D^*)$. ∎

LEMMA 3.5. *Assume that a biconnected series-parallel graph G, with $n(G) \geq 6$ has no diamond. If G has a substructure of type* (b) *in Figure* 3.6(b)*, then $bend(G) = bend(G')$ for $G' = G - uv$. If G has a substructure of type* (c) *in Figure* 3.6(c)*, then $bend(G) = bend(G') + 1$ for $G' = G - v - uw$. (See Figures* 3.7(b) *and* (c).*)*

Proof. Since G has no diamond, by Lemma 3.3, G has either a substructure (b) or (c).

Suppose first that G has a substructure (b). Let $G' = G - uv$, as illustrated in Figure 3.7(b). Then G' is a 2-legged series-parallel graph with terminals u and v. Clearly $n(G') = n(G) \geq 6$. Since G has no diamond, the subgraph G' of G has no diamond. Therefore G' is not a diamond graph, and hence by Lemma 3.2(a), G' has

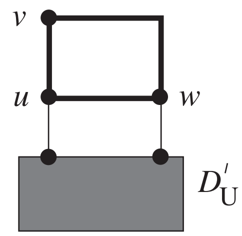


FIG. 3.10. *Construction of D from D'_U.*

an optimal U-shaped drawing D'_{U}. D'_{U} can be extended to a drawing D of G simply by drawing uv as a straight line segment. Since $bend(D) = bend(D'_{\mathrm{U}}) = bend(G') \leq bend(G)$, D is an optimal drawing of G. Thus $bend(G) = bend(D) = bend(G')$.

Suppose next that G has a substructure (c). Let $G' = G - v - uw$, as illustrated in Figure 3.7(c). Then G' is a 2-legged series-parallel graph with terminals u and w and is not a diamond graph. Therefore, by Lemma 3.2(a), G' has an optimal U-shaped drawing D'_{U}. D'_{U} can be extended to a drawing D of G by drawing the complete graph K_3 of three vertices u, v, w as a rectangle with one bend, as illustrated in Figure 3.10. Any orthogonal drawing of K_3 needs a bend, and hence $bend(D) = bend(D'_{\mathrm{U}}) + 1 = bend(G') + 1 \leq bend(G)$. Thus D is an optimal drawing of G, and hence $bend(G) = bend(G') + 1$. □

From the proofs of Lemmas 3.4 and 3.5 we have the following algorithm **Biconnected**(G, D) to find an optimal drawing D of a biconnected series-parallel graph G.

Biconnected(G, D);
begin
One may assume that $n(G) \geq 6$ (otherwise, one can easily find an optimal drawing D of G in linear time);
{By Lemma 3.3, G has one of the three substructures (a)–(c) in Figure 3.6.}
Case 1: G has a diamond C;
Let $G' = G/C$; {G' is a biconnected series-parallel graph.}
Biconnected(G', D');
Extend an optimal drawing D' of G' to an optimal drawing D of G simply by replacing v_C by a rectanglar drawing of C, as illustrated in Figures 3.8(d) and (e); {cf. Lemma 3.4}
Case 2: G has no diamond but has a substructure (b);
Let $G' = G - uv$;
{G' is a 2-legged series-parallel graph with terminals u and v and is not a diamond graph.}
Find an optimal U-shaped drawing D'_{U} of G' by **NonDiamond**; {cf. Lemma 3.2}
Extend D'_{U} to an optimal drawing D of G by drawing uv as a straight line segment; {cf. Lemma 3.5}
Case 3: G has neither a diamond nor a substructure (b) but has a substructure (c);
Let $G' = G - v - uw$;
{G' is a 2-legged series-parallel graph with terminals u and w and is not a diamond graph.}
Find an optimal U-shaped drawing D'_{U} of G' by **NonDiamond**;

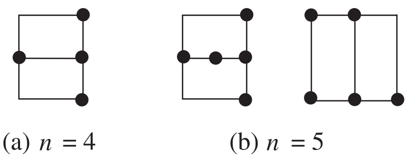


FIG. 3.11. *Biconnected series-parallel graphs with $n = 4$ or $n = 5$.*

Extend D'_{U} to an optimal drawing D of G by drawing $K_3 = uvw$ as a rectangle with one bend, as illustrated in Figure 3.10; {cf. Lemma 3.5}

end.

All substructures (a)–(c) can be found total in time $O(n)$ by a standard bookkeeping method to maintain all degrees of vertices together with all paths of length two with an intermediate vertex of degree two. One can thus observe that **Biconnected** can be executed in linear time.

We thus have the following theorem.

THEOREM 3.6. *An optimal orthogonal drawing of a series-parallel biconnected graph G of $\Delta \leq 3$ can be found in linear time.*

It should be noted that the (orginal) terminals s and t of G are not always on the outer face of a drawing D obtained by **Biconnected**, and hence D is not always an outer drawing, as known from the example in Figure 2.3.

Di Battista, Liotta, and Vargiu gave an $O(n^3)$ algorithm for biconnected series-parallel simple graphs using a min-cost flow technique and a data structure called an SPQ*R tree [4, 5]. Our linear algorithm is much simpler and faster than their algorithm.

We have the following lemma on the minimum number of bends.

LEMMA 3.7. *If a series-parallel graph G is biconnected and $\Delta \leq 3$, then*

$$bend(G) \leq \lceil n(G)/3 \rceil. \tag{3.5}$$

Proof. We prove (3.5) by induction on $n(G)$. Since G is biconnected, $n(G) \geq 3$.

1°. We first show that (3.5) holds if $3 \leq n(G) \leq 5$. If $n(G) = 3$, then $G = K_3$ and $bend(G) = 1 = \lceil n(G)/3 \rceil$. If $n(G) = 4$ and $bend(G) \geq 1$, then $G = K_4 - e$ in Figure 3.11(a), and hence $bend(G) = 2 \leq \lceil n(G)/3 \rceil$, where $K_4 - e$ is a graph obtained from K_4 by deleting an edge. If $n(G) = 5$ and $bend(G) \geq 1$, then G is one of the two graphs in Figure 3.11(b), and hence $bend(G) \leq 2 \leq \lceil n(G)/3 \rceil$.

2°. Assume that $n(G) \geq 6$, and assume inductively that (3.5) holds for every series-parallel biconnected graph of at most $n(G) - 1$ vertices.

3°. We then show that (3.5) holds for G. There are the following three cases to consider.

Case 1: G has a diamond C.

By Lemma 3.4 we have $bend(G) = bend(G')$ for $G' = G/C$. Since $n(G') = n(G) - 3$ and G' is a series-parallel biconnected graph, by the inductive hypothesis we have $bend(G') \leq \lceil n(G')/3 \rceil$. We thus have $bend(G) = bend(G') < \lceil n(G)/3 \rceil$.

Case 2: G has no diamond but has a substructure (b) in Figure 3.6(b).

By Lemma 3.5 we have $bend(G) = bend(G')$ and $n(G) = n(G')$ for $G' = G - uv$. Since G' is a 2-legged series-parallel graph and is not a diamond graph, by Lemma 3.2(c) we have $bend(G') \leq (n(G')-2)/3$. We thus have $bend(G) = bend(G') \leq \lceil n(G)/3 \rceil$.

Case 3: G has neither a diamond nor a substructure (b) but has a substructure (c) in Figure 3.6(c).

By Lemma 3.5 we have $bend(G) = bend(G') + 1$ and $n(G) = n(G') + 1$ for $G' = G - v - uw$. Since G' is a 2-legged series-parallel and is not a diamond graph, by Lemma 3.2(c) we have $bend(G') \leq (n(G') - 2)/3$. We thus have

$$\begin{aligned} bend(G) &\leq (n(G') - 2)/3 + 1 \\ &= n(G)/3 \\ &\leq \lceil n(G)/3 \rceil. \qquad \square \end{aligned}$$

4. Optimal drawing of nonbiconnected series-parallel graph. It has been left as open by Di Battista, Liotta, and Vargiu [4, 5] to obtain an algorithm for nonbiconnected series-parallel graphs G. In section 4.1 we give a linear algorithm for G, and in section 4.2 we give a best possible upper bound on $bend(G)$.

4.1. Algorithm. By Lemma 3.1, Lemma 3.2, and Theorem 3.6, one may assume that G is neither 2-legged nor biconnected. Therefore either $d(s) \geq 2$ or $d(t) \geq 2$, and hence one may assume without loss of generality that $d(s) \geq 2$.

We now claim that one may further assume that $d(t) \geq 2$, and hence G consists of biconnected components $B_1, B_2, \ldots, B_p$, $p \geq 2$, and $p-1$ copies of K_2, alternately connected in series, as illustrated in Figure 4.1(a), after replacing each induced path contained in none of the biconnected components by a single edge. Suppose that $d(t) = 1$, as illustrated in Figure 4.1(b). Then $G' = G - t$ is a series-parallel graph with terminals s and t_p, and takes the form in Figure 4.1(a). Since $d(t_p, G') = d(t_p, G) - 1 = 2$, there is an angle of π or $3\pi/2$ around vertex t_p in any optimal drawing D' of G'. An optimal drawing D of G can be obtained from D' simply by inserting the edge t_pt in the angle. One may thus assume that $d(t) \geq 2$.

For a series-parallel graph G in Figure 4.1(a), each biconnected component B_i, $1 \leq i \leq p$, is a series-parallel graph with terminals s_i and t_i, where $s_1 = s$ and $t_p = t$. Clearly

$$\sum_{i=1}^{p} bend(B_i) \leq bend(G). \tag{4.1}$$

However, (4.1) does not always hold with equality, as follows. Consider the series-parallel graph G in Figure 4.2(a); G has two biconnected components B_1 and B_2, and hence $p = 2$. B_1 and B_2 have no-bend (optimal) drawings D_1 and D_2, as illustrated in Figures 4.2(b) and (c), respectively. Thus $bend(B_1) = bend(B_2) = 0$, and hence $bend(B_1) + bend(B_2) = 0$. However, terminal t_1 of B_1 is not on the outer face of any no-bend drawing of B_1. Similarly, s_2 is not on the outer face of any no-bend drawing of B_2. Therefore, one cannot connect t_1 in D_1 and s_2 in D_2 by an edge without edge-crossing. Thus G does not have a no-bend drawing. On the other hand, B_2 has a drawing D_2^o with one bend in which s_2 is on the outer face, as illustrated in Figure 4.2(d). Combining D_1 and D_2^o, one can construct a drawing D of G with one bend, as illustrated in Figure 4.2(e). Thus $bend(G) = 1$, and hence

$$bend(B_1) + bend(B_2) < bend(G).$$

In section 4.3, we will prove

$$bend(G) = \sum_{i=1}^{p} bend(B_i) \quad \text{or} \quad \sum_{i=1}^{p} bend(B_i) + 1. \tag{4.2}$$

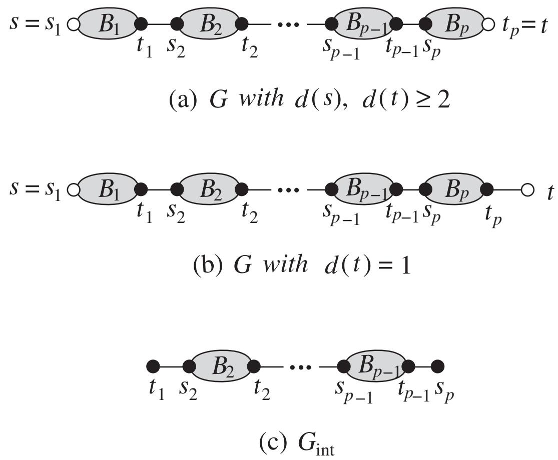


FIG. 4.1. *Nonbiconnected series-parallel graphs.*

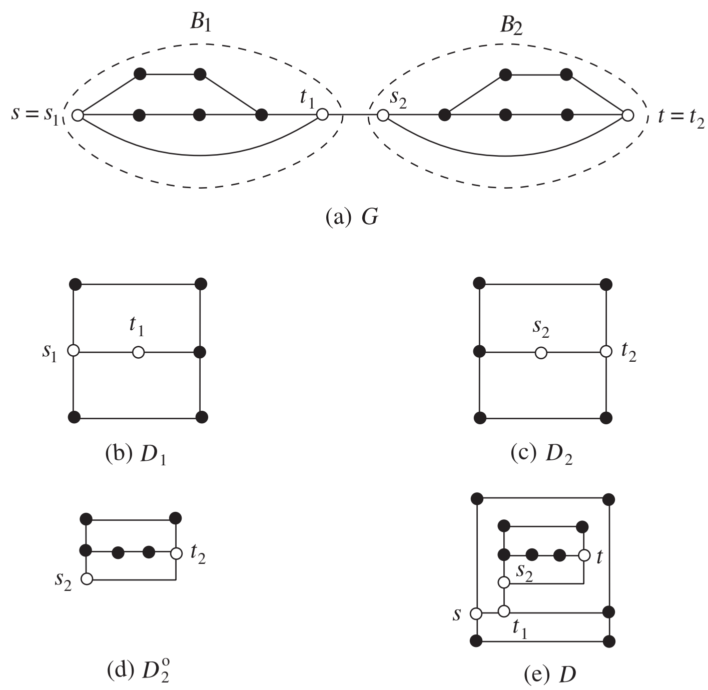


FIG. 4.2. *Construction of an optimal drwing D of G for a case $p = 2$.*

Delete from a series-parallel graph G in Figure 4.1(a) all vertices in B_1 and B_p except t_1 and s_p, and let $G_{\rm int}$ be the resulting series-parallel graph with terminals t_1 and s_p, as illustrated in Figure 4.1(c). Then we claim that $G_{\rm int}$ has an optimal I-shaped drawing. If $p = 2$, then $G_{\rm int} = K_2$, and hence $G_{\rm int}$ has an optimal (no-bend) I-shaped drawing $D_{\rm intI}$. If $p \geq 3$, then G is 2-legged, and hence by Lemmas 3.1 and 3.2, $G_{\rm int}$ has an optimal I-shaped drawing $D_{\rm intI}$, in which both terminals t_1 and s_p are on the outer face.

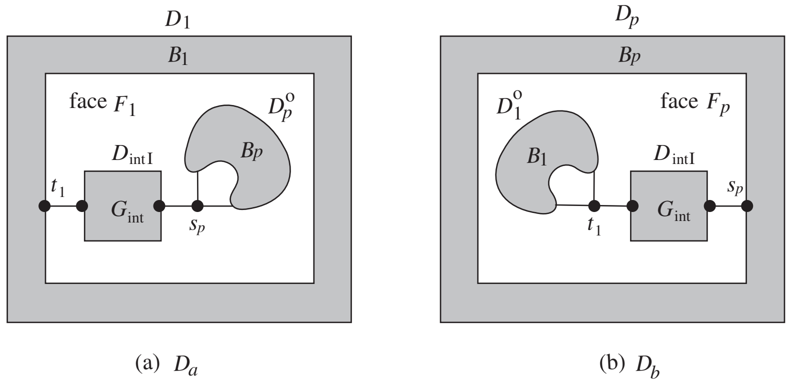


FIG. 4.3. *Two alternative drawings D_a and D_b of G.*

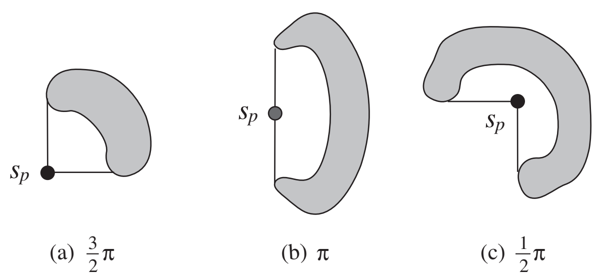


FIG. 4.4. *Three kinds of the outer angle of s_p in D_p^o.*

As known from the example in Figure 4.2, we have the following lemma.

LEMMA 4.1. *Let G be a series-parallel graph taking the form in Figure* 4.1(a). *Then an arbitrary drawing D of G has one of the following two alternatives forms:*

(a) *the terminal s_p of B_p is on the outer face of the drawing $D_p^o = D|B_p$ of B_p (see Figure* 4.3(a)*, for example); and*

(b) *the terminal t_1 of B_1 is on the outer face of the drawing $D_1^o = D|B_1$ of B_1 (see Figure* 4.3(b)*, for example).*

In (a) above, the *outer angle* of s_p, that is, the angle around s_p in the outer face of D_p^o, must be either $3\pi/2$ or π, as illustrated in Figures 4.4(a) and (b); if it were $\pi/2$, as illustrated in Figure 4.4(c), then the edge $t_{p-1}s_p$ could not be drawn in the outer angle. Since $d(t_1, B_1) = 2$, one of the two angles of t_1 in any drawing of B_1 has an angle of $3\pi/2$ or π. Of course, in an arbitrary drawing D of G, both G_{int} and B_p are drawn in the same face F_1 of $D_1 = D|B_1$ having such an angle of t_1, as illustrated in Figure 4.3(a). Note that face F_1 may be outer although it is drawn as an inner face in Figure 4.3(a). Similarly, in (b) above, both G_{int} and B_1 are drawn in the same face F_p of $D_p = D|B_p$, as illustrated in Figure 4.3(b).

Let B be a biconnected series-parallel graph, and let v be a vertex of degree two in B. A drawing D of B is (α, v)*-outer* if v is on the outer face of D and the outer angle of v is α, where $\alpha = \pi/2, \pi$, or $3\pi/2$. An (α, v)-outer drawing D is *optimal* if D has the minimum number of bends among all possible (α, v)-outer drawings of B. The minimum number of bends is denoted by $bend(B, \alpha, v)$. Clearly $bend(B) \le bend(B, \alpha, v)$. However, the equation does not always hold with equality. For example, $bend(B_2) = 0 < 1 = bend(B_2, 3\pi/2, s_2)$ for B_2 in Figure 4.2(a).

Lemma 4.1 and the arguments above imply that we shall find optimal (α, t_1)-outer drawings of B_1 and optimal (α, s_p)-outer drawings of B_p for both $\alpha = 3\pi/2$ or $\alpha = \pi$. However, it suffices to find them only for $\alpha = 3\pi/2$, because $bend(B_1, 3\pi/2, t_1) \leq bend(B_1, \pi, t_1)$ and $bend(B_p, 3\pi/2, s_p) \leq bend(B_p, \pi, s_p)$ as known from the following, Lemma 4.2(a), whose proof will be given in section 4.3.

LEMMA 4.2. *If a biconnected series-parallel graph B has a vertex v of degree two, then*

(a)

$$bend(B, 3\pi/2, v) \leq \min\{bend(B, \pi/2, v), bend(B, \pi, v)\}; \tag{4.3}$$

(b)

$$bend(B, 3\pi/2, v) = bend(B) \quad or \quad bend(B) + 1;$$

and

(c) *an optimal $(3\pi/2, v)$-outer drawing of B can be found in linear time.*

The proof of Lemma 4.2 yields a linear algorithm **OuterDrawing**(B, v, D) to find an optimal $(3\pi/2, v)$-outer drawing of B. Thus one can easily observe that the following algorithm **NonBiconnected**(G, D) finds an optimal drawing of G.

NonBiconnected(G, D**)**;
begin
  Find an optimal drawing D_1 of B_1 and an optimal drawing D_p of B_p by **Biconnected**;
  Find an optimal $(3\pi/2, t_1)$-outer drawing D_1^{o} of B_1 and an optimal $(3\pi/2, s_p)$-outer drawing D_p^{o} of B_p by **OuterDrawing**;
  Find an optimal I-shaped drawing D_{intI} of G_{int}; {cf. Lemmas 3.1 and 3.2}
  Let F_1 be a face of D_1 such that the angle of t_1 in F_1 is π or $3\pi/2$, insert D_{intI} and D_p^{o} in F_1, and let D_a be the resulting drawing of G, as illustrated in Figure 4.3(a);
  Similarly construct a drawing D_b of G, as illustrated in Figure 4.3(b);
  if $bend(D_a) \leq bend(D_b)$, **then return** D_a as D
    else return D_b as D;
end.

One can easily observe that **NonBiconnected** finds an optimal drawing of G in linear time. We thus have the following theorem.

THEOREM 4.3. *An optimal orthogonal drawing of a series-parallel graph G of $\Delta \leq 3$ can be found in linear time.*

4.2. Upper bounds on bends. In this subsection we prove the following theorem.

THEOREM 4.4. *If G is a series-parallel graph of $\Delta \leq 3$, then $bend(G) \leq (n + 4)/3$.*

The bound in Theorem 4.4 is best possible, because $bend(G) = (n + 4)/3$ for a series-parallel graph G consisting of two copies of $K_4 - e$ and several copies of K_2 and K_3 connected in series, as illustrated in Figure 4.5, where $K_4 - e$ is a graph obtained from K_4 by deleting an edge e.

We first have the following lemma.

LEMMA 4.5. *If a biconnected series-parallel graph B has a vertex v of degree two, then*

$$bend(B, 3\pi/2, v) \leq (n(B) + 2)/3.$$

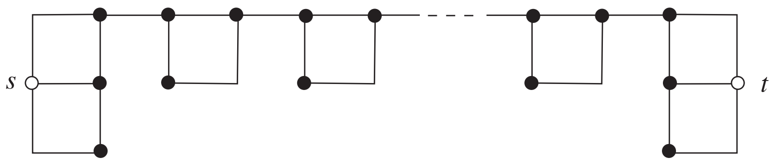


FIG. 4.5. *A series-parallel graph G with $(n+4)/3$ bends.*

Proof. Since B is biconnected, we have $n(B) \geq 3$. Split the vertex v into two vertices s' and t', and let B_v be the resulting 2-legged series-parallel graph, as illustrated in Figure 4.6.

Consider first the case where B_v is a diamond graph. Then clearly $n(B) \geq 6$. By Lemma 3.1(a), B_v has a no-bend L-shaped drawing $D_{v\mathrm{L}}$. From $D_{v\mathrm{L}}$, one can construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with two bends. (See Figures 3.8(g) and (h).) Therefore $\mathit{bend}(B, 3\pi/2, v) \leq \mathit{bend}(D_{3\pi/2}) = 2 \leq (n(B)+2)/3$.

Consider next the case where B_v is not a diamond graph. By Lemma 3.2(a), B_v has an optimal U-shaped drawing $D_{v\mathrm{U}}$, and $\mathit{bend}(D_{v\mathrm{U}}) = \mathit{bend}(B_v)$. From $D_{v\mathrm{U}}$, one can construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with a new bend, and hence $\mathit{bend}(D_{3\pi/2}) = \mathit{bend}(D_{v\mathrm{U}}) + 1$. (See Figures 3.8(e) and (f).) By Lemma 3.2(c), $\mathit{bend}(B_v) \leq (n(B_v)-2)/3$. Clearly $n(B) = n(B_v) - 1$. We thus have

$$\begin{aligned}\mathit{bend}(B, 3\pi/2, v) &\leq \mathit{bend}(D_{3\pi/2})\\ &= \mathit{bend}(D_{v\mathrm{U}}) + 1\\ &= \mathit{bend}(B_v) + 1\\ &\leq (n(B_v)-2)/3 + 1\\ &= (n(B)+2)/3. \qquad \square\end{aligned}$$

We are now ready to present a proof of Theorem 4.4.

Proof of Theorem 4.4. By Lemma 3.7, $\mathit{bend}(G) \leq \lceil n/3 \rceil < (n+4)/3$ for every series-parallel biconnected graph G. We may thus assume that G is not biconnected. Then one may assume that G consists of biconnected components $B_1, B_2, \ldots, B_p$, $p \geq 2$, and several copies of K_2, alternately connected in series, as illustrated in Figure 4.1(a). **NonBiconnected** finds an optimal drawing D of G such that $\mathit{bend}(D) \leq \mathit{bend}(D_a) \leq \mathit{bend}(B_1) + \mathit{bend}(G_{\mathrm{int}}) + \mathit{bend}(B_p, 3\pi/2, s_p)$.

By Lemma 3.7 we have $\mathit{bend}(B_1) \leq \lceil n(B_1)/3 \rceil \leq (n(B_1)+2)/3$. We have $\mathit{bend}(G_{\mathrm{int}}) \leq (n(G_{\mathrm{int}})-2)/3$; if $n(G_{\mathrm{int}}) = 2$, then $\mathit{bend}(G_{\mathrm{int}}) = 0 = (n(G_{\mathrm{int}})-2)/3$; if $n(G_{\mathrm{int}}) \geq 3$ and G_{int} is a diamond graph, then by Lemma 3.1(a) $\mathit{bend}(G_{\mathrm{int}}) = 0 < (n(G_{\mathrm{int}})-2)/3$; if $n(G_{\mathrm{int}}) \geq 3$ and G_{int} is not a diamond graph, then by Lemma 3.2(c) $\mathit{bend}(G_{\mathrm{int}}) \leq (n(G_{\mathrm{int}})-2)/3$. By Lemma 4.5 we have $\mathit{bend}(B_p, 3\pi/2, s_p) \leq (n(B_p)+2)/3$. Clearly $n(G) = n(B_1) + n(G_{\mathrm{int}}) + n(B_p) - 2$. We thus have

$$\begin{aligned}\mathit{bend}(G) &\leq (n(B_1)+2)/3 + (n(G_{\mathrm{int}})-2)/3 + (n(B_p)+2)/3\\ &\leq (n(G)+4)/3. \qquad \square\end{aligned}$$

One can easily observe that each edge has at most one bend in a drawing found by our algorithm.

4.3. Proof of Lemma 4.2. In this subsection, we first give a proof of Lemma 4.2, then present algorithm **OuterDrawing** to find an optimal $(3\pi/2, v)$-outer drawing, and finally prove (4.2).

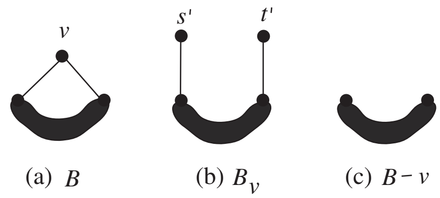


FIG. 4.6. (a) *Biconnected graph* B, (b) 2-*legged graph* B_v, *and* (c) *graph* $B - v$.

Let B be a biconnected series-parallel graph with $\Delta(B) \leq 3$, and let v be a vertex of degree two in B, as illustrated in Figure 4.6(a). Split the vertex v into two vertices s' and t', and let B_v be the resulting 2-legged series-parallel graph with terminals s' and t', as illustrated in Figure 4.6(b).

Consider first the case where B_v is a diamond graph. Then we have the following lemma.

LEMMA 4.6. *If* B_v *is a diamond graph, then* $bend(B, 3\pi/2, v) = bend(B) = 2$.

Proof. Since B_v is a diamond graph, by Lemma 3.1 B_v has a no-bend L-shaped drawing D_{L}, from which one can construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with two bends (see Figures 3.8(g) and (h)), and hence

$$bend(B) \leq bend(B, 3\pi/2, v) \leq bend(D_{3\pi/2}) = 2.$$

Thus it suffices to verify $2 \leq bend(B)$.

Let D^* be an arbitrary optimal drawing of B. Let D_v be a drawing of B_v obtained from D^* without introducing any new bend as follows: erase, from D^*, point v together with very short segments of two edges incident to vertex v, and regard the ends of the erased seqments other than v as vertices s' and t' of B_v. We then have

$$bend(D_v) = bend(D^*) = bend(B). \tag{4.4}$$

The construction of D_v implies that D_v has none of I-, L-, and U-shapes. We now claim $2 \leq bend(D_v)$; if $bend(D_v) = 0$, then by Lemma 3.1(c), D_v must be I-shaped or L-shaped, a contradiction; if $bend(D_v) = 1$, then one can easily observe from the example in Figure 2.6 that D_v is I-, L-, or U-shaped, a contradiction. Thus we have $2 \leq bend(D_v) = bend(B)$. ∎

The proof of Lemma 4.6 immediately yields the following linear algorithm **OuterDiamond**(B, v, D) to find an optimal $(3\pi/2, v)$-outer drawing D of B, with $bend(D) = bend(B) = 2$ if B_v is a diamond graph.

OuterDiamond(B**,** v**,** D**)**;
begin
  Find a no-bend L-shaped drawing $D_{v\mathrm{L}}$ of B_v by **Diamond**;
  Construct from $D_{v\mathrm{L}}$ a $(3\pi/2, v)$-outer drawing D of B with two bends;
  (See Figures 3.8(g) and (h).)
end.

Consider next the case where $B - v$ is a series connection of subgraphs. We then have the following lemma.

LEMMA 4.7. *If* $B - v$ *is a series connection of subgraphs, then* $bend(B, 3\pi/2, v) = bend(B)$.

Proof. Since $\Delta(B) \leq 3$ and $B - v$ is a series connection of subgraphs, B_v has a bridge e other than the two legs, as illustrated in Figure 4.7(a). B_v has two subgraphs

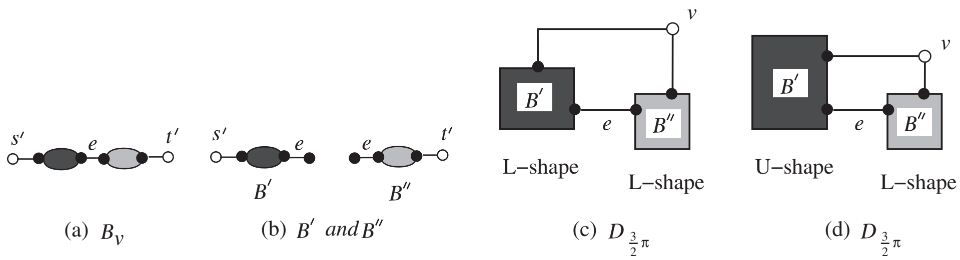


FIG. 4.7. *Illustration of the proof of Lemma* 4.7.

B' and B'', each of which is a 2-legged series-parallel graph and has e as a leg, as illustrated in Figure 4.7(b). Clearly $n(B'), n(B'') \geq 3$.

Consider first the case where both B' and B'' are diamond graphs. Then by Lemma 3.1(a), B' and B'' have no-bend L-shaped drawings D'_{L} and D''_{L}, respectively. Merging D'_{L} and D''_{L}, one can construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with one bend, as illustrated in Figure 4.7(c), and hence $bend(B, 3\pi/2, v) \leq 1$. By Lemma 3.1(c), any no-bend drawings of B' and B'' are not U-shaped but are I-shaped or L-shaped. Therefore, B does not have a no-bend drawing. Thus $D_{3\pi/2}$ is an optimal drawing of B, and hence $bend(B) = bend(B, 3\pi/2, v) = 1$.

Consider next the case of either B' or B'' is not a diamond graph. Assume without loss of generality that B' is not a diamond graph. Then by Lemma 3.2(a), B' has an optimal U-shaped drawing D'_{U}. On the other hand, B'' has an optimal L-shaped drawing D''_{L}; if B'' is a diamond graph, then by Lemma 3.1(a), B'' has an optimal (no-bend) L-shaped drawing; otherwise, by Lemma 3.2(a), B'' has an optimal L-shaped drawing. Merging D'_{U} and D''_{L}, one can construct an optimal $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B without introducing any new bend, as illustrated in Figure 4.7(d). Clearly $D_{3\pi/2}$ is an optimal drawing of B, and hence $bend(B, 3\pi/2, v) = bend(B)$. □

The proof of Lemma 4.7 immediately yields the following linear algorithm **OuterSeries**(B, v, D) to find an optimal $(3\pi/2, v)$-outer drawing D of B, with $bend(D) = bend(B)$ if $B - v$ is a series connection of subgraphs.

OuterSeries(B**,** v**,** D**)**;
begin
  Define B' and B'' as in Figures 4.7(a) and (b);
  if both B' and B'' are diamond graphs,
  then
    Find no-bend L-shaped drawings D'_{L} of B' and D''_{L} of B'' by **Diamond**;
    Construct a drawing D with one bend from D'_{L} and D''_{L} as in Figure 4.7(c);
  else
    Assume that B' is not a diamond graph;
    Find an optimal U-shaped drawing D'_{U} of B' by **NonDiamond**;
    Find an optimal L-shaped drawing D''_{L} of B'' by **Diamond** or **NonDiamond**;
    Construct D from D'_{U} and D''_{L} as in Figure 4.7(d);
end.

We shall thus consider the remaining case where $B - v$ is a parallel connection of subgraph. Roughly speaking, for the case, we reduce the problem of finding an optimal $(3\pi/2, v)$-outer drawing of B to those for smaller graphs B'_{id} and B''_{id} illustrated in Figure 4.8. The detail is given in the following proof of Lemma 4.2(a).

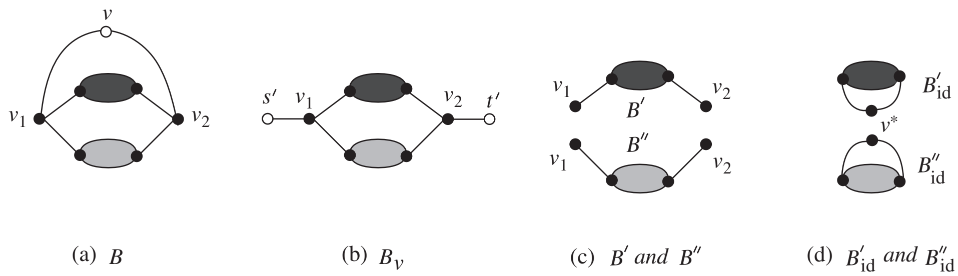


FIG. 4.8. *Illustration of the proof of Lemma* 4.2.

Proof of Lemma 4.2(a). We prove (4.3) by induction on the number $n(B)$ of vertices of B. If $n(B) = 3$, that is, $B = K_3$, then $bend(B, 3\pi/2, v) = 1$, $bend(B, \pi, v) = 2$, and $bend(B, \pi/2, v) = 3$, and hence (4.3) holds. One may thus assume that $n(B) \geq 4$, and inductively assume that (4.3) holds for every biconnected series-parallel graph of at most $n(B) - 1$ vertices. We now prove (4.3) for B. By Lemmas 4.6 and 4.7, one may assume that B_v is not a diamond graph and $B - v$ is a parallel connection of subgraphs. Then $B - v$ is biconnected, and the two vertices v_1 and v_2 that are adjacent to v in B have degree three in B, as illustrated in Figure 4.8(a). Splitting vertices v_1 and v_2 of $B - v$, one can obtain two edge-disjoint series-parallel subgraphs B' and B'' of B, as illustrated in Figure 4.8(c). Clearly $n(B'), n(B'') \geq 2$.

We prove only $bend(B, 3\pi/2, v) \leq bend(B, \pi, v)$, because one can similarly prove $bend(B, 3\pi/2, v) \leq bend(B, \pi/2, v)$. Let D^*_π be an optimal (π, v)-outer drawing of B, then $bend(D^*_\pi) = bend(B, \pi, v)$. It suffices to prove that $bend(B, 3\pi/2, v) \leq bend(D^*_\pi)$. There are the following three cases to consider.

Case 1: D^*_π has a bend on edge vv_1 or vv_2.

Erasing from D^*_π a line segment connecting v and a bend in the edge, one can obtain a drawing D_v of B_v. Since a bend in D^*_π is regarded as a vertex in D_v, we have $bend(B_v) \leq bend(D_v) = bend(D^*_\pi) - 1$.

Since B_v is not a diamond graph, by Lemma 3.2(a) B_v has an optimal U-shape drawing $D_{v\mathrm{U}}$. From $D_{v\mathrm{U}}$, one can construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with a new bend. (See Figures 3.8(e) and (f).) Thus we have

$$\begin{aligned} bend(B, 3\pi/2, v) &\leq bend(D_{3\pi/2}) \\ &= bend(D_{v\mathrm{U}}) + 1 \\ &= bend(B_v) + 1 \\ &\leq bend(D^*_\pi). \end{aligned}$$

Case 2: D^*_π has no bend on edges vv_1 and vv_2, and either $n(B') = 2$ or $n(B'') = 2$.

One may assume without loss of generality that $n(B') = 2$, and hence B' consists of a single edge v_1v_2. Since D^*_π has no bend on the edges v_1v and vv_2, they are drawn on the same straight line segment in D^*_π. Therefore the edge v_1v_2 in B' must have two or more bends in D^*_π, and hence $bend(D^*_\pi|B') \geq 2$. Since B has no multiple edges, $n(B'') \geq 3$ and B'' is a 2-legged series-parallel graph.

If B'' is a diamond graph, then $bend(B'') = 0$ and B'' has a U-shaped drawing D''_U with one bend as known from the example in Figure 2.6. If B'' is not a diamond graph, then by Lemma 3.2(a), B'' has an optimal U-shaped drawing D''_U. In either case $bend(D''_\mathrm{U}) \leq bend(B'')+1$. One can easily extend D''_U to a $(3\pi/2, v)$-outer drawing

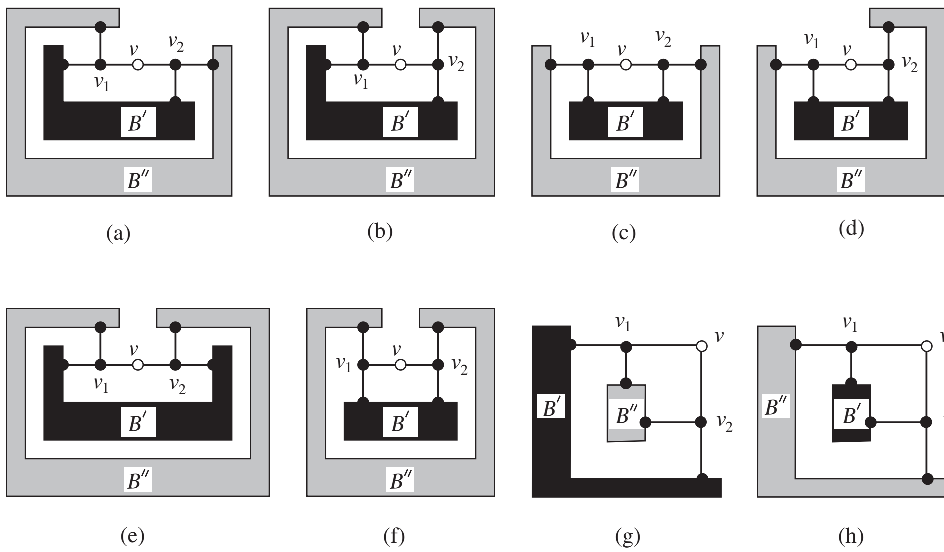


FIG. 4.9. (a)–(f) *All of the possible* (π, v)*-outer drawings* D^*_π*, and* (g), (h) *constructions of* $D_{3\pi/2}$.

$D_{3\pi/2}$ with a new bend. We thus have

$$\begin{aligned} bend(B, 3\pi/2, v) &\le bend(D_{3\pi/2}) \\ &= 1 + bend(D''_{\mathrm{U}}) \\ &\le 1 + bend(B'') + 1 \\ &\le 2 + bend(D^*_\pi|B'') \\ &\le bend(D^*_\pi|B') + bend(D^*_\pi|B'') \\ &= bend(D^*_\pi). \end{aligned}$$

Case 3: Otherwise.

In this case, $n(B'), n(B'') \ge 3$, and hence both B' and B'' are 2-legged series-parallel graphs. Identify v_1 with v_2 as a new vertex v^* in B', and let B'_{id} be the resulting graph, as illustrated in Figure 4.8(d). Similarly define B''_{id}. Then both B'_{id} and B''_{id} are biconnected series-parallel graphs and have a vertex v^* of degree two. Since D^*_π has no bend on the two edges v_1v and vv_2, all of the possible (π, v)-outer drawings of B are those illustrated in Figures 4.9(a)–(f) after interchanging the roles of B' and B'' and the roles of v_1 and v_2, depending on the angles of v_1 and v_2 in D^*_π. There are the following two subcases to consider.

Case 3.1: D^*_π is a drawing illustrated in Figure 4.9(a) or (b).

In this case, one can construct from $D^*_\pi|B'$ a $(3\pi/2, v^*)$-outer drawing of B'_{id} with no new bends, and hence $bend(B'_{\mathrm{id}}, 3\pi/2, v^*) \le bend(D^*_\pi|B')$. Let $D'_{\mathrm{id}3\pi/2}$ be an optimal $(3\pi/2, v^*)$-outer drawing of B'_{id}. Since B'' is 2-legged, B'' has an optimal L-shaped drawing D''_{L}. Merging $D'_{\mathrm{id}3\pi/2}$ and D''_{L}, one can easily construct a $(3\pi/2, v)$-

outer drawing $D_{3\pi/2}$ of B, as illustrated in Figure 4.9(g). We thus have

$$\begin{aligned} bend(B, 3\pi/2, v) &\le bend(D_{3\pi/2}) \\ &= bend(D'_{\mathrm{id}3\pi/2}) + bend(D''_{\mathrm{L}}) \\ &= bend(B'_{\mathrm{id}}, 3\pi/2, v^*) + bend(B'') \\ &\le bend(D^*_\pi|B') + bend(D^*_\pi|B'') \\ &= bend(D^*_\pi). \end{aligned}$$

Case 3.2: D^*_π is a drawing illustrated in Figures 4.9(c)–(f).

We first prove that if D^*_π is a drawing in Figure 4.9(c) or (d), then

$$bend(B''_{\mathrm{id}}, 3\pi/2, v^*) \le bend(D^*_\pi|B''). \tag{4.5}$$

One can construct from $D^*_\pi|B''$ an (α, v^*)-outer drawing of B''_{id} with no new bend, where $\alpha = \pi$ for Figure 4.9(c) and $\alpha = \pi/2$ for Figure 4.9(d). We thus have $bend(B''_{\mathrm{id}}, \alpha, v^*) \le bend(D^*_\pi|B'')$. Since B''_{id} is a biconnected series-parallel graph with $d(v^*, B''_{\mathrm{id}}) = 2$ and $n(B''_{\mathrm{id}}) < n(B)$, by the inductive hypothesis we have $bend(B''_{\mathrm{id}}, 3\pi/2, v^*) \le bend(B''_{\mathrm{id}}, \alpha, v^*)$. Thus we have (4.5).

We also have the following fact, whose proof will be given in section 4.4.

FACT 1. *If D^*_π is a drawing illustrated in Figures* 4.9(e) *and* (f)*, then $bend(B''_{\mathrm{id}}, 3\pi/2, v^*) \le bend(D^*_\pi|B'')$.*

Let $D''_{\mathrm{id}3\pi/2}$ be an optimal $(3\pi/2, v^*)$-outer drawing of B''_{id}. Let D'_{L} be an optimal L-shaped drawing of B'. Merging D'_{L} and $D''_{\mathrm{id}3\pi/2}$, one can easily construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B, as illustrated in Figure 4.9(h). Thus, using (4.5) and Fact 1 we have

$$\begin{aligned} bend(B, 3\pi/2, v) &\le bend(D_{3\pi/2}) \\ &= bend(D'_{\mathrm{L}}) + bend(D''_{\mathrm{id}3\pi/2}) \\ &= bend(B') + bend(B''_{\mathrm{id}}, 3\pi/2, v^*) \\ &\le bend(D^*_\pi|B') + bend(D^*_\pi|B'') \\ &= bend(D^*_\pi). \qquad \square \end{aligned}$$

We then have the following lemma.

LEMMA 4.8. *Suppose that B_v is not a diamond graph and that $B - v$ is a parallel connection of subgraphs. (See Figure* 4.8.*) Then the following* (a) *and* (b) *hold:*

(a) *if $n(B') = 2$ or $n(B'') = 2$, then $bend(B, 3\pi/2, v) = bend(B_v) + 1$; and*

(b) *if $n(B'), n(B'') \ge 3$, then*

$$\begin{aligned} bend(B, 3\pi/2, v) = \min\{&bend(B_v) + 1, \\ &bend(B'_{\mathrm{id}}, 3\pi/2, v^*) + bend(B''), \\ &bend(B') + bend(B''_{\mathrm{id}}, 3\pi/2, v^*)\}. \end{aligned} \tag{4.6}$$

Proof. We first prove

$$bend(B, 3\pi/2, v) \le bend(B_v) + 1. \tag{4.7}$$

Since $n(B) \ge 3$, $n(B_v) \ge 4$. Since B_v is not a diamond graph, by Lemma 3.2(a), B_v has an optimal U-shaped drawing $D_{v\mathrm{U}}$. From $D_{v\mathrm{U}}$, one can easily construct a

$(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with a new bend. (See Figures 3.8(e) and (f).) Therefore,

$$\begin{aligned} bend(B, 3\pi/2, v) &\leq bend(D_{3\pi/2}) \\ &= bend(D_{v\mathrm{U}}) + 1 \\ &= bend(B_v) + 1. \end{aligned}$$

(a) We then prove (a). By (4.7), it suffices to prove

$$bend(B, 3\pi/2, v) \geq bend(B_v) + 1. \tag{4.8}$$

Since B has no multiple edges and either $n(B') = 2$ or $n(B'') = 2$, one may assume without loss of generality that $n(B') = 2$, and hence $n(B'') \geq 3$. Let $D^*_{3\pi/2}$ be an optimal $(3\pi/2, v)$-outer drawing of B, then $bend(B, 3\pi/2, v) = bend(D^*_{3\pi/2})$. If $D^*_{3\pi/2}$ has a bend on an edge incident to v, then $bend(D^*_{3\pi/2}) - 1 \geq bend(B_v)$, and hence $bend(B, 3\pi/2, v) = bend(D^*_{3\pi/2}) \geq bend(B_v) + 1$. One may thus assume that $D^*_{3\pi/2}$ has no bend on the two edges of B incident to v. Then $D^*_{3\pi/2}$ has one or more bends on the edge in B', and hence $bend(D^*_{3\pi/2}) \geq 1$. Let $\overline{B''}$ be the complement of B'' in B. Then $\overline{B''}$ is $K_3 = v_1vv_2$, and hence

$$bend(B^*_{3\pi/2}|\overline{B''}) \geq bend(\overline{B''}) \geq 1. \tag{4.9}$$

By (2.2) we have

$$bend(D^*_{3\pi/2}) = bend(D^*_{3\pi/2}|B'') + bend(D^*_{3\pi/2}|\overline{B''}). \tag{4.10}$$

There are the following two cases to consider.

Case 1: B'' is a diamond graph.

We first claim that $bend(D^*_{3\pi/2}) \geq 2$. Suppose for a contradiction that $bend(D^*_{3\pi/2}) = 1$. Then by (4.9) and (4.10) we have

$$bend(D^*_{3\pi/2}|\overline{B''}) = 1$$

and

$$bend(D^*_{3\pi/2}|B'') = 0.$$

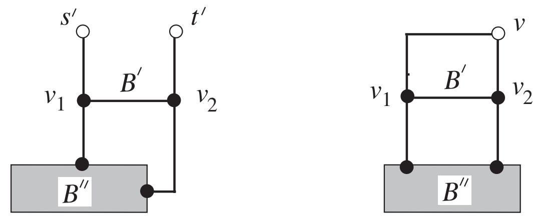
Therefore, by Lemma 3.1(c), the no-bend drawing $D^*_{3\pi/2}|B''$ of B'' must be I- or L-shaped. Since $D^*_{3\pi/2}$ has no bend on edges vv_1 and vv_2, $D^*_{3\pi/2}|B''$ must be L-shaped. Then $D^*_{3\pi/2}$ must have three or more bends on the edge v_1v_2 in B', and hence $bend(D^*_{3\pi/2}|\overline{B''}) \geq 3$, contrary to $bend(D^*_{3\pi/2}|\overline{B''}) = 1$.

Since B'' is a diamond graph, by Lemma 3.1(a), B'' has a no-bend L-shaped drawing D''_{L}. From D''_{L}, one can easily construct a drawing D_v of B_v with one bend, as illustrated in Figure 4.10(a), and hence $bend(B_v) \leq bend(D_v) = 1$. Thus we have $bend(B, 3\pi/2, v) = bend(D^*_{3\pi/2}) \geq 2 \geq bend(B_v) + 1$.

Case 2: B'' is not a diamond graph.

Since $bend(D^*_{3\pi/2}|B'') \geq bend(B'')$, by (4.9) and (4.10) we have

$$bend(D^*_{3\pi/2}) \geq bend(B'') + 1. \tag{4.11}$$



(a) D_v (b) $D_{3\pi/2}$

FIG. 4.10. *Illustration for the proof of Lemma* 4.8(a).

Since B'' is not a diamond graph, by Lemma 3.2(a), B'' has an optimal U-shaped drawing D''_{U}. From D''_{U}, one can easily construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with one new bend, as illustrated in Figure 4.10(b). We then have

$$\begin{aligned} bend(D_{3\pi/2}) &= bend(D''_{\mathrm{U}}) + 1 \\ &= bend(B'') + 1. \end{aligned} \tag{4.12}$$

Erasing from $D_{3\pi/2}$ the line segment connecting v and a bend, one can obtain a drawing D_v of B_v, and hence

$$\begin{aligned} bend(B_v) &\le bend(D_v) \\ &= bend(D_{3\pi/2}) - 1. \end{aligned} \tag{4.13}$$

By (4.11)–(4.13) we have

$$\begin{aligned} bend(B, 3\pi/2, v) &= bend(D^*_{3\pi/2}) \\ &\ge bend(B'') + 1 \\ &= bend(D_{3\pi/2}) \\ &\ge bend(B_v) + 1. \end{aligned}$$

(b) We then prove (b). We first prove that

$$\begin{aligned} bend(B, 3\pi/2, v) \le \min\{&bend(B_v) + 1, \\ &bend(B'_{\mathrm{id}}, 3\pi/2, v^*) + bend(B''), \\ &bend(B') + bend(B''_{\mathrm{id}}, 3\pi/2, v^*)\}. \end{aligned} \tag{4.14}$$

Since (4.7) holds, it suffices to prove that

$$bend(B, 3\pi/2, v) \le bend(B'_{\mathrm{id}}, 3\pi/2, v^*) + bend(B'') \tag{4.15}$$

and

$$bend(B, 3\pi/2, v) \le bend(B') + bend(B''_{\mathrm{id}}, 3\pi/2, v^*). \tag{4.16}$$

We verify only (4.15), because one can similarly verify (4.16).

Let $D'_{\mathrm{id}3\pi/2}$ be an optimal $(3\pi/2, v^*)$-outer drawing of B'_{id}. Since $n(B'') \ge 3$, there is an optimal L-shaped drawing D''_{L} of B''. Merging $D'_{\mathrm{id}3\pi/2}$ and D''_{L}, one can

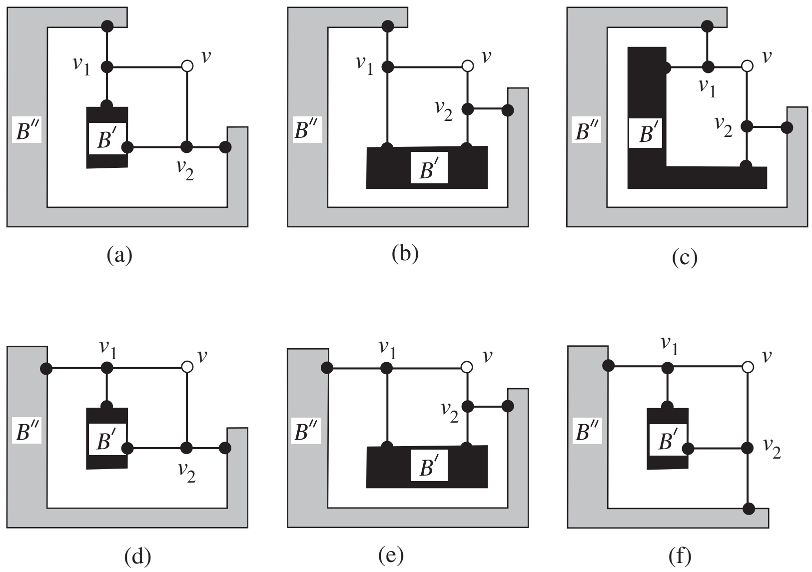


FIG. 4.11. *All of the possible* $(3\pi/2, v)$*-outer drawings* $D^*_{3\pi/2}$ *of* B.

easily construct a $(3\pi/2, v)$-outer drawing $D_{3\pi/2}$ of B with no new bend, as illustrated in Figure 4.9(g). We thus have

$$\begin{aligned} bend(B, 3\pi/2, v) &\leq bend(D_{3\pi/2}) \\ &= bend(D'_{\mathrm{id}3\pi/2}) + bend(D''_{\mathrm{L}}) \\ &= bend(B'_{\mathrm{id}}, 3\pi/2, v^*) + bend(B''). \end{aligned}$$

We next prove that

$$\begin{aligned} bend(B, 3\pi/2, v) \geq \min\{&bend(B_v) + 1, \\ &bend(B'_{\mathrm{id}}, 3\pi/2, v^*) + bend(B''), \\ &bend(B') + bend(B''_{\mathrm{id}}, 3\pi/2, v^*)\}. \end{aligned} \tag{4.17}$$

Let $D^*_{3\pi/2}$ be an optimal $(3\pi/2, v)$-outer drawing of B, then $bend(B, 3\pi/2, v) = bend(D^*_{3\pi/2})$. If $D^*_{3\pi/2}$ has a bend on an edge incident to v, then $bend(D^*_{3\pi/2}) - 1 \geq bend(B_v)$, and hence $bend(B, 3\pi/2, v) = bend(D^*_{3\pi/2}) \geq bend(B_v) + 1$ and (4.17) holds. One may thus assume that $D^*_{3\pi/2}$ has no bend on the two edges of B incident to v. Then all of the possible $(3\pi/2, v)$-outer drawings of B are those illustrated in Figures 4.11(a)–(f) after interchanging the roles of B' and B'' and the roles of v_1 and v_2.

We now claim

$$bend(B''_{\mathrm{id}}, 3\pi/2, v^*) \leq bend(D^*_{3\pi/2}|B''). \tag{4.18}$$

If $D^*_{3\pi/2}$ is a drawing illustrated in Figure 4.11(f), then from $D^*_{3\pi/2}|B''$, one can easily construct a $(3\pi/2, v^*)$-outer drawing of B''_{id} with no new bend, and hence (4.18) holds. One may thus assume that $D^*_{3\pi/2}$ is a drawing illustrated in Figures 4.11(a)–(e). Then one can construct from $D^*_{3\pi/2}|B''$ an (α, v^*)-outer drawing of B''_{id} with no new bend, where $\alpha = \pi/2$ for Figures 4.11(a)–(c) and $\alpha = \pi$ for Figures 4.11(d) and (e). We thus have $bend(B''_{\rm id}, \alpha, v^*) \le bend(D^*_{3\pi/2}|B'')$. By Lemma 4.2(a), $bend(B''_{\rm id}, 3\pi/2, v^*) \le bend(B''_{\rm id}, \alpha, v^*)$. These two equations imply (4.18).

By (4.18) we have

$$\begin{aligned} bend(B, 3\pi/2, v) &= bend(D^*_{3\pi/2}) \\ &= bend(D^*_{3\pi/2}|B') + bend(D^*_{3\pi/2}|B'') \\ &\ge bend(B') + bend(B''_{\rm id}, 3\pi/2, v^*), \end{aligned}$$

and hence (4.17) holds true. □

By Lemmas 4.6, 4.7, and 4.8 and the proofs of (4.7) and (4.14) we have the following recursive algorithm **OuterDrawing**$(B,\ v, D, D_{v\rm I}, D_{v\rm L}, D_{v\rm U})$ to find an optimal $(3\pi/2, v)$-outer drawing D of a biconnected series-parallel graph B with a vertex v of degree two. It finds an optimal I-shaped drawing $D_{v\rm I}$, an optimal L-shaped drawing $D_{v\rm L}$, and an optimal U-shaped drawing $D_{v\rm U}$ of B_v, too.

OuterDrawing($B, v, D, D_{v\rm I}, D_{v\rm L}, D_{v\rm U}$**)**;
begin
Case 1: B_v is a diamond graph.
OuterDiamond(B, v, D**)**; {cf. Lemma 4.6}
Find a no-bend I-shaped drawing $D_{v\rm I}$ and a no-bend L-shaped drawing $D_{v\rm L}$ of B_v by **Diamond**;
Insert a dummy vertex of degree two in an edge of B_v, as illustrated in Figure 2.6(e);
Obtain a U-shaped drawing $D_{v\rm U}$ of B_v with one bend by regarding the dummy vertex as a bend;
Case 2: B_v is not a diamond graph, and $B - v$ is a series connection of subgraphs.
OuterSeries(B, v, D**)**; {cf. Lemma 4.7}
Find an optimal I-shaped drawing $D_{v\rm I}$, an optimal L-shaped drawing $D_{v\rm L}$, and an optimal U-shaped drawing $D_{v\rm U}$ of B_v by **NonDiamond**;
Case 3: B_v is not a diamond graph, and $B - v$ is a parallel connection of subgraphs.
{cf. the proofs of (4.7) and (4.14).}
Case 3.1: Either $n(B') = 2$ or $n(B'') = 2$.
Find an optimal I-shaped drawing $D_{v\rm I}$, an optimal L-shaped drawing $D_{v\rm L}$, and an optimal U-shaped drawing $D_{v\rm U}$ of B_v by **NonDiamond**;
Extend $D_{v\rm U}$ to a $(3\pi/2, v)$-outer drawing D of B with a new bend;
{See Figures 3.8(e) and (f).}
Case 3.2: $n(B'), n(B'') \ge 3$.
{One may assume without loss of generality that B' is not a diamond graph.}
OuterDrawing$(B'_{\rm id}, v^*, D'_{\rm id}, D'_{v^*\rm I}, D'_{v^*\rm L}, D'_{v^*\rm U})$;
OuterDrawing$(B''_{\rm id}, v^*, D''_{\rm id}, D''_{v^*\rm I}, D''_{v^*\rm L}, D''_{v^*\rm U})$;
Combine a U-shaped drawing $D'_{v^*\rm U}$ of $B' (= B'_{{\rm id}v^*})$ and an I-shaped drawing $D''_{v^*\rm I}$ of $B'' (= B''_{{\rm id}v^*})$ to an optimal I-shaped drawing $D_{v\rm I}$, an optimal L-shaped drawing $D_{v\rm L}$, and an optimal U-shaped drawing $D_{v\rm U}$ of B_v, as illustrated in Figure 3.4(e);
{cf. Lemma 3.2}
Let $x = \min\{bend(D_{v\rm U}) + 1, bend(D'_{\rm id}) + bend(D''_{\rm L}), bend(D'_{\rm L}) + bend(D''_{\rm id})\}$, where $D'_{\rm L} = D'_{v^*\rm L}$ and $D''_{\rm L} = D''_{v^*\rm L}$ are optimal L-shaped drawings of $B' = B'_{{\rm id}v^*}$ and $B'' = B''_{{\rm id}v^*}$, respectively;

Case 3.2.1: $x = bend(D_{v\mathrm{U}}) + 1$.
Extend $D_{v\mathrm{U}}$ to a $(3\pi/2, v)$-outer drawing D of B with a new bend;
{See Figure 3.8(e).}
Case 3.2.2: $x = bend(D'_{\mathrm{id}}) + bend(D''_{\mathrm{L}})$.
Construct a $(3\pi/2, v)$-outer drawing D of B by merging D'_{id} and D''_{L}, as illustrated in Figure 4.9(g);
Case 3.2.3: $x = bend(D'_{\mathrm{L}}) + bend(D''_{\mathrm{id}})$.
Construct a $(3\pi/2, v)$-outer drawing D of B by merging D'_{L} and D''_{id}, as illustrated in Figure 4.9(h);
end.

Clearly **OuterDrawing** correctly finds an optimal $(3\pi/2, v)$-outer drawing D of B. We now show that **OuterDrawing** runs in linear time. Let $T(B)$ be the execution time of **OuterDrawing(**$B, v, D, D_{v\mathrm{I}}, D_{v\mathrm{L}}, D_{v\mathrm{U}}$**)**. Let $T_1(B)$ be the execution time of **OuterDrawing** when **Case 1** occurs, that is, B_v is a diamond graph. We similarly define $T_2(B)$, $T_{3.1}(B)$, and $T_{3.2}(B)$ for **Cases 2**, **3.1**, and **3.2**, respectively. Then clearly

$$T(B) = \max\{T_1(B), T_2(B), T_{3.1}(B), T_{3.2}(B)\},$$

$$T_1(B), T_2(B), T_{3.1}(B) = O(n(B)),$$

$$T_{3.2}(B) = T(B'_{\mathrm{id}}) + T(B''_{\mathrm{id}}) + O(1),$$

and

$$n(T(B'_{\mathrm{id}})) + n(T(B''_{\mathrm{id}})) < n(B).$$

We hence have $T(B) = O(n(B))$. We have thus completed a proof of Lemma 4.2(c).

Since $bend(B_v) \le bend(B)$, by Lemmas 4.6, 4.7, and 4.8, we immediately have $bend(B, 3\pi/2, v) \le bend(B) + 1$, completing a proof of Lemma 4.2(b).

Let a series-parallel graph G take the form in Figure 4.1(a), and let D_a be a drawing of G, as illustrated in Figure 4.3(a). Then we have

$$\begin{aligned} bend(G) &\le bend(D_a) \\ &= bend(B_1) + bend(G_{\mathrm{int}}) + bend(B_p, 3\pi/2, s_p). \end{aligned}$$

Clearly

$$\begin{aligned} bend(G_{\mathrm{int}}) &= bend(D_{\mathrm{intI}}) \\ &= \sum_{i=2}^{p-1} bend(B_i) \end{aligned}$$

and

$$bend(B_p, 3\pi/2, s_p) \le bend(B_p) + 1.$$

We thus have

$$bend(G) \le \sum_{i=1}^{p} bend(B_i) + 1. \tag{4.19}$$

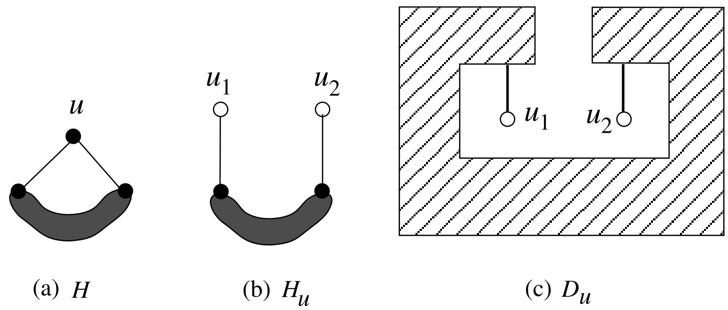


FIG. 4.12. *Biconnected graph H, 2-legged graph H_u, and C-shaped drawing D_u.*

Equations (4.1) and (4.19) immediately imply (4.2).

Omitting the operations in Cases 3.2.2 and 3.2.3 in **OuterDrawing**, one can obtain a nonrescursive algorithm **ApproOuterDrawing** to find a $(3\pi/2, v)$-outer drawing D of B such that D is not always an optimal $(3\pi/2, v)$-outer drawing, but $bend(D) \leq bend(B) + 1$. Replacing **OuterDrawing** in **NonBiconnected** by **ApproOuterDrawing** and finding only the drawing D_a in Figure 4.3, one can obtain a simple linear algorithm to find a drawing D of G such that

$$bend(D) \leq \sum_{i=1}^{p} bend(B_i) + 1.$$

4.4. Proof of Fact 1. In this subsection we give a proof of Fact 1 in Case 3.2 of the proof of Lemma 4.2(a).

A 2-legged series-parallel graph B'' and a biconnected series-parallel graph B''_{id} in Fact 1 are illustrated in Figures 4.8(c) and (d), respectively. B'' can be obtained from B''_{id} by spiltting a vertex v^* of degree two to two vertices v_1 and v_2. The drawing D^*_π in Fact 1 is illustrated in Figure 4.9(e) or (f). We call a drawing of a 2-legged series-parallel graph illustrated in Figure 4.12(c) a *C-shaped drawing.* Then $D^*_\pi|B''$ in Figure 4.9(e) or (f) is a C-shaped drawing of B''. Thus it suffices to prove the following fact. (Regard B''_{id} as H, B'' as H_u, and v^* as u.)

FACT 2. *Let H be a biconnected series-parallel graph with a vertex u of degree two, let H_u be a 2-legged series-parallel graph obtained from H by splitting u to two vertices u_1 and u_2. If D_u is a C-shaped drawing of H_u, then*

$$bend(H, 3\pi/2, u) \leq bend(D_u). \tag{4.20}$$

(See Figure 4.12.)

Proof. We prove (4.20) by induction on the number $n(H)$ of vertices of H. If $n(H) = 3$, that is, $H = K_3$, then $bend(H, 3\pi/2, v) = 1$, $bend(D_u) \geq 4$, and hence (4.20) holds true. One may thus assume that $n(H) \geq 4$, and inductively assume that (4.20) holds for every biconnected series-parallel graph of at most $n(H) - 1$ vertices. We now prove that (4.20) holds for H. There are the following two cases to consider.

Case 1: $H - u$ is a series connection of subgraphs.

In this case, H_u has a bridge e, and has two subgraphs H' and H'', each of which is a 2-legged series-parallel graph and has e as a leg. (See Figures 4.7(a) and (b).) Then we have the following two subcases to consider.

Case 1.1: Both H' and H'' are diamond graphs.

In this case, H' and H'' have no-bend L-shaped drawings D'_{L} and D''_{L}, respectively. Merging D'_{L} and D''_{L}, one can easily construct a $(3\pi/2, u)$-outer drawing $D^*_{3\pi/2}$ of H with one bend, and hence $\mathit{bend}(D^*_{3\pi/2}) = 1$. (See Figure 4.7(c).) Since both H' and H'' are diamond graphs, one can easily observe that $D^*_{3\pi/2}$ is an optimal $(3\pi/2, u)$-outer drawing of H and that a C-shaped drawing D_u of H_u needs one or more bends. We thus have $\mathit{bend}(H_u, 3\pi/2, u) = \mathit{bend}(D^*_{3\pi/2}) = 1 \leq \mathit{bend}(D_u)$.

Case 1.2: Either H' or H'' is not a diamond graph.

In this case, one may assume without loss of generality that H' is not a diamond graph. Then H' has an optimal U-shaped drawing D'_{U}. H'' has an optimal L-shaped drawing D''_{L}. Merging D'_{U} and D''_{L}, one can easily construct a $(3\pi/2, u)$-outer drawing $D_{3\pi/2}$ of H with no new bends. (See Figure 4.7(d).) Thus

$$\begin{aligned}
\mathit{bend}(H, 3\pi/2, u) &\leq \mathit{bend}(D_{3\pi/2}) \\
&= \mathit{bend}(D'_{\mathrm{U}}) + \mathit{bend}(D''_{\mathrm{L}}) \\
&= \mathit{bend}(H') + \mathit{bend}(H'') \\
&\leq \mathit{bend}(D_u|H') + \mathit{bend}(D_u|H'') \\
&= \mathit{bend}(D_u).
\end{aligned}$$

Case 2: $H - u$ is a parallel connection of subgraphs.

Consider first the case where H_u is a diamond graph. By Lemma 4.6, $\mathit{bend}(H, 3\pi/2, u) = \mathit{bend}(H)$. Since H_u has a U-shaped drawing with one bend, we have $\mathit{bend}(H) \leq 2$. Since D_u is a C-shaped drawing of a diamond graph H_u, one can easily observe that $\mathit{bend}(D_u) \geq 2$. We thus have

$$\mathit{bend}(H, 3\pi/2, u) = \mathit{bend}(H) \leq 2 \leq \mathit{bend}(D_u).$$

Consider next the case where H_u is not a diamond graph. Regarding H as B in Figure 4.8, we define H', H'', H'_{id}, and H''_{id}, as illustrated in Figure 4.8. Then we have the following three subcases.

Case 2.1: D_u has a bend on a leg of H_u.

In this case, erasing a line segment connecting a terminal and a bend from D_u, one can obtain a drawing $\widetilde{D_u}$ of H_u such that $\widetilde{D_u}$ is not always C-shaped and $\mathit{bend}(H_u) \leq \mathit{bend}(\widetilde{D_u}) = \mathit{bend}(D_u) - 1$. Since H_u is not a diamond graph, H_u has an optimal U-shape drawing $D_{u\mathrm{U}}$. From $D_{u\mathrm{U}}$, one can construct a $(3\pi/2, u)$-outer drawing $D_{3\pi/2}$ of H with a new bend. (See Figures 3.8(e) and (f).) We thus have

$$\begin{aligned}
\mathit{bend}(H, 3\pi/2, u) &\leq \mathit{bend}(D_{3\pi/2}) \\
&= \mathit{bend}(D_{u\mathrm{U}}) + 1 \\
&= \mathit{bend}(H_u) + 1 \\
&\leq \mathit{bend}(\widetilde{D_u}) + 1 \\
&= \mathit{bend}(D_u).
\end{aligned}$$

Case 2.2: D_u has no bend on the two legs of H_u, and either $n(H') = 2$ or $n(H'') = 2$.

One may assume without loss of generality that $n(H') = 2$, and hence H' consists of a single edge. Then H'' is a 2-legged series-parallel graph. Since the C-shaped drawing D_u has no bend on the two legs of H_u, we have $\mathit{bend}(D_u|H') \geq 4$.

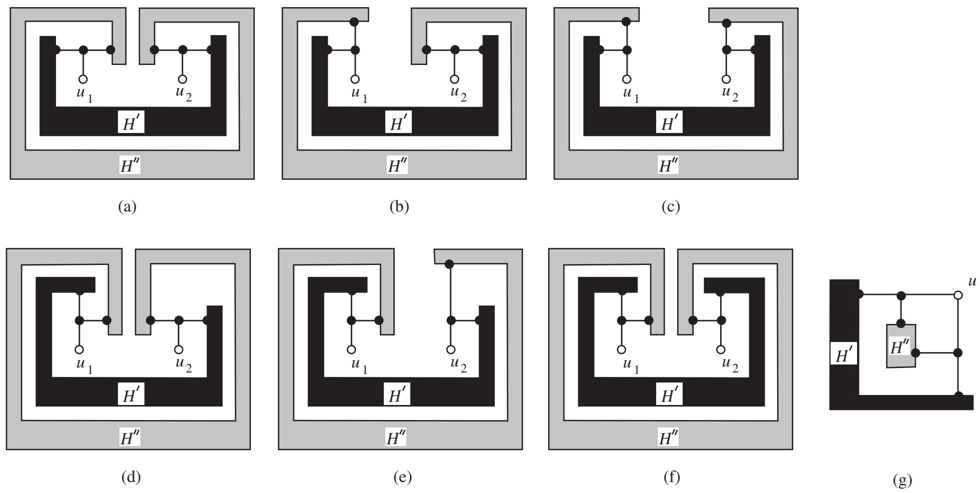


FIG. 4.13. (a)–(f) *All of the possible C-shaped drawings* D_u *of* H_u, *and* (g) *construction of a* $(3\pi/2, u)$*-outer drawing* $D_{3\pi/2}$ *of* H.

Consider first the case where H'' is a diamond graph. Then H_u has a U-shaped drawing $D_{u\mathrm{U}}$ with one bend. From $D_{u\mathrm{U}}$, one can construct a $(3\pi/2, u)$-outer drawing $D_{3\pi/2}$ of H with two bends. Thus $bend(H, 3\pi/2, u) \le bend(D_{3\pi/2}) = 2 < bend(D_u|H') \le bend(D_u)$.

Consider next the case where H'' is not a diamond graph. Then by Lemma 3.2(a), H'' has an optimal U-shaped drawing D''_{U}. From D''_{U}, one can construct a $(3\pi/2, u)$-outer drawing $D_{3\pi/2}$ of H with a new bend. We thus have

$$\begin{aligned} bend(H, 3\pi/2, u) &\le bend(D_{3\pi/2}) \\ &= 1 + bend(D''_{\mathrm{U}}) \\ &= 1 + bend(H'') \\ &< bend(D_u|H') + bend(D_u|H'') \\ &= bend(D_u). \end{aligned}$$

Case 2.3: D_u has no bend on the two legs of H_u, and $n(H'), n(H'') \ge 3$.

In this case H' and H'' are 2-legged series-parallel graphs. All of the possible C-shaped drawings D_u of H_u for this case are those illustrated in Figures 4.13(a)–(f).

Let $D'_{\mathrm{id}3\pi/2}$ be an optimal $(3\pi/2, u^*)$-outer drawing of H'_{id}. Let D''_{L} be an optimal L-shaped drawing of H''. Merging $D'_{\mathrm{id}3\pi/2}$ and D''_{L}, one can easily construct a $(3\pi/2, u)$-outer drawing $D_{3\pi/2}$ of H, as illustrated in Figure 4.13(g). We thus have

$$\begin{aligned} bend(H, 3\pi/2, u) &\le bend(D_{3\pi/2}) \\ &= bend(D'_{\mathrm{id}3\pi/2}) + bend(D''_{\mathrm{L}}) \\ &= bend(H'_{\mathrm{id}}, 3\pi/2, u^*) + bend(H''). \end{aligned} \tag{4.21}$$

We now claim

$$bend(H'_{\mathrm{id}}, 3\pi/2, u^*) \le bend(D_u|H'). \tag{4.22}$$

If D_u is a drawing illustrated in Figure 4.13(f), then $D_u|H'$ is a C-shaped drawing of H', and hence the inductive hypothesis of the proof of Fact 2 implies (4.22). One may thus assume that D_u is a drawing illustrated in Figures 4.13(a)–(e). Then one can construct from $D_u|H'$ an (α, u^*)-outer drawing with no new bend, where $\alpha = \pi$ for Figures 4.13(a)–(c) and $\alpha = \pi/2$ for Figures 4.13(d) and (e). Hence

$$bend(H'_{\mathrm{id}}, \alpha, u^*) \leq bend(D_u|H'). \tag{4.23}$$

Since $n(H'_{\mathrm{id}}) < n(B)$ for the graph B in Fact 1, by the inductive hypothesis of the proof of Lemma 4.2(a) we have

$$bend(H'_{\mathrm{id}}, 3\pi/2, u^*) \leq bend(H'_{\mathrm{id}}, \alpha, u^*). \tag{4.24}$$

Equations (4.23) and (4.24) imply (4.22).

By (4.21) and (4.22) we have

$$\begin{aligned} bend(H, 3\pi/2, u) &\leq bend(D_u|H') + bend(D_u|H'') \\ &= bend(D_u). \qquad \square \end{aligned}$$

5. Conclusions. In this paper, we gave a linear algorithm to find an optimal orthogonal drawing of a series-parallel graph G of $\Delta \leq 3$ in the variable embeddings setting. Our algorithm works well even if G has multiple edges or is not biconnected and is simpler and faster than the previously known one for biconnected series-parallel simple graphs [4, 5]. One can easily extend our algorithm so that it finds an optimal orthogonal drawing of a partial 2-tree of $\Delta \leq 3$. Note that the so-called block-cutvertex graph of a partial 2-tree is a tree although the block-cutvertex graph of a series-parallel graph is a path. We gave a best possible bound on $bend(G)$: $bend(G) \leq (n(G) + 4)/3$.

In an *orthogonal grid drawing*, every vertex has an integer coordinate. The *size* of an orthogonal grid drawing is the sum of the width and height of the minimum axis-parallel rectangle enclosing the drawing. Using an argument which is similar to that in sections 3 and 4 but is more lengthy, one can prove that every series-parallel graph G of $\Delta \leq 3$ has an optimal orthogonal grid drawing of size $\leq 3N/4$, and every biconnected series-parallel graph G of $\Delta \leq 3$ has an optimal orthogonal grid drawing of size $\leq 2N/3 + 1$, where $N = n(G) + bend(G)$. The proof is omitted in the paper.

It is left as a future work to obtain a linear algorithm for a larger class of planar graphs.

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