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On the *f*-Coloring of Multigraphs

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***Abstract*—An f-coloring of a multigraph $G=(V,E)$ is a coloring of edges E such that each color appears at each vertex $v \in V$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f-chromatic index $q_f^*(G)$ of G. Various scheduling problems on networks are reduced to finding an f-coloring of a multigraph. This paper gives an upper bound on the f-chromatic index. Denote by $d(v)$ the degree of vertex $v \in V$, and define $d_f(G)=\max_{v \in V}\lceil d(v)/f(v)\rceil$. Denote by $E(H)$ and $V(H)$ the edge and vertex sets of a subgraph H of G respectively, and define**

$$r_f(G)=\max_{H \subset G}\left\lceil |E(H)|/\left\lfloor \Sigma_{v \in V(H)} f(v)/2\right\rfloor\right\rceil$$

where H runs over all subgraphs of G having at least three vertices. Then our bound is

$$q_f^*(G) \leqslant \max\left\{r_f(G),\left\lfloor\left(9d_f(G)+6\right)/8\right\rfloor\right\}.$$

The proof is constructive, and yields a polynomial-time algorithm to f-color G with at most $\left\lfloor\left(9q_f^*(G)+6\right)/8\right\rfloor$ colors.

I. Introduction

In this paper we deal with a *multigraph* G which may have multiple edges but no self-loops, and simply call G a *graph*. A graph G with a vertex set V and edge-set E is denoted by $G=(V,E)$. In an ordinary edge coloring, each vertex has at most one edge colored with the same color. The minimum number of colors needed for an ordinary coloring of G is called the *chromatic index* of G and denoted by $q^*(G)$ [4]. Hakimi and Kariv generalized the coloring and obtained many interesting results [10]. For a function f which assigns a positive integer $f(v)$ to each vertex $v \in V$, an *f-coloring* of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with the same color. (They call the f-coloring a "proper edge-coloring".) The minimum number of colors needed to f-color G is called an *f-chromatic index* of G, and denoted by $q_f^*(G)$.

The f-coloring has applications to scheduling problems like the file transfer problem in a computer network [2], [3], [13]. In the model a vertex of a graph G represents a computer, and an edge does a file which one wish to transfer between the two computers corresponding to its ends. The integer $f(v)$ is the number of communication ports available at a computer v. The edges colored with the same color represent files that can be transferred in the network simultaneously. Thus an f-coloring of G using $q_f^*(G)$ colors corresponds to a scheduling of file transfers with the minimum finishing time.

Since the ordinary edge-coloring problem is NP-hard [12], the f-coloring problem which asks to f-color a given graph G with $q_f^*(G)$ colors is also NP-hard in general. Therefore it is unlikely that there is a polynomial-time algorithm for the f-coloring problem [1], [5]. Thus a good approximation algorithm would be useful for the problem.

We denote the degree of vertex v by $d(v)$, and the maximum degree of G by $d(G)$. We write $H \subset G$ if H is a subgraph of G. $V(H)$ denotes the set of vertices of H, and $E(H)$ the set of edges of H. Goldberg [7] has obtained the following upper bound on the ordinary chromatic index $q^*(G)$:

$$q^*(G) \leqslant \max\{r(G),\lfloor(9d(G)+6)/8\rfloor\}.$$

Here $r(G)$ is a trivial lower bound on $q^*(G)$:

$$r(G)=\max_{H \subset G}\lceil |E(H)|/\lfloor |V(H)|/2\rfloor\rceil$$

where H runs over all subgraphs of G having at least three vertices. Throughout the paper $\lfloor x\rfloor$ means the maximum integer no greater than x and $\lceil x\rceil$ means the minimum integer no less than x. A slightly better upper bound on $q^*(G)$ has been obtained [14].

The following $d_f(G)$ and $r_f(G)$ are trivial lower bounds on the f-chromatic index $q_f^*(G)$. Let

$$d_f(G)=\max_{v \in V(G)}\lceil d(v)/f(v)\rceil$$

then $q_f^*(G) \geqslant d_f(G)$ because at least $\lceil d(v)/f(v)\rceil$ colors are necessary to f-color the edges incident to v. Let

$$r_f(G)=\max_{H \subset G}\left\lceil |E(H)|/\left\lfloor \Sigma_{v \in V(H)} f(v)/2\right\rfloor\right\rceil$$

then $q_f^*(G) \geqslant r_f(G)$ since at most $\left\lfloor \Sigma_{v \in V(H)} f(v)/2\right\rfloor$ edges in a subgraph H can be colored with the same color.

Hakimi and Kariv [10] conjecture that Goldberg's bound for $q^*(G)$ would be generalized to the case of f-coloring, and the following inequality would hold

$$q_f^*(G) \leqslant \max\left\{r_f(G),\left\lfloor\left(9d_f(G)+6\right)/8\right\rfloor\right\}.$$

This paper proves this conjecture affirmatively. The proof is constructive and immediately yields a polynomial algorithm for f-coloring any given graph G with at most

$$q_f(G)=\max\left\{r_f(G),\left\lfloor\left(9d_f(G)+6\right)/8\right\rfloor\right\}$$

colors. Since $q_f(G) \leqslant (9q_f^*(G)+6)/8$, the asymptotic performance ratio [5] of the algorithm is $9/8$. Although

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Hakimi and Kariv present another upper bound for $q_f^*(G)$ [10], neither theirs nor ours can be derived from the other.

II. Preliminaries

Let $q_f(G)=\max\{r_f(G), \lfloor(9d_f(G)+6)/8\rfloor\}$. We denote by Q the set of $q_f(G)$ colors used to f-color a graph G. An edge colored with color $c \in Q$ is called a *c-edge*. Denote by $d(v,c,G)$ the number of c-edges of G incident to vertex v, and define $m(v,c,G)=f(v)-d(v,c,G)$. G is f-colored if and only if $m(v,c,G)\geqslant 0$ for every $v\in V$ and $c\in Q$. Color c is *available* at vertex v if $m(v,c,G)\geqslant 1$. Define $M(v)=\{c\in Q\colon\ m(v,c,G)\geqslant 1\}$. We often write simply $m(v,c)$ and $d(v,c)$ for $m(v,c,G)$ and $d(v,c,G)$ if no confusion may occur. An edge joining vertices v and w is denoted by vw.

"Switching an alternating path" is a standard coloring technique [4], [6]–[11], [15]. We also use it with some modification. A "walk" is used instead of a path. A *walk* W is a sequence of distinct edges $v_0v_1, v_1v_2,\cdots,v_{k-1}v_k$, where the vertices $v_0,v_1,\cdots,v_k$ are not necessarily distinct. Walk W is often denoted simply by $v_0v_1\cdots v_k$. We call v_0 the *start vertex* of W and v_k the *end vertex*. The *length* of W is the number k of edges in W, and denoted by $|W|$. If $v_0=v_k$, then W is called a *cycle*. For two distinct colors a, $b\in Q$, a walk W of length one or more is called an *ab-alternating walk* if W satisfies the following conditions:

(1) The edges of W are colored alternately with a and b with the first edge $e_1=v_0v_1$ being colored b (i.e., the ith edge $e_i=v_{i-1}v_i$ is colored b if i is odd, and with a if i is even);

(2) $m(v_0,a)\geqslant 1$ if $v_0\neq v_k$,
$m(v_0,a)\geqslant 2$ if $v_0=v_k$ and $|W|$ is odd;

(3) $m(v_k,b)\geqslant 1$ if $v_0\neq v_k$ and $|W|$ is even,
$m(v_k,a)\geqslant 1$ if $v_0\neq v_k$ and $|W|$ is odd.

Note that any circle W of even length whose edges are colored with a and b alternately is an ab-alternating walk (cycle). If G is f-colored and W is an ab-alternating walk, then interchanging the colors a and b of the edges in walk W preserves an f-coloring of G. This operation is called *switching* W. When W is switched, $m(v_i,a)$ and $m(v_i,b)$ remain as they were if $i\neq 0,k$, while $m(v_0,b)\geqslant 1$ if W is not a cycle of even length.

We denote by $W(a,b,v_0)$ an ab-alternating walk which starts with vertex v_0 and is not a cycle of even length. One can observe the following lemma.

Lemma 1: Let a graph $G=(V,E)$ be f-colored, and let $v_0\in V$ and a, $b\in Q$. Then the following (a) and (b) hold.

(a) If a b-edge e is incident to v_0, then there is an ab or ba-alternating walk passing through e.

(b) If $m(v_0,a)\geqslant 1$ and $m(v_0,b)=0$, then there is a walk $W(a,b,v_0)$.

Proof: (a) Extend a walk $W=e$ forward and backward, with choosing a- and b-edges alternately, as long as possible. Then the resulting walk W satisfies the termination conditions (2) and (3) above, and is an ab- or ba-alternating walk passing through e.

(b) Since $m(v_0,a)\geqslant 1$ and $m(v_0,b)=0$, $d(v_0,b)-d(v_0,a)\geqslant 1$ and hence there is a b-edge e incident to v_0. Extend a walk $W=e$ forward, with choosing a- and b-edges alternately, until (2) and (3) hold. In particular when the walk W returns to v_0 with an a-edge, continue to extend W by choosing a b-edge incident to v_0. (Since $d(v_0,b)-d(v_0,a)\geqslant 1$, there exists such a b-edge which has not been included in W so far.) Thus one can obtain a walk $W(a,b,v_0)$ which is not a cycle of even length. Q.E.D.

Denote by $W_i(a,b,v)$ an ab- or ba-alternating walk passing through vertex v. Note that $W_i(a,b,v)$ may not start with vertex v and may be an alternating cycle of even length.

III. The Main Theorem and an Outline of the Proof

The following is a main result of this paper.

Theorem 1: Every multigraph $G=(V,E)$ satisfies the following:

$$q_f^*(G)\leqslant \max\left\{r_f(G), \lfloor(9d_f(G)+6)/8\rfloor\right\}.$$

Suppose to the contrary that for a graph G

$$q_f^*(G) > q_f(G)=\max\left\{r_f(G), \lfloor(9d_f(G)+6)/8\rfloor\right\}$$

and let $G=(V,E)$ be such a graph with a minimum number of edges. Clearly $E\neq\phi$. Let $e=xy$ be an arbitrary edge of G, and let $G'=G-e$ be the graph obtained from G by deleting edge e. Then by the assumption G' can be f-colored with $q_f(G)(\geqslant q_f(G'))$ colors. We will prove that the f-coloring of G' can be extended to an f-coloring of G using $q_f(G)$ colors, that is, a contradiction occurs.

The outline of the proof is as follows. As shown later, there is at least one available color at every vertex, that is, $M(v)\neq\phi$ for every $v\in V$. If $M(x)\cap M(y)\neq\phi$, then we color the uncolored edge $e=xy$ of G with a color in $M(x)\cap M(y)$ to complete an f-coloring of G. If $M(x)\cap M(y)=\phi$, then we change the f-coloring of $G'=G-e$ by switching alternating walks in order to make $M(x)\cap M(y)\neq\phi$. If there is either a walk $W(a,b,x)$ not ending at y or a walk $W(b,a,y)$ not ending at x for $a\in M(x)$ and $b\in M(y)$, then one can easily make $M(x)\cap M(y)\neq\phi$ by switching such a walk. Even if every $W(a,b,x)$ (or $W(b,a,y)$) ends at y (or x), one can systematically change the f-coloring of G' to make $M(x)\cap M(y)\neq\phi$ if $W(a,b,x)$ (or $W(b,a,y)$) has length greater than six. For the remaining case in which every $W(a,b,x)$ (or $W(b,a,y)$) ends at y (or x) and has length at most six, we need rather lengthy arguments to make $M(x)\cap M(y)\neq\phi$. The following two sections are devoted to the detail of the proof.

IV. The Proof

The following Lemmas 2–6 can be easily derived.

Lemma 2: $d_f(G)\geqslant 2$, and consequently, $q_f(G)\geqslant d_f(G)+1$.

Proof: If $d_f(G) \leqslant 1$, then obviously $q_f^*(G) = d_f(G) \leqslant q_f(G)$. Therefore, $d_f(G) \geqslant 2$, and consequently,

$$q_f(G) \geqslant \lfloor (9d_f(G)+6)/8 \rfloor \geqslant d_f(G)+1.$$

Q.E.D.

Lemma 3: $\Sigma_{c \in Q} m(v,c) \geqslant f(v)$ for every vertex $v \in V$, and in particular $\Sigma_{c \in Q} m(v,c) \geqslant f(v)+1$ if $v = x$ or y.

Proof: Since $d(v) \leqslant d_f(G) f(v)$ and $q_f(G) \geqslant d_f(G)+1$, $\Sigma_{c \in Q} m(v, c) \geqslant q_f(G) f(v) - d(v) \geqslant (q_f(G) - d_f(G)) f(v) \geqslant f(v)$. Noting that the uncolored edge e joins x and y, one can easily verify the claim for the case $v = x$ or y. Q.E.D.

Thus every vertex has at least one available color. In particular both x and y have at least two available colors or one available color c with $m(x,c) \geqslant 2$ or $m(y,c) \geqslant 2$.

Lemma 4: The ends x and y of the uncolored edge e have no common available color, that is, $M(x) \cap M(y) = \phi$.

Proof: If x and y have a common missing color c, then coloring edge e with c yields an f-coloring of G using at most $q_f(G)$ colors. Q.E.D.

Lemma 5: Every walk $W(a,b,x)$ ends at y and every walk $W(b,a,y)$ ends at x for any $a \in M(x)$ and $b \in M(y)$.

Proof: By Lemma 1(b) there are walks $W(a,b,x)$ and $W(b,a,y)$. If $W(a,b,x)$ does not end at y or $W(b,a,y)$ does not end at x, switching $W(a,b,x)$ or $W(b,a,y)$ would make x and y have a common available color a or b, contrary to Lemma 4. Q.E.D.

Thus the walk traversing $W(a,b,x)$ backward is $W(b,a,y)$.

Lemma 6: Let $a \in M(x)$ and $b \in M(y)$, then $m(v,a) = m(v,b) = 0$ for every vertex $v \neq x, y$ on any walk $W(a,b,x)$.

Proof: Otherwise, either one of walks $W(a,b,x)$ or one of walks $W(b,a,y)$ traversing the walk $W(a,b,x)$ reversely ends at v, contrary to Lemma 5. Q.E.D.

Lemma 7: If $a \in M(x)$ and $b \in M(y)$, then
(a) $m(x,a) = m(y,b) = 1$; and furthermore
(b) for any walk $W(a,b,x)$ and any $c \in Q$

$$\Sigma_{v \in V(W(a,b,x))} m(v,c) \leqslant 1.$$

Proof: (a) If $m(x,a) \geqslant 2$, then switching a walk $W(a,b,x)$ would make $a \in M(x) \cap M(y)$. If $m(x,b) \geqslant 2$, then switching a walk $W(b,a,y)$ would make $b \in M(x) \cap M(y)$. Either case contradicts Lemma 4.

(b) Suppose first that there are two consecutive vertices u and v on $W(a,b,x)$ such that $m(u,c) \geqslant 1$ and $m(v,c) \geqslant 1$. Then by Lemmas 4 and 6 $c \neq a$, b and u or v, say u, is distinct from x and y. Recolor edge uv on $W(a,b,x)$ with color c, then u lies on one of walks $W(a,b,x)$ and either $m(u,a) > 1$ or $m(u,b) \geqslant 1$, contrary to Lemma 6.

Suppose next that there are two nonconsecutive vertices u and v on $W(a,b,x)$ such that u precedes v in $W(a,b,x)$, $m(u,c) \geqslant 1$ and $m(v,c) \geqslant 1$. Let $u' \neq v$ be the vertex succeeding u on $W(a,b,x)$, and let $c' \neq a, b$ be an available color at u'. In light of the paragraph above, $c \neq c'$. Since $m(x,a) = m(y,b) = 1$, by Lemma 3 there exists such a color c' even if $u' = x$ or y. Switch a walk $W(c',c,u')$; then $W(a,b,x)$ is not destroyed since $c, c' \neq a, b$. If $W(c',c,u')$ does not end at u, then the consecutive vertices u and u' on $W(a,b,x)$ have a common available color c. If $W(c',c,u')$ ends at u, then u' and v have a common available color c and the number of edges between u' and v on $W(a,b,x)$ is less than the number of edges between u and v. Repeating this operation, one can eventually produce two consecutive vertices on $W(a,b,x)$ with a common available color, a contradiction.

Suppose finally that there is a vertex v on a walk $W(a,b,x)$ such that $m(v,c) \geqslant 2$. Then $c \neq a, b$. Let v' be a vertex adjacent to v on $W(a,b,x)$. By Lemma 3 v' has an available color $c' \neq a, b, c$. Switching a walk $W(c',c,v')$ would produce a common available color c at the consecutive vertices v and v' on $W(a,b,x)$ even if $W(c',c,v')$ ends at v, a contradiction. Q.E.D.

Define $f(S) = \Sigma_{v \in S} f(v)$ for $S \subset V$, then we have the following lemma.

Lemma 8: Assume that $S \subset V$ satisfies $\Sigma_{v \in S} m(v,c) \leqslant 1$ for every $c \in Q$. Then the following (a) and (b) hold.
(a) If x, $y \in S$, then $f(S) \leqslant 8$; and
(b) If x or y is in S, then $f(S) \leqslant 9$.

Proof: (a) Since $\Sigma_{v \in S} m(v,c) \leqslant 1$ for every $c \in Q$,

$$q_f(G) \geqslant \Sigma_{v \in S} \Sigma_{c \in Q} m(v,c).$$

For any vertex $v \neq x$, y,

$$\Sigma_{c \in Q} m(v,c) = q_f(G) f(v) - d(v) \geqslant (q_f(G) - d_f(G)) f(v).$$

On the other hand,

$$\Sigma_{c \in Q} m(x,c) \geqslant (q_f(G) - d_f(G)) f(x) + 1$$

and

$$\Sigma_{c \in Q} m(y,c) \geqslant (q_f(G) - d_f(G)) f(y) + 1.$$

Therefore, we have

$$q_f(G) \geqslant \Sigma_{v \in S} \Sigma_{c \in Q} m(v,c) \geqslant (q_f(G) - d_f(G)) f(S) + 2$$

and hence,

$$f(S) \leqslant (q_f(G) - 2)/(q_f(G) - d_f(G)). \tag{1}$$

Therefore, $f(S) \leqslant 1$ if $d_f(G) = 2$. Thus one may assume

$$d_f(G) \geqslant 3. \tag{2}$$

Since $q_f(G) \geqslant \lfloor (9d_f(G)+6)/8 \rfloor$

$$q_f(G) \geqslant (9d_f(G) - 1)/8. \tag{3}$$

The inequalities (1), (2) and (3) yield $f(S) \leqslant 8$.
(b) Similar to (a) above. Q.E.D.

We say that an edge vv' of G *leaves a subgraph* H *of* G *at vertex* v if $v \in V(H)$ and $v' \notin V(H)$. Lemma 9 below follows from the lemmas above.

Lemma 9: For any $a \in M(x)$ and $b \in M(y)$, the following (a)–(d) hold.

(a) Let H be the subgraph of G induced by the edges of all walks $W(a, b, x)$ and $W(b, a, y)$, then all a- and b-edges of G incident to vertices in H belong to H. In particular no a- or b-edge of G leaves H.

(b) $m(x, a, H) = m(y, b, H) = 1$ and $m(x, b, H) = m(y, a, H) = 0$, that is, $d(x, a, H)+1 = d(x, b, H) = f(x)$ and $d(y, a, H) = d(y, b, H)+1 = f(y)$.

(c) $m(v, a, H) = m(v, b, H) = 0$ for any $v \in V(H) - \{x, y\}$, that is, $d(v, a, H) = d(v, b, H) = f(v)$.

(d) There exists a single walk $W(a, b, x)$ which passes through all edges of H and ends at y. (Such a walk $W(a, b, x)$ is called the *ab-critical walk* and denoted by $W(a, b)$.)

Proof:

(a) Suppose that there is an a- or b-edge e' which is incident to a vertex in H but does not belong to H. Then there is a walk $W(a, b, x)$ or $W(b, a, y)$ passing through e', contrary to the definition of H.

(b) and (c) are immediate from (a) above and Lemmas 6 and 7.

(d) Noting (b) and (c) and using a standard argument on an Eulerian cycle, one can easily know that H can be decomposed into a single ab-alternating walk starting at x and ending at y and several ab-alternating cycles of even length. A single walk $W(a, b)$ can be constructed by appropriately linking the walk and cycles. Q.E.D.

Consider the subgraph of G' induced by all the a- and b-edges. By Lemma 9(a) the ab-critical walk $W(a, b)$ is indeed the connected component (containing x and y) of the subgraph. We denote by $V(a, b)$ the set of vertices in $W(a, b)$ and by $|W(a, b)|$ the length of $W(a, b)$. (When an f-coloring is modified, it may become that there is no ab-critical walk, that is, there exists either a walk $W(a, b, x)$ not ending at y or a walk $W(b, a, y)$ not ending at x. In such a case $|W(a, b)|$ is defined to be infinite for the sake of convenience.) We have the following lemma.

Lemma 10: Let $a \in M(x)$, $b \in M(y)$, and $S = V(a, b)$, then the following (a)–(c) hold:

(a) $|W(a, b)| = 2$, 4 or 6;

(b) $|W(a, b)| = f(S) - 1$;

(c) $f(S)$ is an odd number.

Proof: By Lemmas 7 and 8(a), $f(S) \leqslant 8$. By Lemmas 9(a)–(c), $W(a, b)$ contains $(f(S)-1)/2$ a-edges and $(f(S)-1)/2$ b-edges. Therefore, $f(S)$ is an odd number and $|W(a, b)| = f(S) - 1$ is an even number. Thus $|W(a, b)|$ is either 2, 4, or 6. Q.E.D.

Thus we shall consider three cases $|W(a, b)| = 2$, 4, and 6. We further have the following lemmas.

Lemma 11: Let $S = V(a, b)$ and let H' be the subgraph of G induced by the vertex set S. (Thus $H \subset H' \subset G$.) Then there exists a color $c \in Q$ such that $\Sigma_{v \in S} m(v, c, H') \geqslant 3$. Furthermore, the color c satisfies either (a) or (b) below:

(a) $\Sigma_{v \in S} m(v, c, G) = 0$ and at least three c-edges leave H';

(b) $\Sigma_{v \in S} m(v, c, G) = 1$ and at least two c-edges leave H'.

Proof: Let $t = \lfloor f(S)/2 \rfloor$. Since

$$q_f(G) \geqslant r_f(G) \geqslant |E(H')|/t$$

$tq_f(G) \geqslant |E(H')|$. Each of $q_f(G)$ colors is assigned to at most t edges in H'. If each of $q_f(G)$ colors were assigned to exactly t edges in H', then since there is an uncolored edge e in H', $|E(H')| = tq_f(G)+1$, a contradiction. Hence there must exist a color $c \in Q$ that is assigned to at most $t-1$ edges of H'. Since $f(S)$ is an odd number by Lemma 10(c),

$$\Sigma_{v \in S} m(v, c, H') \geqslant f(S) - 2(t-1) = 3.$$

Since $\Sigma_{v \in S} m(v, c, G) \leqslant 1$ by Lemma 7, the color c satisfies either (a) or (b). Q.E.D.

Lemma 12: Interchanging the uncolored edge e with another edge if necessary, one can alter the f-coloring of $G' = G - e$ so that $m(x, c, G) = 1$ for a color c and at least two c-edges leave H'.

Proof: We first show that the f-coloring satisfying (a) in Lemma 11 can be altered so that (b) holds. We then show that the f-coloring satisfying (b) can be altered so that $m(x, c, G) = 1$.

Let $S = V(a, b)$, and suppose that $\Sigma_{v \in S} m(v, c, G) = 0$ and there are at least three c-edges leaving H'. Let uu' be one of the three c-edges leaving H' where $u \in S$ and $u' \notin S$. Noting Lemmas 3 and 7 one can easily know that even if $u = x$ or y there exists a color $g \in M(u)$ such that $m(u, g, G) = 1$ and $g \neq a, b, c$.

By Lemma 7(b) we have

$$1 = \Sigma_{v \in S} m(v, g, G) = f(S) - \Sigma_{v \in S} d(v, g, G).$$

By Lemma 10(c), $f(S)$ is odd. Furthermore, $\Sigma_{v \in S} d(v, g, H')$ is even. Therefore, if there is a g-edge leaving H', then there are two or more g-edges leaving H', and hence (b) in Lemma 11 would hold with respect to color g. Thus one may assume that no g-edge leaves H'.

If a walk $W(g, c, u)$ contained exactly one c-edge uu' leaving H', then switching $W(g, c, u)$ would make (b) hold. Thus every walk $W(g, c, u)$ must contain at least two c-edges leaving H'.

Any walk $W(g, c, u)$ does not end at a vertex in S because $m(w, c, G) = 0$ for every vertex $w \in S$, $m(u, g, G) = 1$ and $m(w, g, G) = 0$ for every vertex $w \in S - \{u\}$. Since no g-edge leaves H', every walk $W(g, c, u)$ must contain three or more c-edges leaving H'. Let vv' be the last c-edge leaving H' in a walk $W(g, c, u)$ where $v \in S$ and $v' \notin S$. Then $v \neq u$: otherwise, there would exist a walk $W(g, c, u)$ containing exactly one c-edge leaving H'. Let W' be the latter half of the walk $W(g, c, u)$ starting with edge vv'. There is a color $h \in M(v)$ such that $h \neq a, b, c, g$. No h-edges leave H' by the same argument as g.

For each vertex $w \in S - \{v\}$ $m(w, h, G) = 0$, and for each vertex $w \in S - \{u\}$ $m(w, g, G) = 0$. Furthermore, no

h- or g-edge leaves H'. Therefore every walk $W(h, g, v)$ must end at u. Switch a walk $W(h, g, v)$, then color g is available at v and W' now becomes a walk $W(g, c, v)$. Switch this walk $W(g, c, v)$, then c is now available at v and there remain at least two c-edges leaving H', so (b) holds. Thus we have shown that the f-coloring satisfying (a) can be altered so that (b) holds.

Next we will show that the f-coloring satisfying (b) can be altered so that $m(x, c, G) = 1$. Assume that $m(u, c, G) = 1$ for $u \in S - \{x\}$. Let $W(a, b) = e_1, e_2, \cdots, e_k$ and $e_i = u'u$. Assume that e_i is colored a. (The proof for the case e_i is colored b is similar.) Erase the color a of e_i, then switch the subwalk $e_1, e_2, \cdots, e_{i-1}$ of $W(a, b)$, and finally, color edge $e = xy$ with color b. We now regard the ends u and u' of the uncolored edge e_i as x and y, respectively. Then $a,\ c \in M(x)$, $b \in M(y)$, $W(a, b) = e_{i+1},\ e_{i+2}, \cdots, e_k, e, e_1, \cdots, e_{i-1}$ and at least two c-edges leave H'. Q.E.D.

Assume that $W(a, b)$ is the longest critical walk with respect to all choices of $a \in M(x)$ and $b \in M(y)$ and that the f-coloring is altered as in Lemma 12. Then one may assume that there is the cb-critical walk $W(c, b)$ and $|W(c, b)| \leqslant |W(a, b)|$. We now have the following lemma.

Lemma 13: $|W(a, b)| = 4$ or 6.

Proof: By Lemma 10(a), $|W(a, b)| = 2$, 4, or 6. Suppose that $|W(a, b)| = 2$, and let v be the intermediate vertex of the walk $W(a, b)$. Since $f(V(a, b)) = 3$ by Lemma 10(b), $f(x) = f(y) = f(v) = 1$. By Lemma 12 there are two c-edges leaving H': one is incident to y, and the other to v. Since $W(c, b)$ contains the b-edge xv and the two c-edges leaving H' at y and v, $|W(c, b)| \geqslant 4$, contrary to the assumption $|W(c, b)| \leqslant |W(a, b)|$. Q.E.D.

In the succeeding section we deal with the remaining case $|W(a, b)| = 4$ or 6.

V. Case $|W(a, b)| = 4$ or 6

We first have the following lemma.

Lemma 14: Let $c_1,\ c_3 \in M(x)$ and $c_2,\ c_4 \in M(y)$ where $c_1,\ c_3 \neq c_2,\ c_4$ but possibly $c_1 = c_3$ or $c_2 = c_4$. Let $W(c_1, c_2)$ and $W(c_3, c_4)$ be two critical walks, and let $T = V(c_1, c_2) \cup V(c_3, c_4)$. Then

(a) $\Sigma_{v \in T} m(v, g) \leqslant 1$ for any color $g \in Q$; and

(b) $\Sigma_{v \in T} f(v) \leqslant 8$.

Proof: (a) Suppose that $\Sigma_{v \in T} m(v, g) \geqslant 2$ for a color $g \in Q$. Then by Lemma 7 one may assume that

(i) $g \neq c_1, c_2, c_3, c_4$,

(ii) $m(u, g) = m(v, g) = 1$ for two vertices $u \in V(c_1, c_2) - V(c_3, c_4)$ and $v \in V(c_3, c_4) - V(c_1, c_2)$,

(iii) $m(w, g) = 0$ for every vertex $w \in T - \{u, v\}$.

Let u' be the vertex on $W(c_1, c_2)$ succeeding to x, then we claim that there is a color $g' \in M(u')$ such that $g' \neq c_1, c_2, c_3, c_4$. By Lemma 7 any color $g' \in M(u')$ is different from c_1 and c_3, and if $u' \neq y$ then $g' \neq c_2, c_4$. If $u' = y$, then $f(y) \geqslant 2$ (since two c_1-edges meet at y) and hence $\Sigma_{c \in Q} m(y, c) \geqslant 3$ by Lemma 3, so there is a color $g' \in M(y)$ with $g' \neq c_2, c_4$.

Similarly let v' be the vertex on $W(c_3, c_4)$ succeeding to x, then there is a color $g'' \in M(v')$ such that $g'' \neq c_1, c_2, c_3, c_4$.

We next claim that $u' \neq v'$. If $u' = v'$, then $u' \neq u$, v and there is a walk $W(g', g, u')$ not ending at u or v, say u. Switching the walk $W(g', g, u')$ would make $g \in M(u') \cap M(u)$ for two vertices u' and u on $W(c_1, c_2)$, contrary to Lemma 7.

We then claim one may assume that $g \in M(u') \cap M(v')$. If $u \neq u'$, then every walk $W(g', g, u')$ must end at u: otherwise, switching a walk $W(g', g, u')$ would make two vertices u' and u on $W(c_1, c_2)$ have a common missing color g, contrary to Lemma 7. Similarly if $v \neq v'$, then every walk $W(g'', g, v')$ must end at v. If $u \neq u'$, then switch a walk $W(g', g, u')$. If $v \neq v'$, then switch a walk $W(g'', g, v')$. Now we have $g \in M(u') \cap M(v')$.

We finally show that an f-coloring of all of G can be obtained, that is, a contradiction is derived. First consider the case when a walk $W(c_1, g, x)$ ends at v'. Switch it, then $g \in M(u') \cap M(x)$. Recolor the c_2-edge xu' on $W(c_1, c_2)$ with color g, and color edge e with c_2, completing the f-coloring of all of G. Next consider the case when a walk $W(c_1, g, x)$ does not end at v'. Switch it, then $g \in M(v') \cap M(x)$. Recolor the c_4-edge xv' on $W(c_3, c_4)$ with color g and color edge e with c_4, completing the f-coloring of all of G.

(b) By (a) above and Lemma 8(a) $\Sigma_{v \in T} f(v) \leqslant 8$. Q.E.D.

Let H' be the subgraph of G induced by the vertex set $S = V(a, b)$. By Lemma 12 one may assume that $m(x, c, G) = 1$ and there are at least two c-edges leaving H'. We will prove that if $|W(a, b)| = 6$ a contradiction occurs. We now have:

Lemma 15: If $|W(a, b)| = 6$, then $V(c, b) \subset S$.

Proof: Suppose $V(c, b) \not\subset S$. Since no b-edge leaves H', $V(c, b) - S$ contains at least two vertices. Since $|W(a, b)| = 6$, by Lemma 10(b) $f(S) = 7$. Therefore $\Sigma_{v \in S \cup V(c, b)} f(v) \geqslant 9$, contrary to Lemma 14(b). Q.E.D.

Let $W(a, b) = xv_1v_2v_3v_4v_5y$, where vertices $x, v_1, v_2, v_3, v_4, v_5, y$ may not be distinct. Since $x,\ y,\ v_1 \in V(c, b)$ and $V(c, b) \subset S$, by Lemma 9(a) no c-edge leaves H' from x, v_1 or y. Therefore the two c-edges leave H' at v_2, v_3, v_4 or v_5. We now have the following lemmas.

Lemma 16: If $|W(a, b)| = 6$, then there is no bc- or cb-alternating walk W that contains a c-edge leaving H' and exactly one of the three b-edges in $W(a, b)$.

Proof: Otherwise, the walk W contains either b-edge v_2v_3 or v_4v_5 together with two c-edges leave H' at its ends, and W does not pass through any other vertices in S. Switching W would make $|W(a, b)| \geqslant 8$, contrary to Lemma 13. Q.E.D.

Lemma 17: If $|W(a, b)| = 6$, then there is no ac- or ca-alternating walk W that contains a c-edge leaving H' and exactly one of the three a-edge in $W(a, b)$.

Proof: Otherwise, switching W would make $|W(a, b)| \geqslant 8$. Q.E.D.

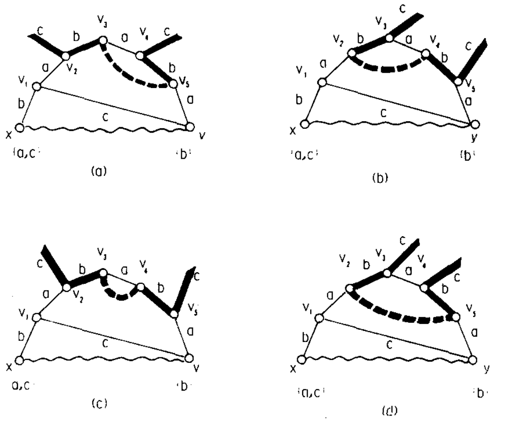

Fig. 1. The four cases of $W_i(b, c, v_2)$.

Lemma 18: If $|W(a, b)| = 6$, then $|W(c, b)| = 2$, and every bc- or cb-alternating walk $W_i(b, c, v_2)$ passing through v_2 must contain the two b-edges v_2v_3 and v_4v_5 in $W(a, b)$.

Proof: Note that $|W(c, b)| \leqslant |W(a, b)| = 6$. Suppose first that $|W(c, b)| = 6$. Then, since $V(c, b) \subset S$, $W(c, b)$ contains all the three b-edges in $W(a, b)$, and consequently, $V(c, b) = S$. Therefore by Lemma 9(a) no c-edge leaves H', a contradiction.

Suppose next that $|W(c, b)| = 4$. Then $W(c, b)$ contains exactly two of the three b-edges in $W(a, b)$, and hence there is a bc- or cb-alternating walk W that contains exactly one b-edge in $W(a, b)$ and c-edges leaving H', contrary to Lemma 16.

Thus $|W(c, b)| = 2$ and $W(c, b) = xv_1y$. By Lemma 16 every bc- or cb-alternating walk containing b-edge v_2v_3 or v_4v_5 must contain both. Therefore every walk $W_i(b, c, v_2)$ contains both of the b-edges together with two or four c-edges leaving H'. Q.E.D.

We are now ready to prove that a contradiction occurs when $|W(a, b)| = 6$.

Lemma 19: $|W(a, b)| \neq 6$.

Proof: Vertices v_2, v_3, v_4, and v_5 appear in a walk $W_i(b, c, v_2)$ in one of the four orders depicted in Figs. 1(a)–(d), where W_i is drawn in thick lines and a dashed line represents a subwalk of W_i having one or more edges. Since $|W(c, b)| = 2$, $f(x) = f(y) = f(v_1) = 1$ and hence x, v_1 and y are all distinct from the other vertices in S, although possibly $v_2 = v_4$, $v_2 = v_5$ or $v_3 = v_5$.

Case 1: Fig. 1(a) or (b).

Switch the walk $W_i(b, c, v_2)$, and subsequently switch the new ac-alternating cycle

$$W_i(a, c, v_1) = v_1v_2v_3v_4v_5yv_1.$$

Then $|W(b, c)| \geqslant 8$, a contradiction.

Case 2: Fig. 1(c) or (d).

Suppose that there is a c-edge v_2v_5. Then $v_1v_2v_5yv_1$ is an ac-alternating cycle, and there are two c-edges leaving H' at v_3 and v_4. Therefore, there exists a walk $W_i(a, c, v_3)$ which contains exactly one a-edge v_3v_4 in $W(a, b)$ and c-edges leaving H', contrary to Lemma 17. Thus there is no c-edge v_2v_5. Hence there are two c-edges v_2v_2' and v_5v_5' leaving H', where v_2', $v_5' \notin S$.

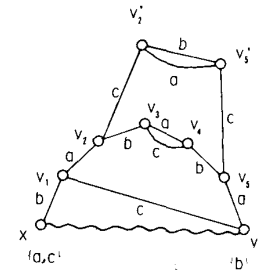

Fig. 2. Case $|W(a, b)| = 6$.

We claim $v_2' \neq v_5'$. Suppose that $v_2' = v_5'$. Switch a walk $W_i(b, c, v_2)$, then a new ab-critical path, if exists, passes through more than six edges including xv_1, v_1v_2, v_2v_2', $v_2'v_5$, and v_5y. Thus $|W(a, b)| > 6$, a contradiction. Therefore, $v_2' \neq v_5'$.

Similarly one can observe that there is an a-edge $v_2'v_5'$: otherwise, switching a walk $W_i(b, c, v_2)$ would make $|W(a, b)| > 6$.

There is a b-edge $v_2'v_5'$: otherwise, switching a walk $W_i(b, c, v_2)$ and subsequently switching the ac-alternating cycle $W_i(a, c, v_2) = v_2v_3v_4v_5yv_1v_2$ would make $|W(c, b)| > 6$.

Furthermore, there is a c-edge v_3v_4: otherwise, two c-edges leave H' at v_3 and v_4, and hence switching a walk $W_i(a, c, v_1)$ would make $|W(c, b)| > 6$.

Thus G has a subgraph depicted in Fig. 2. It should be noted that switching the bc-alternating cycle $W_i(b, c, v_2) = v_2v_2'v_5'v_5v_4v_3v_2$ makes $W(a, b)$ pass through v_2' and v_5'. Let $T = S \cup \{v_2', v_5'\}$, then $f(T) \geqslant 9$. Therefore by Lemma 8(a) there is a color $g \in Q$ such that $\Sigma_{v \in T} m(v, g) \geqslant 2$. Then we have $g \neq a, b, c$: otherwise, $g \in M(v_2')$ or $g \in M(v_5')$, and hence switching the cycle $W_i(b, c, v_2)$ would make

$$\Sigma_{v \in V(a, b)} m(v, g) > 2$$

for a new critical walk $W(a, b)$ passing through v_2' and v_5', contradicting Lemma 7(b). Furthermore, one may assume that $m(u, g) = m(v, g) = 1$ for two vertices $u = v_2'$ or v_5' and $v = v_3$ or v_4. Let h be an available color at y other than b, then $h \neq a, b, c, g$. If a walk $W(h, g, y)$ does not end at v, then switching it makes y and v on $W(a, b)$ have a common available color g, a contradiction. If a walk $W(h, g, y)$ ends at v, then switching it and subsequently switching the cycle $W_i(b, c, v_2)$ makes y and u on the new critical walk $W(a, b)$ have a common available color g, a contradiction. Q.E.D.

Thus we shall finally prove that a contradiction occurs when $|W(a, b)| = 4$. By Lemma 10(b) $f(S) = 5$. Let $W(a, b) = xv_1v_2v_3y$, where vertices x, v_1, v_2, v_3, y are not necessarily distinct from each other. Since $|W(c, b)| \leqslant |W(a, b)| = 4$, $|W(c, b)| = 2$ or 4. We now have:

Lemma 20: $|W(c, b)| = 4$.

Proof: Suppose $|W(c, b)| = 2$, then $f(x) = f(y) = f(v_1) = 1$ and two c-edges leave H' at v_2 and v_3. Every walk $W_i(b, c, v_2)$ is edge-disjoint with $W(c, b)$. Switching a

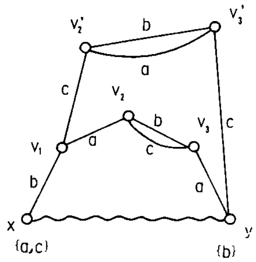

Fig. 3. Case $|W(a,b)|=4$.

walk $W_i(b,c,v_2)$ would make $|W(a,b)|\geqslant 6$, a contradiction. Q.E.D.

Since there are c-edges leaving $H', W(c,b)$ contains b-edge xv_1 but does not contain b-edge v_2v_3. Therefore $W(c,b)=xv_1v_2'v_3'y$ where $v_2',\ v_3'\notin S$ and $v_2'\neq v_3'$. Since $f(V(c,b))=5$ by Lemma 10(b), $f(v_2')=f(v_3')=1$. Possibly $v_1=y$, but all other vertices in $T=S\cup\{v_2',v_3'\}$ must be distinct from each other. Clearly $f(T)=7$. Denote by H'' the subgraph of G induced by the vertex set T. Thus $H'\subset H''\subset G$.

Lemma 21: There are a c-edge v_2v_3 and an a-edge $v_2'v_3'$.

Proof: Suppose that there is no c-edge v_2v_3. Then two c-edges v_2v_2'' and v_3v_3'' leave H' at v_2 and v_3, where $v_2'', v_3''\notin S$. Since a walk $W_i(b,c,v_2)$ is edge-disjoint with $W(b,c)$, switching a walk $W_i(b,c,v_2)$ would make $|W(a,b)|\geqslant 6$.

Suppose that there is no a-edge $v_2'v_3'$. Then, since $m(v_2',a)=m(v_3',a)=0$, there are two a-edges leaving H'' at v_2' or v_3'. Switching a walk $W_i(a,b,v_2')$ would make $|W(c,b)|\geqslant 6$. Q.E.D.

Thus G has a subgraph depicted in Fig. 3. Possibly $v_1=y$, but all other vertices of the subgraph are distinct from each other. Clearly $f(x)=f(v_2)=f(v_3)=f(v_2')=f(v_3')=1$. On the other hand $f(y)=f(v_1)=1$ if $y\neq v_1$; and $f(y)=2$ if $y=v_1$. One can derive a contradiction through an argument similar to the case of an ordinary edge-coloring [7], [11]. The detail is given in the Appendix for the patient reader.

VI. Algorithmic Remarks

The proof given in the preceding sections yields a polynomial-time algorithm which f-colors any given graph $G=(V,E)$ with at most $q_f(G)$ colors, where

$$q_f(G)=\max\left\{r_f(G),\left\lfloor\left(9d_f(G)+6\right)/8\right\rfloor\right\}.$$

In what follows we give some remarks on the algorithm.

The algorithm is iterative in a sense that it colors the edges of G one by one. As in [11] it is not necessary to compute $q_f(G)$ explicitly. Initially $q=\lfloor(9d_f(G)+6)/8\rfloor$ colors are available, and a new color is introduced only if $q_f(G)$ turns out to exceed the number q of the currently used colors. The algorithm colors an edge xy in four steps, each corresponding to the cases $|W(a,b)|=2,4,6$ or the case $|W(a,b)|\geqslant 8$. Each of the first three steps increases $|W(a,b)|$ two or more, and reduces to one of the latter cases. One can easily know that each of these steps repeats the switching of an alternating walk at most constant times. When we eventually have $|W(a,b)|\geqslant 8$, the last step makes x and y have a common available color. The step requires at most 10 switchings of alternating walks, since by Lemma 8(b), $W(a,b)$ contains two vertices having a common missing color among the first 10 vertices starting from x and each switching used in the proof of Lemma 7 decreases the distance between the two vertices at least one. Hence the algorithm colors one edge by repeating the switching of an alternating walk at most constant times.

Using data structures mentioned in [11] one can easily switch an alternating walk in $O(f(V)+d(G))$ time. Therefore the algorithm runs in $O(|E|(f(V)+d(G)))$ time. Furthermore it uses $O(|E|)$ storage space. If there is a vertex $v\in V$ such that $d(v)\leqslant f(v)$, an f-coloring of G can be immediately extended from an f-coloring of the graph $G-v$ obtained from G by deleting v. Thus one may assume that $f(v)\leqslant d(v)$ for every $v\in V$ and hence $f(V)\leqslant 2|E|$.

Since $d_f(G)$, $r_f(G)\leqslant q_f^*(G)$, the algorithm uses at most $\lfloor(9q_f^*(G)+6)/8\rfloor$ colors. Therefore, the number of colors used by the algorithm exceeds $9/8$ times of the minimum number $q_f^*(G)$ no more than one. Thus the asymptotic performance ratio of the algorithm is $9/8$.

VII. Conclusions

In this paper we obtained an upper bound on the f-chromatic index $q_f^*(G)$ of a multigraph G. The proof is constructive, and yields an $O(|E|^2)$ algorithm to f-color G with at most $\lfloor(9q_f^*(G)+6)/8\rfloor$ colors. Using the algorithm one can find near optimal solutions to scheduling problems like the file transfer problem on a computer network.

Finally we mention some conjectures. Our upper bound is a generalization of Goldberg's upper bound on the ordinary chromatic index $q^*(G)$. Another upper bound on $q^*(G)$ slightly better than Goldberg's has been given [14]. We conjecture that the bound also can be generalized to the case of the f-chromatic index. Concerning to the ordinary edge-coloring, Goldberg and Seymour had a conjecture that the bound $q^*(G)\leqslant\max\{d(G)+1,\ r(G)\}$ would hold for any multigraph G [8], [15]. (By Vizing's theorem [4], this inequality holds for any simple graph G.) We have a conjecture that the bound

$$q_f^*(G)\leqslant\max\left\{d_f(G)+1,r_f(G)\right\}$$

would hold for any multigraph G.

Appendix

The Rest of Proof

Similarly to Lemma 11 we obtain the following lemma.

Lemma 22: Let $T=V(a,b)\cup V(c,b)$, and let H'' be the subgraph induced by the vertex set T. Then there exists a color $g\in Q$ such that $\Sigma_{v\in T}m(v,g,H'')\geqslant 3$.

By Lemma 14 $\Sigma_{v \in T} m(v, g, G) \leqslant 1$. Now either (i) or (ii) below holds:

(i) $\Sigma_{v \in T} m(v, g, G) = 1$ and at least two g-edges leave H'';

(ii) $\Sigma_{v \in T} m(v, g, G) = 0$ and at least three g-edges leave H''.

Case (i): We separate this case into three subcases.

Case (i)-1: $m(v_1, g, G) = 1$, or $m(y, g, G) = 1$:

We first show that the case of $m(v_1, g, G) = 1$ can be reduced to the case of $m(y, g, G) = 1$. Suppose $m(v_1, g, G) = 1$. Then erase the color of b-edge xv_1, and subsequently color edge $e = xy$ with color b. Now the roles of y and v_1 have been interchanged.

Thus we may assume that $m(y, g, G) = 1$. Since the longest critical walk $W(a, b)$ has length four, $|W(a, g)| \leqslant 4$ and $|W(c, g)| \leqslant 4$. Since $f(T) = 7$, $V(a, g) \cup V(c, g) \subset T$: otherwise, $f(T \cup V(a, g) \cup V(c, g)) \geqslant 9$ and hence by Lemma 8(a) there is a color $h \in Q$ such that $\Sigma_{v \in T \cup V(a, g) \cup V(c, g)} m(v, h) \geqslant 2$, contradicting Lemma 14(a). Hence no g-edges leave H'' at x, v_3, or v_3'. Thus the two g-edges leave H'' at v_1, v_2, or v_2', and consequently $W(a, g)$ does not contain a-edge $v_1 v_2$. (Note that $f(v_2) = f(v_2') = 1$.) Switching a walk $W_i(a, g, v_1)$ makes $|W(a, b)| \geqslant 6$, a contradiction.

Case (i)-2: $m(x, g, G) = 1$:

In this case $|W(g, b)| \leqslant 4$ and $V(g, b) \subset T$. Since no g-edges leave H'' at x, y or v_1, the two g-edges leave H'' at v_2, v_3, v_2' or v_3'. Due to the symmetry of colors a and c in H'', one may assume that a g-edge leaves H'' at v_2 or v_3. The walk $W_i(b, g, v_2)$ that contains the b-edge $v_2 v_3$ and the g-edge leaving H'' does not contain b-edge xv_1. Switching $W_i(b, g, v_2)$ would make $|W(a, b)| \geqslant 6$.

Case (i)-3: $m(v_2, g, G) = 1$, $m(v_3, g, G) = 1$, $m(v_2', g, G) = 1$ or $m(v_3', g, G) = 1$:

Due to the symmetry of colors a and c, one may assume that $m(v_2', g, G) = 1$ or $m(v_3', g, G) = 1$. Furthermore, by an argument similar to Case (i)-1, one may assume that $m(v_3', g, G) = 1$. Let J be the set of vertices in a walk $W(g, b, v_3')$. We first show that the walk $W(g, b, v_3')$ ends at y and satisfies $|W(g, b, v_3')| \leqslant 4$ and $\Sigma_{w \in J \cup V(b, c)} m(w, h, G) \leqslant 1$ for any color $h \in Q$. Otherwise, erasing the color of c-edge $v_3' y$ and subsequently coloring edge e with color c would make $|W(g, b)| \geqslant 6$ or $\Sigma_{w \in V(g, b) \cup V(b, c)} m(w, h, G) \geqslant 2$, a contradiction.

We next show that no g-edge leaves H'' at v_2' or y. Suppose to the contrary that a g-edge leaves H'' at v_2' or y. Then $\Sigma_{w \in J \cup T} f(w) \geqslant 9$. Therefore by Lemma 8 there are two vertices $u \in J - T$ and $v \in T - J$ having a common available color $h \neq a, b, c$. Since $\Sigma_{w \in J \cup V(b, c)} m(w, h, G) \leqslant 1$, v is either v_2 or v_3. Let d be an available color at y other than b. Switch a walk $W(d, h, y)$. If $W(d, h, y)$ did not end at v, then (by switching it) there are two vertices v and y on $W(a, b)$ having a common available color h, contrary to Lemma 7. If $W(d, h, y)$ ended at v, then erasing the color of c-edge $v_3' y$ and coloring edge e with color c would produce two vertices y and u having a common available color h on the new critical walk $W(g, b)$, contrary to Lemma 7.

We next show that no g-edge leaves H'' at x or v_1. Otherwise, switching the walk $W(g, b, v_1')$, erasing color c from c-edge $v_3' y$, and coloring edge e with c would make $|W(c, b)| \geqslant 6$.

Thus the two g-edges leave H'' at v_2 and v_3, and there is no other g-edge leaving H''. Therefore only the b-edge $v_2 v_3$ in H'' is contained in $W_i(b, g, v_2)$. Then switching $W_i(b, g, v_2)$ would make $|W(a, b)| \geqslant 6$, a contradiction.

Case (ii): In this case at least three g-edges leave H''. Let uu' be the g-edge leaving H'' such that $u \in T - \{x\}$ and $u' \notin T$. Let $h \neq a, b, c, g$ be an available color at u. One may assume that no h-edge leaves H'': otherwise, Case (i) above would hold with respect to color h. Let vv' be the last g-edge leaving H'' in a walk $W(h, g, u)$, where $v \in T$ and $v' \notin T$.

We first show that $u \neq v$. Otherwise, switch a walk $W(h, g, u)$ containing exactly one g-edge leaving H''. Then color g is available at u and there remain at least two g-edges leaving H''. Thus Case (i) would apply.

We next show that $v = x$. Suppose $v \neq x$. Let W' be the latter half of the walk $W(h, g, u)$ starting with edge vv'. Let i be an available color at v such that $i \neq a, b, c, g, h$. One may assume that no i-edge leaves H'' and every walk $W(i, h, v)$ ends at u. Switch a walk $W(i, h, v)$; then color h is available at v and W' becomes a walk $W(h, g, v)$. Switch the walk $W(h, g, v)$, then color g is available at v and there remain at least two g-edges leaving H''. Thus Case (i) would apply.

We next show that none of the walks $W(a, g, x)$ and $W(c, g, x)$ contains any vertex of H'' except x. Otherwise, due to the symmetry of colors a and c in H'', one may assume that a walk $W(a, g, x)$ contains a vertex of H'' different from x. Let ww' be the last g-edge on the walk $W(a, g, x)$ which leaves H'', where $w \in T$ and $w' \notin T$. Let $i \in M(w)$ with $i \neq a$, b, c, g, and let W' be the latter half of the walk $W(a, g, x)$ starting with edge ww'. No a- or i-edge leaves H''. Recolor all a-edges in H'' with color i and all i-edges in H'' with color a. Then the previous ab-critical walk $W(a, b)$ becomes the ib-critical walk $W(i, b)$, and W' becomes a walk $W(a, g, w)$ since $a \in M(w)$. Switch the walk $W(a, g, w)$, then $|W(i, b)| = 4$, $|W(c, b)| = 4$, $g \in M(w)$ and at least two g-edges leave H''. Thus Case (i) would apply.

We then show that no g-edges leaves H'' at v_2, v_3, v_2' or v_3'. Otherwise, due to the symmetry of colors a and c in H'', one may assume that a g-edge leaves H'' at v_2' or v_3'. Switch a walk $W(a, g, x)$, then g is available at x. Clearly $|W(g, b)| \leqslant 4$, and hence $W(g, b)$ must not contain b-edge $v_2' v_3'$. Switching a walk $W_i(g, b, v_2')$ would make $|W(c, b)| \geqslant 6$, a contradiction.

Thus there are three g-edges xx', yy', $v_1 v_1'$ leaving H'', where x', y', $v_1' \notin T$. One may assume that $v_1' \neq y'$ and there is a b-edge $v_1' y'$: otherwise, switching a walk $W(a, g, x)$ would make $|W(g, b)| \geqslant 6$.

Since $\Sigma_{v \in T \cup \{v_1', y'\}} f(v) \geqslant 9$, there are two vertices $u_1 \in T$ and $u_2 \in \{v_1', y'\}$ having a common available color. If

$u_1 \in V(a, b)$, then switching $W(c, g, x)$ makes $V(g, b) \cup V(a, b)$ contain two vertices u_1 and u_2 having a common available color. If $u_1 \in V(c, b)$, then switching $W(a, g, x)$ makes $V(g, b) \cup V(c, b)$ contain two vertices having a common available color. Either case contradicts Lemma 14.

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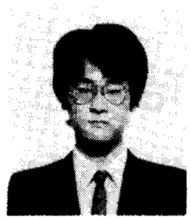

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