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journal or publication title	IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans
volume	32
number	6
page range	670-679
year	2002
URL	http://hdl.handle.net/10097/46852

doi: 10.1109/TSMCA.2002.807045

Group Decision With Inconsistent Knowledge

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***Abstract*—In this paper, dual exponential possibility distributions, namely, upper and lower exponential possibility distributions, are identified from the given data to characterize a decision-maker's knowledge. A decision group's knowledge can be represented by a set of such dual possibility distributions. The inherent diversity of knowledge among decision-makers is characterized by a conflict index. A conflict resolution model is proposed based on the conflict index, which integrates multiple possibility distributions identified into a new one to represent compromised knowledge of a decision group. As an application, a portfolio selection problem with multiple decision-makers is considered.**

***Index Terms*—Conflict analysis, fuzzy sets, group decision, portfolio selection, possibility theory.**

I. Introduction

A conflict is a situation in which two or more decision-makers are in dispute over some issue. Conflicts are, no doubt, one of the most typical attributes of human nature. Conflict analysis and resolution play an important role in business, economical, governmental, political and lawsuit disputes, labor-management negotiations, and military operations.

Generally speaking, the main research work in the filed of conflict resolution is based on game theory founded by von Neumann and Morgenstern [23]. For example, Howard [13] began the stability study of a state in metagames and hypergames. The research was further improved and extended to graph models by Fang *et al.* [5]. Moreover, many papers on arbitration and fair division have been published [1], [2], [18], [26].

The rough set-based method is a new one to analyze the structure of conflict [14], [15]. Three kinds of binary relations between participants, called conflict, neutrality, and alliance, are defined. Based on these relations, the participants in debate are divided into several coalitions and a strategy for conflict resolution, called intimidation, is investigated.

Different from the above research, this paper considers how to obtain more reasonable knowledge from conflicting knowledge provided by multiple decision-makers. This kind of conflict originates from the inherent diversity of knowledge and cognition on some issue under debate among multiple decision-makers in a decision group. The aim of this paper is to investigate the structure of such a conflict and to resolve it to obtain more reliable knowledge to be used for decision-making [12].

Manuscript received August 9, 2001; revised October 30, 2002. This work was supported by the Japanese Ministry of Education, Sports, Science and Technology (Research-in-aid Programs 12680444 and 14730083). This paper was recommended by Associate Editor K. W. Hipel.

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Digital Object Identifier 10.1109/TSMCA.2002.807045

This paper is organized as follows. In Section II, upper and lower possibility distributions for representing human knowledge, initially proposed in [20], are redefined by the inconsistency index of these dual possibility distributions. Upper possibility distributions can be regarded as optimistic viewpoints and lower distributions as pessimistic ones in the sense that upper possibility distributions always give higher possibility grades than the lower ones. As a result, multiple experts' knowledge can be characterized by a set of dual exponential possibility distributions. In Section III, a conflict index is defined to measure the difference of each pair of possibility distributions. Based on the defined conflict index, conflict situation in a decision group is investigated so that stable subgroups, key members, outliers, and a core of the decision group can be found out. A conflict resolution model is proposed where the possibility distributions are preprocessed so that possibility distributions with the higher conflict indices with the other possibility distributions are regarded as ouliers to be eliminated. A new possibility distribution can be obtained to represent more reliable knowledge. The researches on how to obtain fused information from multiple information sources, which are inconsistent with each other in nature, such as multiple sensors and a multi-expert pool, have been done from the viewpoint of information fusion based on probability and possibility theories, respectively. The probability networks, such as Bayes networks and Markov networks, are well-known probability methods for information fusion where information is presented as a conditional probability distribution and fusion procedure is based on Bayes formula [16]. The Dempster–Shafer theory of evidence (DS) is an important tool of information fusion to deal with nonadditional probability phenomena where fusion procedure is based on Dempster's rule of combination [17]. Dubois and Prade [3], [4] and Yager and Kelman proposed some information fusion models based on possibility theory [24], [25]. The approaches related to information fusion for decision analysis have been researched by Guo *et al.* [6]–[11]. This paper proposes a new method for information integration from the viewpoint of conflict analysis in its own right. In Section IV, as an application example, a portfolio selection problem with a group of experts is considered. In Section V, a numerical example is given to show the proposed method. Finally, some concluding remarks are included.

II. Identification of Dual Possibility Distributions From Given Data

Generally speaking, the vagueness and ambiguity of human understanding, the ignorance of cognition, and the diversity of evaluation are always contained in human knowledge. A possibility distribution (a kind of fuzzy membership function) is a kind of representation of knowledge where the center reflects the most possible case and the spread reflects the others with relatively low possibilities. The area of a possibility distribution can be regarded as a sort of measure of fuzziness. In a complex case, it is difficult to directly get a possibility distribution. However, we can easily obtain possibility grades of discrete data from some person reflecting his judgement. For example, in portfolio selection problems, experts can choose some typical patterns from the past security data and give them associated possibility grades to reflect their judgment on the situation of stock markets in the future. The higher the possibility grades of security data, the more similar to the future. Now let us consider how to obtain the possibility distribution from the given data.

A. The Concepts of Lower and Upper Possibility Distributions

Suppose that a data set $\{(\mathbf{x}_i, h_i)|i = 1, \ldots, m\}$ is given. Here $\mathbf{x}_i = [x_{i1}, \ldots, x_{in}]^t$ is an n-dimensional vector to characterize some specified event, h_i is an associated possibility grade given by some person to reflect his judgement on what the possibility grade of the ith sample is for this event, and m is the number of samples. The data set $(\mathbf{x}_i, h_i)$ $(i = 1, \ldots, m)$ can be approximated by a dual data sets $(\mathbf{x}_i, h_{li})$ and $(\mathbf{x}_i, h_{ui})$ $(i = 1, \ldots, m)$ with the condition $h_{li} \le h_i \le h_{ui}$. Assume that the values h_{li} and h_{ui} are from a class of functions $G(\mathbf{x}, \theta)$ with parameter vector θ. Let $G(\mathbf{x}_i, \theta_l)$ and $G(\mathbf{x}_i, \theta_u)$ correspond to h_{li} and h_{ui} $(i = 1, \ldots, m)$, respectively, and simply be denoted as $\pi_l(\mathbf{x}_i)$ and $\pi_u(\mathbf{x}_i)$. Given the data set $\{(\mathbf{x}_i, h_i)|i = 1, \ldots, m\}$, the objective of estimation is to obtain two optimal parameter vectors θ_u^* and θ_l^* from the parameter space to approximate $(\mathbf{x}_i, h_i)$ from the upper and lower directions according to some given measure. Moreover, it is needed that the relation $G(\mathbf{x}, \theta_l^*) \le G(\mathbf{x}, \theta_u^*)$ holds for any arbitrary n-dimensional vector $\mathbf{x}$.

Suppose that function $G(\mathbf{x}, \theta)$ is an exponential function $\exp\{-(\mathbf{x} - \mathbf{a})^t\mathbf{D}^{-1}(\mathbf{x} - \mathbf{a})\}$. Then the following formulas hold:

$$\pi_l(\mathbf{x}_i) = \exp\{-(\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_l^{-1}(\mathbf{x}_i - \mathbf{a})\} \quad i = 1, \ldots, m \tag{1}$$

$$\pi_u(\mathbf{x}_i) = \exp\{-(\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_u^{-1}(\mathbf{x}_i - \mathbf{a})\} \quad i = 1, \ldots, m \tag{2}$$

$$\pi_l(\mathbf{x}_i) \le h_i \le \pi_u(\mathbf{x}_i) \text{ and } \pi_l(\mathbf{x}) \le \pi_u(\mathbf{x}) \tag{3}$$

where $\mathbf{a} = [a_1, a_2, \ldots, a_n]^t$ is a center vector, and $\mathbf{D}_l$ and $\mathbf{D}_u$ are positive definite matrices, denoted as $\mathbf{D}_l > 0$ and $\mathbf{D}_u > 0$, respectively. It can be seen that in the above exponential functions vector $\mathbf{a}$ and matrices $\mathbf{D}_u$ and $\mathbf{D}_l$ are parameters to be solved. Different parameters $\mathbf{a}$, $\mathbf{D}_u$, and $\mathbf{D}_l$ lead to different values $\pi_l(\mathbf{x}_i)$ and $\pi_u(\mathbf{x}_i)$ which approximate the given possibility degree h_i to the different extent.

Definition 1: The inconsistency index of the two approximations (1) and (2), denoted as κ, is defined as follows:

$$\begin{aligned}\kappa &= -\ln \sqrt[m]{\prod_{i=1}^{m} \frac{\pi_l(\mathbf{x}_i)}{\pi_u(\mathbf{x}_i)}} = \frac{\left(\sum_{i=1}^{m} (\ln \pi_u(\mathbf{x}_i) - \ln \pi_l(\mathbf{x}_i))\right)}{m} \\ &= \frac{1}{m}\left(\sum_{i=1}^{m} (\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_l^{-1}(\mathbf{x}_i - \mathbf{a}) - \sum_{i=1}^{m} (\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_u^{-1}(\mathbf{x}_i - \mathbf{a})\right).\end{aligned} \tag{4}$$

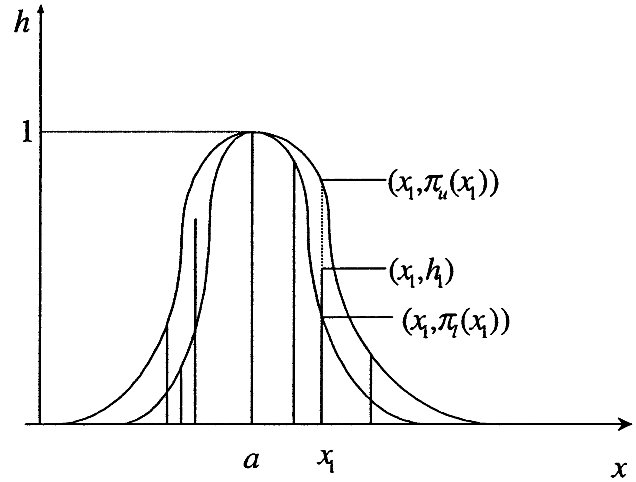

Fig. 1. Concepts of lower and upper possibility distributions. (The lower curve is the lower possibility distribution and the upper curve is the upper possibility distribution).

It is known from Definition 1 that the smaller the parameter κ is, the closer to h_i the values $\pi_l(\mathbf{x}_i)$ and $\pi_u(\mathbf{x}_i)$ are from the lower and upper directions, respectively.

Definition 2: Denote the optimal solutions of $\mathbf{a}$, $\mathbf{D}_u$, and $\mathbf{D}_l$ as $\mathbf{a}_*$, $\mathbf{D}_{*_u}$, and $\mathbf{D}_{*_l}$, respectively, which minimize κ with constraint (3). The following functions:

$$\pi_{*_l}(\mathbf{x}) = \exp\{-(\mathbf{x} - \mathbf{a}_*)^t\mathbf{D}_{*_l}^{-1}(\mathbf{x} - \mathbf{a}_*)\} \tag{5}$$

$$\pi_{*_u}(\mathbf{x}) = \exp\{-(\mathbf{x} - \mathbf{a}_*)^t\mathbf{D}_{*_u}^{-1}(\mathbf{x} - \mathbf{a}_*)\} \tag{6}$$

are called the lower and upper exponential possibility distributions of the possibility vector $\mathbf{X}$, respectively. For simplicity, afterwards we write $\pi_l(\mathbf{x})$ and $\pi_u(\mathbf{x})$ instead of $\pi_{*_l}(\mathbf{x})$ and $\pi_{*_u}(\mathbf{x})$, respectively, and denote the possibility vector $\mathbf{X}$ with exponential possibility distribution $\exp\{-(\mathbf{x}-\mathbf{a})^t\mathbf{D}^{-1}(\mathbf{x}-\mathbf{a})\}$ as $\mathbf{X} \sim (\mathbf{a}, \mathbf{D})_e$. The concept of the lower and upper possibility distributions is illustrated in Fig. 1. It can be seen from Fig. 1 that the given possibility degrees are completely included into the boundary of the lower and upper possibility distributions. The lower distribution can be regarded as a pessimistic viewpoint and the upper possibility distribution as an optimistic one in the sense that the lower possibility distribution always gives a smaller possibility grade than the upper one. The difference between the dual possibility distributions reflects the inconsistency of knowledge.

B. Identification of Upper and Lower Possibility Distributions

A model to identify the lower and upper possibility distributions is built to minimize the inconsistency index as follows:

$$\begin{aligned}\min_{\mathbf{a}, \mathbf{D}_u, \mathbf{D}_l} &\sum_{i=1}^{m} (\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_l^{-1}(\mathbf{x}_i - \mathbf{a}) - \sum_{i=1}^{m} (\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_u^{-1}(\mathbf{x}_i - \mathbf{a}) \\ \text{s.t. } &(\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_l^{-1}(\mathbf{x}_i - \mathbf{a}) \ge -\ln h_i, \ i = 1, \ldots, m \\ &(\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_u^{-1}(\mathbf{x}_i - \mathbf{a}) \le -\ln h_i, \ i = 1, \ldots, m \\ &\mathbf{D}_u - \mathbf{D}_l \ge 0, \ \mathbf{D}_l > 0\end{aligned} \tag{7}$$

where the objective function is from (4), $(\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_l^{-1}(\mathbf{x}_i - \mathbf{a}) \ge -\ln h_i$ is from $\pi_l(\mathbf{x}_i) \le h_i$, $(\mathbf{x}_i - \mathbf{a})^t\mathbf{D}_u^{-1}(\mathbf{x}_i - \mathbf{a}) \le -\ln h_i$ is from $\pi_u(\mathbf{x}_i) \ge h_i$, $\mathbf{D}_u - \mathbf{D}_l \ge 0$ is from $\pi_l(\mathbf{x}) \le \pi_u(\mathbf{x})$, and $\mathbf{D}_l > 0$ is due to $\mathbf{D}_l$ being a positive definite matrix.

It should be noted that the optimization problem (7) is equivalent to the integrated model proposed in [20] in form. However, they arise from very different considerations. The latter was an integration of two optimization problems to obtain lower and upper possibility distributions simultaneously. The former is to seek an optimal center vector $\mathbf{a}$ and optimal positive definite matrices $\mathbf{D}_u$ and $\mathbf{D}_l$ to minimize inconsistency index κ defined in formula (4). In the following, let us consider how to obtain center vector $\mathbf{a}$ and positive matrices $\mathbf{D}_l$ and $\mathbf{D}_u$.

It is straightforward that the lower and upper possibility distributions should have the same center vectors. Otherwise, the relation $\pi_u(\mathbf{x}) \geq \pi_l(\mathbf{x})$ cannot always hold. Because a vector $\mathbf{x}$ with the highest possibility grade should be closest to center vector $\mathbf{a}$ among all $\mathbf{x}_i$ ($i = 1, \ldots, m$), center vector $\mathbf{a}$ can be estimated as

$$\mathbf{a} = \mathbf{x}_{i^*} \tag{8}$$

where $\mathbf{x}_{i^*}$ denotes the vector whose grade is $h_{i^*} = \max_{k=1,\ldots,m} h_k$. The associated possibility grade of $\mathbf{x}_{i^*}$ is revised to be 1 because it becomes center vector. Taking the transformation $\mathbf{y} = \mathbf{x} - \mathbf{a}$, the problem (8) is changed into the following one:

$$\begin{aligned} \min_{\mathbf{D}_u, \mathbf{D}_l} \; & \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{D}_l^{-1} \mathbf{y}_i - \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{D}_u^{-1} \mathbf{y}_i \\ \text{s.t. } & \mathbf{y}_i^t \mathbf{D}_l^{-1} \mathbf{y}_i \geq -\ln h_i, \; i = 1, \ldots, m \\ & \mathbf{y}_i^t \mathbf{D}_u^{-1} \mathbf{y}_i \leq -\ln h_i, \; i = 1, \ldots, m \\ & \mathbf{D}_u - \mathbf{D}_l \geq 0 \\ & \mathbf{D}_l > 0. \end{aligned} \tag{9}$$

Formula (9) is a nonlinear optimization problem due to the last two constraints. To cope with this difficulty, we use principle component analysis (PCA) to rotate the given data $(\mathbf{y}_i, h_i)$ to obtain a positive definite matrix easily. Data $\mathbf{y}_i$ ($i = 1, \ldots, m$) can be transformed by a linear transformation matrix $\mathbf{T}$ whose columns are eigenvectors of matrix $\Sigma = [\sigma_{ij}]$, where σ_{ij} is defined as

$$\sigma_{ij} = \frac{\left\{ \sum_{k=1}^{m} (x_{ik} - a_i)(x_{jk} - a_j) h_k \right\}}{\sum_{k=1}^{m} h_k} \tag{10}$$

where x_{ik} is the ith element of the kth sample $\mathbf{x}_k$. Using $\mathbf{T}$, data $\mathbf{y}_i$ is transformed into $\mathbf{z}_i = \mathbf{T}^t \mathbf{y}_i$. Then formulas (1) and (2) can be rewritten as follows:

$$\pi_l(\mathbf{z}_i) = \exp\{-\mathbf{z}_i^t \mathbf{T}^t \mathbf{D}_l^{-1} \mathbf{T} \mathbf{z}_i\}, \; i = 1, \ldots, m \tag{11}$$

$$\pi_u(\mathbf{z}_i) = \exp\{-\mathbf{z}_i^t \mathbf{T}^t \mathbf{D}_u^{-1} \mathbf{T} \mathbf{z}_i\}, \; i = 1, \ldots, m. \tag{12}$$

Since $\mathbf{T}$ is obtained by PCA, $\mathbf{T}^t \mathbf{D}_u^{-1} \mathbf{T}$ and $\mathbf{T}^t \mathbf{D}_l^{-1} \mathbf{T}$ can be assumed to be diagonal matrices as follows:

$$\mathbf{C}_u = \mathbf{T}^t \mathbf{D}_u^{-1} \mathbf{T} = \begin{pmatrix} c_{u1} & & 0 \\ & \ddots & \\ 0 & & c_{un} \end{pmatrix} \tag{13}$$

$$\mathbf{C}_l = \mathbf{T}^t \mathbf{D}_l^{-1} \mathbf{T} = \begin{pmatrix} c_{l1} & & 0 \\ & \ddots & \\ 0 & & c_{\ln} \end{pmatrix}. \tag{14}$$

As a result, the model (9) can be rewritten as the following LP problem:

$$\begin{aligned} \min_{\mathbf{C}_l, \mathbf{C}_u} \; & \sum_{i=1}^{m} \mathbf{z}_i^t \mathbf{C}_l \mathbf{z}_i - \sum_{i=1}^{m} \mathbf{z}_i^t \mathbf{C}_u \mathbf{z}_i \\ \text{s.t. } & \mathbf{z}_i^t \mathbf{C}_l \mathbf{z}_i \geq -\ln h_i, \; i = 1, \ldots, m \\ & \mathbf{z}_i^t \mathbf{C}_u \mathbf{z}_i \leq -\ln h_i, \; i = 1, \ldots, m \\ & c_{lj} \geq c_{uj}, \; j = 1, \ldots, n \\ & c_{uj} \geq \varepsilon, \; j = 1, \ldots, n \end{aligned} \tag{15}$$

where the condition $c_{lj} \geq c_{uj} \geq \varepsilon > 0$ makes the matrix $\mathbf{D}_u - \mathbf{D}_l$ semi-positive definite and matrices $\mathbf{D}_u$ and $\mathbf{D}_l$ positive. Denote the optimal solutions of (15) as $\mathbf{C}_u^*$ and $\mathbf{C}_l^*$. Thus, we have

$$\begin{aligned} \mathbf{D}_u^* &= \mathbf{T} \mathbf{C}_u^{*^{-1}} \mathbf{T}^t \\ \mathbf{D}_l^* &= \mathbf{T} \mathbf{C}_l^{*^{-1}} \mathbf{T}^t. \end{aligned} \tag{16}$$

For simplicity, afterwards we write $\mathbf{D}_u$ and $\mathbf{D}_l$ instead of $\mathbf{D}_u^*$ and $\mathbf{D}_l^*$.

Given a possibility vector $\mathbf{X} \sim (\mathbf{a}, \mathbf{D})_e$, the possibility distribution of a possibilistic variable Y with $Y = \mathbf{r}^t \mathbf{X}$, denoted as $\pi_B(y)$, is defined by the extension principle [22] as follows:

$$\pi_B(y) = \max_{\{\mathbf{x} | y = \mathbf{r}^t \mathbf{x}\}} \exp\{-(\mathbf{x} - \mathbf{a})^t \mathbf{D}^{-1} (\mathbf{x} - \mathbf{a})\} \tag{17}$$

where $\mathbf{r}$ is an n-dimensional vector. Solving the optimization problem (17), the possibility distribution of y can be obtained as

$$\pi_B(y) = \exp\{-(y - \mathbf{r}^t \mathbf{a})^2 (\mathbf{r}^t \mathbf{D} \mathbf{r})^{-1}\} \tag{18}$$

where $\mathbf{r}^t \mathbf{a}$ is the center value and $\mathbf{r}^t \mathbf{D} \mathbf{r}$ is the spread value of Y. $Y \sim (\mathbf{r}^t \mathbf{a}, \mathbf{r}^t \mathbf{D} \mathbf{r})_e$ is called the one-dimensional (1-D) realization of $\mathbf{X} \sim (\mathbf{a}, \mathbf{D})_e$.

Theorem 1: Assuming that the given data $\mathbf{y}_i, h_i$, $i = 1, \ldots, m$, are obtained from an exponential possibility distribution $(\mathbf{0}, \mathbf{A}^\nabla)_e$, the inconsistency index of upper and lower possibilistic distributions is 0.

Proof: The given data $\mathbf{y}_i, h_i, i = 1, \ldots, m$, are obtained from an exponential possibility distribution $(\mathbf{0}, \mathbf{A}^\nabla)_e$ which means that the following equations hold:

$$\exp\{-\mathbf{y}_i^t \mathbf{A}^{\nabla^{-1}} \mathbf{y}_i\} = h_i, \; i = 1, \ldots, m. \tag{19}$$

Let us consider the following optimization problem for finding out the upper possibility matrix $\mathbf{A}_u$ and the lower possibility matrix $\mathbf{A}_l$ from the above given data:

$$\begin{aligned} \min_{\mathbf{A}_l, \mathbf{A}_u} \; & \kappa = \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{A}_l^{-1} \mathbf{y}_i - \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{A}_u^{-1} \mathbf{y}_i \\ \text{subject to } & \mathbf{y}_i^t \mathbf{A}_i^t \mathbf{y}_i \geq -\ln h_i, \; i = 1, \ldots, m \\ & \mathbf{y}_i^t \mathbf{A}_u^{-1} \mathbf{y}_i \leq -\ln h_i, \; i = 1, \ldots, m. \end{aligned} \tag{20}$$

The optimization problem (20) can be separated into the following two optimization problems:

$$\begin{aligned} \max_{\mathbf{A}_u} \; & J_1(\mathbf{A}_u) = \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{A}_u^{-1} \mathbf{y}_i \\ \text{s.t. } & \mathbf{y}_i^t \mathbf{A}_u^{-1} \mathbf{y}_i \leq -\ln h_i, \; i = 1, \ldots, m \end{aligned} \tag{21}$$

and

$$\min_{\mathbf{A}_l} \ J_2(\mathbf{A}_l) = \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{A}_l^{-1} \mathbf{y}_i$$
$$\text{s.t. } \mathbf{y}_i^t \mathbf{A}_l^{-1} \mathbf{y}_i \geq -\ln h_i, \ i = 1, \ldots, m. \tag{22}$$

Since any data $\mathbf{y}_i, h_i, i = 1, \ldots, m$, is obtained from the exponential possibility distribution $(0, \mathbf{A}^\nabla)_e$, it should satisfy (19). Therefore, $\mathbf{A}^\nabla$ is an admissible solution of (21) and (22). Assume that there is another matrix $\mathbf{A}^*$ such as $J_l(\mathbf{A}^*) > J_l(\mathbf{A}^\nabla)$ in (21). Then, for some i, we have

$$\mathbf{y}_i^t \mathbf{A}^{*^{-1}} \mathbf{y}_t > \mathbf{y}_t^t \mathbf{A}^{\nabla^{-1}} \mathbf{y}_t = -\ln h_i \tag{23}$$

which shows that $\mathbf{A}^*$ is not admissible. Thus, $\mathbf{A}^\nabla$ is the optimal solution of (21). In the same way, we can prove that the optimal solution of (22) is also $\mathbf{A}^\nabla$. Therefore, both $\mathbf{A}_u$ and $\mathbf{A}_l$ are $\mathbf{A}^\nabla$ so that the inconsistency index $\kappa = 0$. □

This theorem means that inconsistency index κ can reflect how the given possibility grades can be approximated by the two obtained exponential functions. The smaller the index κ is, the better the given possibility degrees are characterized by the identified dual possibility distributions.

III. Conflict Analysis Among Multiple Decision-Makers

The data set from s decision-makers is denoted as

$$\{\langle(\mathbf{x}_1, h_1^1), (\mathbf{x}_2, h_2^1), \ldots, (\mathbf{x}_m, h_m^1)\rangle$$
$$, \ldots, \langle(\mathbf{x}_1, h_1^s), (\mathbf{x}_2, h_2^s), \ldots, (\mathbf{x}_m, h_m^s)\rangle\}$$

where $\mathbf{x}_k = [x_{1k}, \ldots, x_{nk}]^t$ is the kth sample, h_k^i, $i = 1, \ldots, s; k = 1, \ldots, m$, is any given possibility grade by the ith person (the superscript of h_k^i) to reflect his judgement on what the possibility grade of the kth sample (the subscript of h_k^i) is for some specified event, and m is the number of samples. For example, $\langle(\mathbf{x}_1, h_1^1), (\mathbf{x}_2, h_2^1), \ldots, (\mathbf{x}_m, h_m^1)\rangle$ is used to represent the judgment from expert 1. Using the above-mentioned identification method, s dual possibility distributions can be obtained to reflect the inherent diversity in human thought of s persons. The set formed by s dual possibility distributions, denoted as $U = \{\langle \mathbf{X}_i \sim (\mathbf{a}_i, \mathbf{D}_{ui})_e, (\mathbf{a}_i, \mathbf{D}_{li})_e\rangle | i = 1, \ldots, s\}$, is called an information block where $\langle \mathbf{X}_i \sim (\mathbf{a}_i, \mathbf{D}_{ui})_e, (\mathbf{a}_i, \mathbf{D}_{li})_e\rangle$ is obtained from data set $\{(\mathbf{x}_1, h_1^i), (\mathbf{x}_2, h_2^i), \ldots, (\mathbf{x}_m, h_m^i)\}$.

A. Conflict Index of Exponential Possibility Distributions

Definition 3: Given two possibility variables $X_i \sim (a_i, r_i)_e$ and $X_j \sim (a_j, r_j)_e$, the conflict index of X_i and X_j, denoted as $\tau(X_i, X_j)$, is defined as

$$\tau(X_i, X_j) = -\ln\left(\max_x \pi_i(x) \cdot \pi_j(x)\right). \tag{24}$$

It can be understood the conflict index of the two exponential possibility variables is defined based on their possibility measure where product operator takes the place of min operator with considering exponential functions. The higher the possibility measure, the lower their conflict index.

Theorem 2: Given two possibility variables $X_i \sim (a_i, r_i)_e$ and $X_j \sim (a_j, r_j)_e$, their conflict index $\tau(X_i, X_j)$ is as follows:

$$\tau(X_i, X_j) = \frac{(a_i - a_j)^2}{(r_i + r_j)}. \tag{25}$$

Proof: The problem $\max_x \pi_i \cdot \pi_j$ in Definition 3 leads to the following optimization problem:

$$\min_x \ f(x) = \frac{(x - a_i)^2}{r_i} + \frac{(x - a_j)^2}{r_j} \tag{26}$$

The optimal solution of (26) is

$$x^* = \frac{(a_i r_j + a_j r_i)}{(r_i + r_j)}. \tag{27}$$

Substituting (27) into (24) leads to (25). □

It is seen from (25) that the closer the centers and the wider the spreads of the two possibility distributions, the lower the conflict index. If they have the same centers, the conflict index will be 0 because of the inclusion relation between them. Such a conclusion is very close to human intuition to reflect the conflict situation of uncertainty knowledge.

Definition 4: Given two n-dimensional possibility vectors $\mathbf{X}_i \sim (\mathbf{a}_i, \mathbf{D}_i)_e$ and $\mathbf{X}_j \sim (\mathbf{a}_j, \mathbf{D}_j)_e$, their conflict index, denoted as $\tau(\mathbf{X}_i, \mathbf{X}_j)$, is defined as follows:

$$\tau(\mathbf{X}_i, \mathbf{X}_j) = \max_{\mathbf{r}_{ij}} \tau(\mathbf{r}_{ij}^t \mathbf{X}_i, \mathbf{r}_{ij}^t \mathbf{X}_j) \tag{28}$$

where $\mathbf{r}_{ij}$ is an n-dimensional vector. It is known from this definition that the conflict index of two high-dimensional possibility vectors is defined by the maximum conflict index of their 1-D realization, which can be regarded as the most pessimistic viewpoint for measuring their inconsistency degree .

Theorem 3: Given two n-dimensional possibility vectors $\mathbf{X}_i \sim (\mathbf{a}_i, \mathbf{D}_i)_e$ and $\mathbf{X}_j \sim (\mathbf{a}_j, \mathbf{D}_j)_e$, the conflict index $\tau(\mathbf{X}_i, \mathbf{X}_j)$ is as follows:

$$\tau(\mathbf{X}_i, \mathbf{X}_j) = \lambda_{\max}(\mathbf{Q}) \tag{29}$$

where $\lambda_{\max}(\mathbf{Q})$ is the maximum eigenvalue of the following matrix $\mathbf{Q}$:

$$\mathbf{Q} = (\mathbf{R}^{-1})^t(\mathbf{a}_i - \mathbf{a}_j)(\mathbf{a}_i - \mathbf{a}_j)^t \mathbf{R}^{-1} \tag{30}$$
$$\mathbf{D}_i + \mathbf{D}_j = \mathbf{R}^t \mathbf{R}. \tag{31}$$

Proof: Considering formula (18) and Theorem 2, $\tau(\mathbf{r}_{ij}^t \mathbf{X}_i, \mathbf{r}_{ij}^t \mathbf{X}_j)$ is obtained as

$$\tau(\mathbf{r}_{ij}^t \mathbf{X}_i, \mathbf{r}_{ij}^t \mathbf{X}_j) = \frac{\mathbf{r}_{ij}^t(\mathbf{a}_i - \mathbf{a}_j)(\mathbf{a}_i - \mathbf{a}_j)^t \mathbf{r}_{ij}}{\mathbf{r}_{ij}^t(\mathbf{D}_i + \mathbf{D}_j)\mathbf{r}_{ij}} \tag{32}$$

so that $\tau(\mathbf{X}_i, \mathbf{X}_j)$ is as follows:

$$\tau(\mathbf{X}_i, \mathbf{X}_j) = \max_{\mathbf{r}_{ij}} \frac{\mathbf{r}_{ij}^t(\mathbf{a}_i - \mathbf{a}_j)(\mathbf{a}_i - \mathbf{a}_j)^t \mathbf{r}_{ij}}{\mathbf{r}_{ij}^t(\mathbf{D}_i + \mathbf{D}_j)\mathbf{r}_{ij}}. \tag{33}$$

Using (30) and (31) and taking the transformation $\mathbf{z}_{ij} = \mathbf{R}\mathbf{r}_{ij}$, (33) is transformed into

$$\tau(\mathbf{X}_i, \mathbf{X}_j) = \max_{\mathbf{r}_{ij}} \frac{\mathbf{z}_{ij}^t \mathbf{Q} \mathbf{z}_{ij}}{\mathbf{z}_{ij}^t \mathbf{z}_{ij}}. \tag{34}$$

According to the well-known Rayleigh–Ritz theorem, the optimal value of (34) is the maximum eigenvalue of $\mathbf{Q}$, namely, $\lambda_{\max}(\mathbf{Q})$. □

Lemma 1: It holds that

$$(\mathbf{D}_A^{-1} + \mathbf{D}_B^{-1})^{-1} = \mathbf{D}_A - \mathbf{D}_A(\mathbf{D}_A + \mathbf{D}_B)^{-1}\mathbf{D}_A. \tag{35}$$

Proof:

$$\begin{aligned}&(\mathbf{D}_A+\mathbf{D}_B)(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A=\mathbf{D}_A\\ &\Leftrightarrow \mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A+\mathbf{D}_B(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A=\mathbf{D}_A\\ &\Leftrightarrow (\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A+\mathbf{D}_B^{-1}\mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A\\ &=\mathbf{D}_B^{-1}\mathbf{D}_A\\ &\Leftrightarrow \mathbf{I}-(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A+\mathbf{D}_B^{-1}\mathbf{D}_A\\ &\quad -\mathbf{D}_B^{-1}\mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A=\mathbf{I}\\ &\Leftrightarrow \mathbf{D}_A^{-1}\mathbf{D}_A-(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A+\mathbf{D}_B^{-1}\mathbf{D}_A\\ &\quad -\mathbf{D}_B^{-1}\mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A=\mathbf{I}\\ &\Leftrightarrow (\mathbf{D}_A^{-1}+\mathbf{D}_B^{-1})(\mathbf{D}_A-\mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A)=\mathbf{I}. \end{aligned} \tag{36}$$

It means that

$$(\mathbf{D}_A^{-1}+\mathbf{D}_B^{-1})^{-1}=\mathbf{D}_A-\mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A \tag{37}$$

which proves this lemma. □

Lemma 2: It holds that

$$\begin{aligned}&(\mathbf{a}-\mathbf{b})^t(\mathbf{D}_A^{-1}+\mathbf{D}_B^{-1})^{-1}(\mathbf{a}-\mathbf{b})=-(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})^t\\ &\times(\mathbf{D}_A+\mathbf{D}_B)^{-1}(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})+\mathbf{a}^t\mathbf{D}_A\mathbf{a}+\mathbf{b}^t\mathbf{D}_B\mathbf{b}.\end{aligned} \tag{38}$$

Proof: Using Lemma 1, the following equalities hold:

$$\begin{aligned}&(\mathbf{a}-\mathbf{b})^t(\mathbf{D}_A^{-1}+\mathbf{D}_B^{-1})^{-1}(\mathbf{a}-\mathbf{b})\\ &=(\mathbf{a}-\mathbf{b})^t(\mathbf{D}_A-\mathbf{D}_A(\mathbf{D}_A+\mathbf{D}_B)^{-1}\mathbf{D}_A)(\mathbf{a}-\mathbf{b})\\ &=(\mathbf{a}-\mathbf{b})^t\mathbf{D}_A(\mathbf{a}-\mathbf{b})-[\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b}-(\mathbf{D}_A+\mathbf{D}_B)\mathbf{b}]^t\\ &\quad\times(\mathbf{D}_A+\mathbf{D}_B)^{-1}[\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b}-(\mathbf{D}_A+\mathbf{D}_B)\mathbf{b}]\\ &=(\mathbf{a}-\mathbf{b})^t\mathbf{D}_A(\mathbf{a}-\mathbf{b})-(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})^t(\mathbf{D}_A+\mathbf{D}_B)^{-1}\\ &\quad\times(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})+2\mathbf{b}^t(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})-\mathbf{b}^t(\mathbf{D}_A+\mathbf{D}_B)\mathbf{b}\\ &=-(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})^t(\mathbf{D}_A+\mathbf{D}_B)^{-1}(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})\\ &\quad+\mathbf{a}^t\mathbf{D}_A\mathbf{a}-2\mathbf{b}^t\mathbf{D}_A\mathbf{a}+\mathbf{b}^t\mathbf{D}_A\mathbf{b}\\ &\quad+2\mathbf{b}^t\mathbf{D}_A\mathbf{a}+2\mathbf{b}^t\mathbf{D}_B\mathbf{b}-\mathbf{b}^t\mathbf{D}_A\mathbf{b}-\mathbf{b}^t\mathbf{D}_B\mathbf{b}\\ &=-(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})^t(\mathbf{D}_A+\mathbf{D}_B)^{-1}(\mathbf{D}_A\mathbf{a}+\mathbf{D}_B\mathbf{b})\\ &\quad+\mathbf{a}^t\mathbf{D}_A\mathbf{a}+\mathbf{b}^t\mathbf{D}_B\mathbf{b}.\end{aligned} \tag{39}$$

It proves this lemma.

Lemma 3: It holds that

$$\lambda_{\max}(\mathbf{c}\mathbf{c}^t)=\mathbf{c}^t\mathbf{c} \tag{40}$$

where $\mathbf{c}$ is an arbitrary vector, and $\lambda_{\max}(\mathbf{c}\mathbf{c}^t)$ is the maximum eigenvalue of $\mathbf{c}\mathbf{c}^t$.

Proof: From the following relation:

$$(\mathbf{c}\mathbf{c}^t)\mathbf{c}=\mathbf{c}(\mathbf{c}^t\mathbf{c})=(\mathbf{c}^t\mathbf{c})\mathbf{c} \tag{41}$$

it can be seen that $\mathbf{c}^t\mathbf{c}=\sum_{i=1,\ldots,n}c_i^2$ is one egienvalue of the matrix $\mathbf{c}\mathbf{c}^t$ and the corresponding eigenvector is $\mathbf{c}$. Taking $\mathbf{c}=[c_1,\ldots,c_n]^t$, it holds that

$$\mathbf{c}\mathbf{c}^t=\begin{bmatrix}c_1^2 & c_1c_2 & \cdots & c_1c_n\\ c_1c_2 & c_2^2 & \cdots & c_2c_n\\ \vdots & \vdots & \ddots & \vdots\\ c_nc_1 & c_nc_2 & \cdots & c_n^2\end{bmatrix}. \tag{42}$$

Using the corollary of Gersgorin (see the Appendix), all eigenvalues of $\mathbf{c}\mathbf{c}^t$ lie in the region

$$\bigcup_{i=1,\ldots,n}\left\{z\in C: |z-c_i^2|\le\frac{1}{|c_i|}\sum_{j=1,\ldots,n,j\ne i}|c_j||c_ic_j| = \sum_{j=1,\ldots,n,j\ne i}c_j^2\right\} \tag{43}$$

which means that

$$\lambda(\mathbf{c}\mathbf{c}^t)\le\sum_{j=1,\ldots,n}c_j^2 \tag{44}$$

so that $\mathbf{c}^t\mathbf{c}=\sum_{j=1,\ldots,n}c_j^2$ is the maximum eigenvalue of $\mathbf{c}\mathbf{c}^t$. □

Theorem 4: It holds that

$$\begin{aligned}\tau(\mathbf{X}_i,\mathbf{X}_j)=&\lambda_{\max}(\mathbf{Q})\\ =&-(\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{D}_j^{-1}\mathbf{a}_j)^t(\mathbf{D}_i^{-1}+\mathbf{D}_j^{-1})^{-1}\\ &\cdot(\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{D}_j^{-1}\mathbf{a}_j)+\mathbf{a}_i^t\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{a}_j^t\mathbf{D}_j^{-1}\mathbf{a}_j.\end{aligned} \tag{45}$$

Proof: Let $\mathbf{c}=(\mathbf{R}^{-1})^t(\mathbf{a}_i-\mathbf{a}_j)$ so that $\mathbf{Q}=\mathbf{c}\mathbf{c}^t$. Using Lemmas 1 and 2 leads to

$$\begin{aligned}\lambda_{\max}(\mathbf{Q})=&\mathbf{c}^t\mathbf{c}=(\mathbf{a}_i-\mathbf{a}_j)^t\mathbf{R}^{-1}(\mathbf{R}^{-1})^t(\mathbf{a}_i-\mathbf{a}_j)\\ =&(\mathbf{a}_i-\mathbf{a}_j)^t(\mathbf{D}_i+\mathbf{D}_j)^{-1}(\mathbf{a}_i-\mathbf{a}_j)\\ =&-(\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{D}_j^{-1}\mathbf{a}_j)^t(\mathbf{D}_i^{-1}+\mathbf{D}_j^{-1})^{-1}\\ &\cdot(\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{D}_j^{-1}\mathbf{a}_j)+\mathbf{a}_i^t\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{a}_j^t\mathbf{D}_j^{-1}\mathbf{a}_j.\end{aligned} \tag{46}$$

It proves this theorem. □

If center vectors of two possibility distributions are the same, that is, $\mathbf{a}_i=\mathbf{a}_j$, the value of (32) will be always 0 for any vector $\mathbf{r}_{ij}$. In this case, their conflict index is 0. It should be noted that in [22] the possibility measure of two n-dimensional possibility variables $\mathbf{X}_i\sim(\mathbf{a}_i,\mathbf{D}_i)_e$ and $\mathbf{X}_j\sim(\mathbf{a}_j,\mathbf{D}_j)_e$ is defined as $\max_{\mathbf{x}}\pi_i(\mathbf{x})\cdot\pi_j(\mathbf{x})$. Defining $\varsigma(\mathbf{X}_i,\mathbf{X}_j)=-\ln(\max_{\mathbf{x}}\pi_i(\mathbf{x})\cdot\pi_j(\mathbf{x}))$ as a new index for measuring the difference of $\mathbf{X}_i$ and $\mathbf{X}_j$, the following equation:

$$\begin{aligned}\varsigma(\mathbf{X}_i,\mathbf{X}_j)=&-(\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{D}_j^{-1}\mathbf{a}_j)^t(\mathbf{D}_i^{-1}+\mathbf{D}_j^{-1})^{-1}\\ &\cdot(\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{D}_j^{-1}\mathbf{a}_j)+\mathbf{a}_i^t\mathbf{D}_i^{-1}\mathbf{a}_i+\mathbf{a}_j^t\mathbf{D}_j^{-1}\mathbf{a}_j\end{aligned}$$

can also be obtained by directly solving optimization problem $\max_{\mathbf{x}}\pi_i(\mathbf{x})\cdot\pi_j(\mathbf{x})$, which is the same as the result in Theorem 4. It means that $\varsigma(\mathbf{X}_i,\mathbf{X}_j)=\tau(\mathbf{r}_{ij}^{*t}\mathbf{X}_i,\mathbf{r}_{ij}^{*t}\mathbf{X}_j)=\tau(\mathbf{X}_i,\mathbf{X}_j)=\varsigma(\mathbf{r}_{ij}^{*t}\mathbf{X}_i,\mathbf{r}_{ij}^{*t}\mathbf{X}_j)$ where $\mathbf{r}_{ij}^*=\arg\max_{\mathbf{r}_{ij}}\tau(\mathbf{r}_{ij}^t\mathbf{X}_i,\mathbf{r}_{ij}^t\mathbf{X}_j)$.

B. Conflict Resolution Model Based on Conflict Index

In information block $U=\{\mathbf{X}_i\sim\langle(\mathbf{a}_i,\mathbf{D}_{ui})_e,(\mathbf{a}_i,\mathbf{D}_{li})_e\rangle|i=1,\ldots,s\}$, we can calculate the conflict indices of $\mathbf{X}_i$ and $\mathbf{X}_j$ based on the upper and lower possibility distributions, which are called the upper and lower conflict indices of $\mathbf{X}_i$ and $\mathbf{X}_j$ and denoted as $\tau^u(\mathbf{X}_i,\mathbf{X}_j)$ and $\tau^l(\mathbf{X}_i,\mathbf{X}_j)$, respectively. The conflict indices of U based on upper and lower possibility distributions, denoted as τ^u and τ^l, respectively, are as follows:

$$\tau^l = \max_{i,j\in\{1,\ldots,s\}} \tau^l(\mathbf{X}_i, \mathbf{X}_j) \tag{47}$$

$$\tau^u = \max_{i,j\in\{1,\ldots,s\}} \tau^u(\mathbf{X}_i, \mathbf{X}_j). \tag{48}$$

Definition 5: Let δ_u and δ_l be the predetermined thresholds for τ^u and τ^l, respectively. An information block U is optimistically stable if and only if τ^u is not larger than δ_u. An information block U is pessimistically stable if and only if τ^l is not larger than δ_l. An information block U is stable if and only if it is both optimistically and pessimistically stable.

If an information block is unstable, it should be preprocessed to make itself stable. The basic idea for preprocessing is to delete the outliers from the given information block. In other words, a possibility distribution, which has the higher conflict index with the others, can be regarded as an outlier to be deleted.

The following algorithm is used to obtain the stable block from the given information block U with maximum cardinality, which is called as the efficient block of U, denoted as U_e.

Step 1. Represent s upper and s lower conflict indices by two $s \times s$ matrices $Q^u = [q^u_{ij}]$ and $Q^l = [q^l_{ij}]$ where $q^u_{ij} = \tau^u(\mathbf{X}_i, \mathbf{X}_j)$ and $q^l_{ij} = \tau^l(\mathbf{X}_i, \mathbf{X}_j)$ $(i = 1,\ldots,s; j = 1,\ldots,s)$.
Step 2. Transform elements q^u_{ij} (q^l_{ij}) into 0 if $q^u_{ij} > \delta_u$ $(q^l_{ij} > \delta_l)$ else into 1. Obtain a binary matrix Q^* so that $Q^* = [q_{ij}]_{s\times s} = [q^u_{ij} \cdot q^l_{ij}]$.
Step 3. Denote the index of the row of Q^* with the biggest number of the element 1 as i^*. If the numbers of 1 in the i^*th row is larger than one, the efficient block is obtained as the set $U_e = \{\mathbf{X}_j \sim \langle(\mathbf{a}_j, \mathbf{D}_{uj})_e, (\mathbf{a}_j, \mathbf{D}_{lj})_e\rangle | q_{i^*j} = 1, j = 1,\ldots,s\}$. Otherwise there is no efficient block, and the information block U is then called a conflict information block.

Sometimes more than one efficient blocks, for example, r efficient blocks can be obtained, which are denoted as $U_{e1},\ldots,U_{er}$, respectively.

Definition 6: Supporting that $U_{e1},\ldots,U_{er}$ are the r different efficient blocks obtained from a given information block U, the set $U_c = U_{e1} \cap U_{e2} \cap \cdots \cap U_{er}$ is called the core of U.

It is also clear that U is a conflict information block if and only if $\forall i, j \in \{1,\ldots,s\}, i \neq j$, $r^u_{ij} > \delta_u$ or $r^l_{ij} > \delta_l$. Otherwise, there must be an efficient information block. If U is a conflict information block, then any subset of U is also a conflict information block.

Now, let us consider how to obtain a compromised possibility distribution from the obtained efficient block. Without loss of generality, suppose that there is only one efficient information block $U_e = \{\mathbf{X}_i \sim \langle(\mathbf{a}_i, \mathbf{D}_{ui})_e, (\mathbf{a}_i, \mathbf{D}_{li})_e\rangle | i = 1,\ldots,p\}$ obtained from U where p is the cardinality of U_e. The inconsistency degree of $\mathbf{X}_i \sim \langle(\mathbf{a}_i, \mathbf{D}_{ui})_e, (\mathbf{a}_i, \mathbf{D}_{li})_e\rangle$ in U_e, denoted as $\mathrm{Inc}(\mathbf{X}_i)$, can be calculated as follows:

$$\mathrm{Inc}(\mathbf{X}_i) = \sum_{j\in\{1,\ldots,p\};i\neq j} (\tau^u(\mathbf{X}_i, \mathbf{X}_j) + \tau^l(\mathbf{X}_i, \mathbf{X}_j)),\ i=1,\ldots,p. \tag{49}$$

According to values of $\mathrm{Inc}(\mathbf{X}_i)$, data set

$$\{\langle(\mathbf{x}_1, h^1_1), (\mathbf{x}_2, h^1_2),\ldots,(\mathbf{x}_m, h^1_m)\rangle ,\ldots,\langle(\mathbf{x}_1, h^p_1), (\mathbf{x}_2, h^p_2),\ldots,(\mathbf{x}_m, h^p_m)\rangle\}$$

can be reordered as

$$\{\langle(\mathbf{x}_1, h^{1^*}_1), (\mathbf{x}_2, h^{1^*}_2),\ldots,(\mathbf{x}_m, h^{1^*}_m)\rangle ,\ldots,\langle(\mathbf{x}_1, h^{k^*}_1), (\mathbf{x}_2, h^{k^*}_2),\ldots,(\mathbf{x}_m, h^{k^*}_m)\rangle ,\ldots,\langle(\mathbf{x}_1, h^{p^*}_1), (\mathbf{x}_2, h^{p^*}_2),\ldots,(\mathbf{x}_m, h^{p^*}_m)\rangle\}$$

so that $h^{k^*}_i$ $(i = 1,\ldots,m)$ corresponds to the k^*th smallest $\mathrm{Inc}(\mathbf{X}_i)$. The decision-makers with the smallest $\mathrm{Inc}()$ are regarded as the key members of this decision group. The compromised possibility grade of the ith sample $\mathbf{x}_i$ is as follows:

$$h^c_i = \sum_{k^*=1,\ldots,p} w_{k^*} h^{k^*}_i,\ i = 1,\ldots,m \tag{50}$$

where the weight $0 \leq w_{k^*} \leq 1$ is determined by ordered weighted aggregating (OWA) operators [25]. Using the identification method introduced in Section II, the new dual possibility distributions, denoted as $\mathbf{X} \sim \langle(\mathbf{a}_c, \mathbf{D}_{cu})_e, (\mathbf{a}_c, \mathbf{D}_{cl})_e\rangle$, can be obtained from $\mathbf{x}_i, h^c_i, i = 1,\ldots,m$.

IV. Portfolio Selection With a Group of Experts

Portfolio selection problems based on possibility theory have been studied in [10], [11], [20], and [21]. Different from probability models, such as Markowitz's model, by which optimal portfolios are selected based on the statistic characteristics of the past security data, possibility models select optimal ones based on the past security data plus experts' judgment on those data, where possibility grades are used to characterize decision-makers' knowledge. Now let us consider portfolio selection problems with multiple experts. The data set is given as

$$\{\langle(\mathbf{x}_1, h^1_1), (\mathbf{x}_2, h^1_2),\ldots,(\mathbf{x}_m, h^1_m)\rangle ,\ldots,\langle(\mathbf{x}_1, h^s_1), (\mathbf{x}_2, h^s_2),\ldots,(\mathbf{x}_m, h^s_m)\rangle\}$$

where $\mathbf{x}_k = [x_{1k},\ldots,x_{nk}]^t$ is a vector of returns of n securities S_j $(j = 1,\ldots,n)$ at the kth period, h^i_k is an associated possibility grade given by the ith expert to reflect his judgment on the possibility degree that such returns of n securities will appear in the future, and s is the number of experts. Using the method introduced in Sections II and III, a compromised possibility distribution $\mathbf{X} \sim \langle(\mathbf{a}_i, \mathbf{D}_{cu})_e, (\mathbf{a}_i, \mathbf{D}_{cl})_e\rangle$ is obtained to characterize integrated knowledge on stock market prediction from multiple experts.

The portfolio return can be written as

$$Z = \mathbf{r}^t\mathbf{x} = \sum_{j=1,\ldots,n} r_j x_j \tag{51}$$

where r_j denotes the proportion of the total investment funds devoted to security S_j and x_j is its return.

Because return vector $\mathbf{x}$ is governed by the dual possibility distribution $\mathbf{X} \sim \langle(\mathbf{a}_c, \mathbf{D}_{cu})_e, (\mathbf{a}_c, \mathbf{D}_{cl})_e\rangle$, using formula (18) the upper and lower possibility distributions of a possibility portfolio return Z, denoted as $\pi_{Z_u}(z)$ and $\pi_{Z_l}(z)$, respectively, are obtained as follows:

$$\pi_{Z_u}(z) = \exp\{-(z - \mathbf{r}^t\mathbf{a}_c)^2(\mathbf{r}^t\mathbf{D}_{cu}\mathbf{r})^{-1}\} \tag{52}$$
$$\pi_{Z_l}(z) = \exp\{-(z - \mathbf{r}^t\mathbf{a}_c)^2(\mathbf{r}^t\mathbf{D}_{cl}\mathbf{r})^{-1}\} \tag{53}$$

where $\mathbf{r}^t\mathbf{a}_c$ is the center value and $\mathbf{r}^t\mathbf{D}_{cu}\mathbf{r}$ and $\mathbf{r}^t\mathbf{D}_{cl}\mathbf{r}$ are the spreads of a possibility portfolio return Z based on the compromised upper and lower possibility distributions, respectively.

The following two quadratic programming problems are given to obtain optimal portfolios, which minimize the spreads of possibility portfolio returns because the spreads of possibility portfolio returns are regarded as the measure of risk:

$$\begin{aligned} \min_{\mathbf{r}} \;& \mathbf{r}^t\mathbf{D}_{cu}\mathbf{r} \\ \text{s.t } & \mathbf{r}^t\mathbf{a}_c = c \\ & \sum_{i=1}^{n} r_i = 1,\; r_i \geq 0 \end{aligned} \tag{54}$$

$$\begin{aligned} \min_{\mathbf{r}} \;& \mathbf{r}^t\mathbf{D}_{cl}\mathbf{r} \\ \text{s.t. } & \mathbf{r}^t\mathbf{a}_c = c \\ & \sum_{i=1}^{n} r_i = 1,\; r_i \geq 0 \end{aligned} \tag{55}$$

where c is the expected center value of a possibility portfolio return which should comply with the constraint $\min_{i=1,\ldots,n} a_i \leq c \leq \max_{i=1,\ldots,n} a_i$ to guarantee the existence of solutions in (54) and (55). Because $\mathbf{D}_{cu}$ and $\mathbf{D}_{cl}$ are positive definite matrices, (54) and (55) are convex programming problems.

Consider the following optimization problem where short sale is allowed for investment:

$$\begin{aligned} \min_{\mathbf{r}} \;& \mathbf{r}^t\mathbf{D}\mathbf{r} \\ \text{s.t. } & \mathbf{r}^t\mathbf{a} = c \\ & \sum_{i=1}^{n} r_i = 1 \end{aligned} \tag{56}$$

where $\mathbf{D}$ is either $\mathbf{D}_{cu}$ or $\mathbf{D}_{cl}$. The optimal solution $\mathbf{r}^*$ can be obtained by minimizing the following Lagrangian function:

$$L(\mathbf{r}, \lambda_1, \lambda_2) = \mathbf{r}^t\mathbf{D}\mathbf{r} + \lambda_1(c - \mathbf{r}^t\mathbf{a}) + \lambda_2(1 - \mathbf{r}^t\mathbf{1}) \tag{57}$$

where $\mathbf{1} = [1, \ldots, 1]^t$. $L(\mathbf{r}, \lambda_1, \lambda_2)$ is a convex function because of $\mathbf{D} > 0$. The necessary and sufficient conditions for optimality of (57) are

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{r}} &= \mathbf{0} \\ \frac{\partial L}{\partial \lambda_1} &= 0 \\ \frac{\partial L}{\partial \lambda_2} &= 0 \end{aligned} \tag{58}$$

which can be explicitly written as

$$2\mathbf{D}\mathbf{r} - \lambda_1\mathbf{a} - \lambda_2\mathbf{1} = 0 \tag{59}$$
$$c - \mathbf{r}^t\mathbf{a} = 0 \tag{60}$$
$$1 - \mathbf{r}^t\mathbf{1} = 0. \tag{61}$$

From (59), we have

$$\mathbf{r}^t = 0.5(\lambda_1\mathbf{a}^t\mathbf{D}^{-1} + \lambda_2\mathbf{1}^t\mathbf{D}^{-1}). \tag{62}$$

Substituting (62) into (60) and (61) leads to the following equations:

$$c = 0.5(\lambda_1\mathbf{a}^t\mathbf{D}^{-1}\mathbf{a} + \lambda_2\mathbf{1}^t\mathbf{D}^{-1}\mathbf{a}) \tag{63}$$
$$1 = 0.5(\lambda_1\mathbf{a}^t\mathbf{D}^{-1}\mathbf{1} + \lambda_2\mathbf{1}^t\mathbf{D}^{-1}\mathbf{1}). \tag{64}$$

For simplicity, we let

$$\begin{aligned} \alpha &= 0.5\mathbf{a}^t\mathbf{D}^{-1}\mathbf{a},\; \beta = 0.5\mathbf{1}^t\mathbf{D}^{-1}\mathbf{a} \\ \gamma &= 0.5\mathbf{1}^t\mathbf{D}^{-1}\mathbf{1}. \end{aligned} \tag{65}$$

It should be noted that α, β, and γ are constant values. Thus, (63) and (64) can be rewritten as

$$\alpha\lambda_1 + \beta\lambda_2 = c \tag{66}$$
$$\beta\lambda_1 + \gamma\lambda_2 = 1. \tag{67}$$

Assuming that $e = \alpha\gamma - \beta^2$ is not zero, we can solve (66) and (67) to obtain λ_1 and λ_2 as follows:

$$\lambda_1 = \frac{(c\gamma - \beta)}{e} \tag{68}$$
$$\lambda_2 = \frac{(\alpha - c\beta)}{e}. \tag{69}$$

Substituting (68) and (69) into (62) leads to

$$\mathbf{r}^{*t} = \frac{0.5(c\gamma - \beta)\mathbf{a}^t\mathbf{D}^{-1}}{e} + \frac{0.5(\alpha - c\beta)\mathbf{1}^t\mathbf{D}^{-1}}{e}. \tag{70}$$

Thus,

$$\begin{aligned} \mathbf{r}^* &= \left(\left(\frac{0.5\gamma}{e}\right)\mathbf{D}^{-1}\mathbf{a} - \left(\frac{0.5\beta}{e}\right)\mathbf{D}^{-1}\mathbf{1}\right)c \\ &\quad + \left(\frac{0.5\alpha}{e}\right)\mathbf{D}^{-1}\mathbf{I} - \left(\frac{0.5\beta}{e}\right)\mathbf{D}^{-1}\mathbf{a}\Bigg) \\ &= \mathbf{b}c + \mathbf{d} \end{aligned} \tag{71}$$

where

$$\mathbf{b} = \left(\frac{0.5\gamma}{e}\right)\mathbf{D}^{-1}\mathbf{a} - \left(\frac{0.5\beta}{e}\right)\mathbf{D}^{-1}\mathbf{1} \tag{72}$$
$$\mathbf{d} = \left(\frac{0.5\alpha}{e}\right)\mathbf{D}^{-1}\mathbf{I} - \left(\frac{0.5\beta}{e}\right)\mathbf{D}^{-1}\mathbf{a}. \tag{73}$$

Because $\mathbf{b}$ and $\mathbf{d}$ are constant vectors, it follows from (71) that the optimal solution $\mathbf{r}^*$ is a linear function of the given center c.

Considering that $\mathbf{r}^{*t}\mathbf{D}\mathbf{r}^*$ is the smallest spread of the portfolio return denoted as ρ, we have

$$\begin{aligned} \rho &= \mathbf{r}^{*t}\mathbf{D}\mathbf{r}^* = (\mathbf{b}^t c + \mathbf{d}^t)\mathbf{D}(\mathbf{b}c + \mathbf{d}) \\ &= c^2\mathbf{b}^t\mathbf{D}\mathbf{b} + 2c\mathbf{b}^t\mathbf{D}\mathbf{d} + \mathbf{d}^t\mathbf{D}\mathbf{d}. \end{aligned} \tag{74}$$

Since $\mathbf{b}^t\mathbf{D}\mathbf{b}$, $\mathbf{b}^t\mathbf{D}\mathbf{d}$, and $\mathbf{d}^t\mathbf{D}\mathbf{d}$ are constants denoted as t_1, t_2, and t_3, respectively, (74) can be simply written as follows:

$$\rho = t_1c^2 + t_2c + t_3 \tag{75}$$

which means that the spread ρ is a quadratic function of the given center c.

Theorem 5: The spread of the possibility portfolio return based on the compromised lower possibility distribution is not

TABLE I
SECURITY DATA WITH EXPERTS' JUDGMENTS

Years	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5	Sec.1	Sec.2	Sec.3	Sec.4
1977(1)	0.881	0.805	0.192	0.192	0.21	-0.305	-0.173	-0.318	-0.477
1978(2)	0.279	0.273	0.901	0.621	0.69	0.513	0.098	0.285	0.714
1979(3)	0.52	0.602	0.54	0.685	0.672	0.055	0.2	-0.047	0.165
1980(4)	0.623	0.671	0.517	0.542	0.557	-0.126	0.03	0.104	-0.043
1981(5)	0.811	0.877	0.312	0.298	0.277	-0.28	-0.183	-0.171	-0.277
1982(6)	0.522	0.565	0.623	0.58	0.536	-0.003	0.067	-0.039	0.476
1983(7)	0.377	0.431	0.676	0.918	0.782	0.428	0.3	0.149	0.225
1984(8)	0.46	0.452	0.698	0.717	0.726	0.192	0.103	0.26	0.29
1985(9)	0.348	0.374	0.716	0.809	0.877	0.446	0.216	0.419	0.216
1986(10)	0.736	0.797	0.371	0.414	0.41	-0.088	-0.046	-0.078	-0.272
1987(11)	0.598	0.667	0.556	0.622	0.575	-0.127	-0.071	0.169	0.144
1988(12)	0.588	0.673	0.54	0.628	0.582	-0.015	0.056	-0.035	0.107
1989(13)	0.475	0.484	0.709	0.753	0.696	0.305	0.038	0.133	0.321
1990(14)	0.415	0.434	0.535	0.468	0.615	-0.096	0.089	0.732	0.305
1991(15)	0.561	0.582	0.581	0.649	0.639	0.016	0.09	0.021	0.195
1992(16)	0.443	0.503	0.669	0.681	0.671	0.128	0.083	0.131	0.39
1993(17)	0.611	0.689	0.482	0.582	0.522	-0.01	0.035	0.006	-0.072
1994(18)	0.222	0.223	0.661	0.405	0.546	0.154	0.176	0.908	0.715

larger than the one based on the compromised upper possibility distribution.

Proof: Denote the optimal solutions obtained from (54) and (55) as $\mathbf{r}_u^*$ and $\mathbf{r}_l^*$, respectively, with the same center value. According to the feature of the upper and lower possibility distributions, i.e., $\mathbf{D}_{cu} - \mathbf{D}_{cl} \geq 0$, the following inequality holds:

$$\mathbf{r}_u^{*^t}\mathbf{D}_{cu}\mathbf{r}_u^* \geq \mathbf{r}_u^{*^t}\mathbf{D}_{cl}\mathbf{r}_u^*. \tag{76}$$

Because $\mathbf{r}_l^*$ is the optimal solution of (55), we have

$$\mathbf{r}_u^{*^t}\mathbf{D}_{cl}\mathbf{r}_u^* \geq \mathbf{r}_l^{*^t}\mathbf{D}_{cl}\mathbf{r}_l^*. \tag{77}$$

As a result

$$\mathbf{r}_u^{*^t}\mathbf{D}_{cu}\mathbf{r}_u^* \geq \mathbf{r}_l^{*^t}\mathbf{D}_{cl}\mathbf{r}_l^* \tag{78}$$

which proves the theorem. □

The nondominated solutions with different centers of a possibility portfolio in (54) and (55) can form two efficient frontiers.

V. NUMERICAL EXAMPLE

In order to show the above-proposed approaches, a numerical example for portfolio selection is given where four typical securities were considered. Their returns from 1977 to 1994 were collected as references to predict their returns in the next year. Five experts were invited to give their judgment on how similar the economic situation in the next year would be to that in each sample year. For example, in the opinion of expert 1, the economic situation in the next year would have high similarity to that in 1977 so that he gave the possibility grade, for example, 0.881 to 1977. Because the stock market is a sort of mirror to reflect the current economics situation, it was reasonable to predict that the returns of these four securities would reappear in the next year with possibility grade 0.881. Different from probability reflecting the frequency of some happening, possibility grades represent a kind of potential of some happening, which is problem-specific. Table I lists returns of four securities (sec.1, sec.2, sec.3 and sec.4) from 1977 to 1994 and the associated possibility grades given by five experts. The procedure for obtaining possibility distributions from the above given data was as follows. Firstly, obtain the center $\mathbf{a}$ of possibility distribution for each expert by (8). Then obtain the transformation matrix $\mathbf{T}$ by (10) and $\mathbf{z}_i = \mathbf{x}_i - \mathbf{a}$ $(i = 1, \ldots, 18)$. After that, solve the LP problem (15) and obtain distribution matrices $\mathbf{D}_u$ and $\mathbf{D}_l$ by (16). For example, the dual possibility distributions from the experts 1, denoted as $\mathbf{X}_1 \sim \langle(\mathbf{a}_1, \mathbf{D}_{u1})_e, (\mathbf{a}_1, \mathbf{D}_{l1})_e\rangle$, were obtained as follows:

$$\mathbf{a}_1 = [-0.305, -0.173, -0.318, -0.477]^t$$

$$\mathbf{D}_{u1} = \begin{bmatrix} 321.045 & 105.691 & -444.331 & 93.502 \\ 105.691 & 35.305 & -146.258 & 30.933 \\ -444.331 & -146.258 & 618.334 & -128.763 \\ 93.502 & 30.933 & -128.763 & 28.420 \end{bmatrix}$$

$$\mathbf{D}_{l1} = \begin{bmatrix} 320.616 & 105.648 & -444.659 & 93.464 \\ 105.648 & 34.876 & -146.402 & 30.880 \\ -444.659 & -146.402 & 618.023 & -128.961 \\ 93.464 & 30.880 & -128.961 & 27.670 \end{bmatrix}.$$

TABLE II
CONFLICT INDEX MATRIX Q^u

	x1	x2	x3	x4	x5
x1	1	0.007	0.687	0.49	0.574
x2	0.007	1	0.608	0.429	0.506
x3	0.687	0.608	1	0.128	0.155
x4	0.49	0.429	0.128	1	0.063
x5	0.574	0.506	0.155	0.063	1

After obtaining all upper and lower possibility distributions of five experts, the conflict indices were calculated by (45). The obtained conflict index matrices based on upper and lower possibility distributions are listed in Tables II and III, respectively.

TABLE III
CONFLICT INDEX MATRIX Q^l

	x1	x2	x3	x4	x5
x1	1	0.083	1.335	0.941	0.959
x2	0.083	1	1.05	0.681	0.713
x3	1.335	1.05	1	0.385	0.302
x4	0.941	0.681	0.385	1	0.129
x5	0.959	0.713	0.302	0.129	1

TABLE IV
BINARY MATRIX Q^*

	x1	x2	x3	x4	x5
x1	1	1	0	0	0
x2	1	1	0	0	0
x3	0	0	1	1	1
x4	0	0	1	1	1
x5	0	0	1	1	1

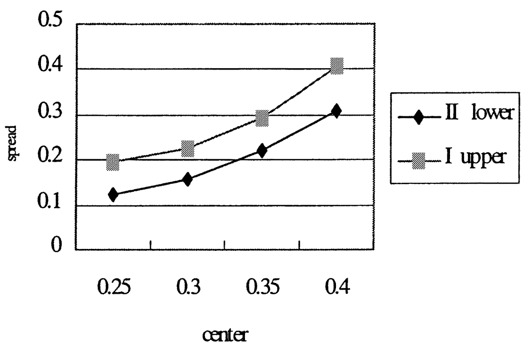

Fig. 2. Portfolio frontiers based on compromise upper and lower possibility distributions.

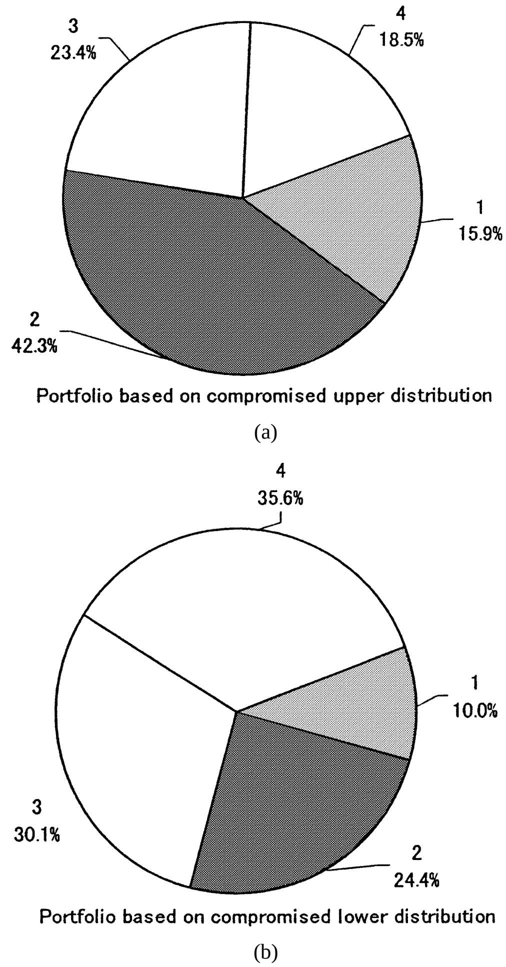

Fig. 3. Portfolios based on compromised (a) upper and (b) lower possibility distributions with $c = 0.3$.

The binary matrix Q^* is given in Table IV with $\delta_u = 0.2$ and $\delta_l = 0.4$. From Table IV, it is known that the efficient block is

$$U_e = \{\mathbf{X}_3 \sim \langle(\mathbf{a}_3, \mathbf{D}_{u3})_e, (\mathbf{a}_3, \mathbf{D}_{l3})_e\rangle, \mathbf{X}_4 \sim \langle(\mathbf{a}_4, \mathbf{D}_{u4})_e, (\mathbf{a}_4, \mathbf{D}_{l4})_e\rangle, \mathbf{X}_5 \sim \langle(\mathbf{a}_5, \mathbf{D}_{u5})_e, (\mathbf{a}_5, \mathbf{D}_{l5})_e\rangle\}.$$

The inconsistency degrees $\mathrm{Inc}(\mathbf{X}_3)$, $\mathrm{Inc}(\mathbf{X}_4)$, and $\mathrm{Inc}(\mathbf{X}_5)$ were obtained as $\mathrm{Inc}(\mathbf{X}_3) = 0.97$, $\mathrm{Inc}(\mathbf{X}_4) = 0.705$, and $\mathrm{Inc}(\mathbf{X}_5) = 0.649$. Because $\mathbf{X}_5$ had the smallest inconsistency degree in the efficient block, $\mathbf{X}_5$ was the key member for conflict resolution. The weight coefficients of h_k^3, h_k^4, and h_k^5 ($k = 1, \ldots, 18$) were set as 1/6, 1/3, and 1/2, respectively. Using the weighted possibility grades, the compromised possibility distribution $\mathbf{X} \sim \langle(\mathbf{a}_c, \mathbf{D}_{cu})_e, (\mathbf{a}_c, \mathbf{D}_{cl})_e\rangle$ were obtained as follows.

$$\mathbf{a}_c = [0.446, 0.216, 0.419, 0.216]^t$$

$$\mathbf{D}_{cu} = \begin{bmatrix} 0.988 & 0.275 & 0.001 & 0.129 \\ 0.275 & 0.277 & 0.075 & 0.020 \\ 0.001 & 0.075 & 1.134 & -0.070 \\ 0.129 & 0.020 & -0.070 & 0.918 \end{bmatrix}$$

$$\mathbf{D}_{cl} = \begin{bmatrix} 0.710 & 0.197 & -0.042 & 0.355 \\ 0.197 & 0.168 & 0.061 & 0.095 \\ -0.042 & 0.061 & 0.993 & -0.256 \\ 0.355 & 0.095 & -0.256 & 0.369 \end{bmatrix}.$$

Using models (54) and (55), the possibility portfolio frontiers based on the compromised upper and lower possibility distributions are shown in Fig. 2. It should be noted that the portfolios in the upper and lower frontiers with the same center were different with each other. For example, the portfolios with $c = 0.3$ are shown in Fig. 3.

VI. CONCLUSION

In this paper, the dual possibility distributions for approximating the given possibility grades are identified by minimizing the inconsistency index of two possibility distributions. The upper possibility distribution reflects an optimistic viewpoint and the lower possibility distribution reflects a pessimistic one. Different from other methods on conflict analysis, this paper focuses on analyzing the conflict situation in a decision group. The conflict among decision-makers arises from the inherent diversity of knowledge and cognition on some issue under debate. A set of dual possibility distributions is used to characterize multisource knowledge from multiple decision-makers. A conflict index between two possibility distributions is defined to reflect their difference degree. Based on the conflict index, conflict situation of knowledge can be investigated and a conflict resolution model is proposed to obtain a compromised possibility distribution, which represents a more reliable knowledge. As an application, a portfolio selection problem with multiple experts is considered. Because the compromised possibility distribution has higher credibility than a single one, it is natural to believe that the more reasonable decision can be made.

APPENDIX

Theorem (Gersgorin): Let $A = [a_{ij}] \in C^{n\times n}$, all the eigenvalue of A are located in the union of the n discs

$$\bigcup_{i=1,\ldots,n}\left\{z\in C:||z-a_{ii}|\leq \sum_{j=1,\ldots,n,j\neq i}|a_{ij}|\right\}.$$

Corollary: Let $A = [a_{ij}] \in C^{n\times n}$ and let $p_1, p_2, \ldots, p_n$ be positive real numbers. Then all the eigenvalues of A lie in the region

$$\bigcup_{i=1,\ldots,n}\left\{z\in C:|z-a_{ii}|\leq \frac{1}{p_i}\sum_{j=1,\ldots n,j\neq i}p_j|a_{ij}|\right\}.$$

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