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On the Absence of Limit Cycles in State-Space Digital Filters With Minimum L_2-Sensitivity

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***Abstract*—This brief proposes a systematic approach to synthesis of limit cycle free state-space digital filters with minimum L_2-sensitivity. We synthesize the minimum L_2-sensitivity realization adopting the balanced realization as an initial realization. The coordinate transformation matrix which transforms the balanced realization into the minimum L_2-sensitivity realization is expressed as the product of a positive definite symmetric matrix and arbitrary orthogonal matrix. We show that the controllability and observability Gramians of the minimum L_2-sensitivity realization satisfy a sufficient condition for the absence of limit cycles when we select an appropriate orthogonal matrix. As a result, the minimum L_2-sensitivity realization without limit cycles can be synthesized by selecting an appropriate orthogonal matrix.**

***Index Terms*—Controllability Gramian, limit cycles, minimum L_2-sensitivity realization, observability Gramian, state-space digital filters.**

I. Introduction

On the fixed-point implementation of digital filters, undesirable finite-word-length (FWL) effects arise. Limit cycles occur in recursive digital filters implemented with FWL due to the nonlinear action of adder overflow and quantization of the products.

Some filter realizations are known to be free of limit cycles. For example, the balanced realization and the minimum roundoff noise realization are the minimum L_1/L_2-sensitivity realizations without L_2-scaling constraints and subject to L_2-scaling constraints, respectively, and do not generate limit cycles [1], [2]. However, it would be more natural to use L_2-sensitivity than to use L_1/L_2-sensitivity as a coefficient sensitivity since L_2-sensitivity measure is formulated without any approximation while L_1/L_2-sensitivity is formulated with approximation. It has not been investigated whether the minimum L_2-sensitivity realization generates limit cycles or not. Therefore, it is worth investigating the limit cycles of the minimum L_2-sensitivity realization.

To the L_2-sensitivity minimization problem, Yan *et al.* [3] and Hinamoto *et al.* [4] proposed solutions using iterative calculations. Both of the solutions in [3] and [4] try to solve nonlinear equations by successive approximation. On the other hand, our group proposed a closed form solution to the L_2-sensitivity minimization problem of second-order state-space digital filters [5].

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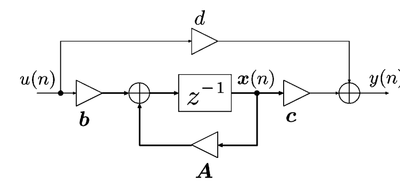

Fig. 1. Block diagram of a state-space digital filter.

The authors in [3]–[5], however, have not investigated the limit cycles of the minimum L_2-sensitivity realization.

In this brief, we shall prove the absence of limit cycles of state-space digital filters with minimum L_2-sensitivity. The minimum L_2-sensitivity realizations have freedom for orthogonal transformations. In other words, minimum L_2-sensitivity realizations are not unique. We select the minimum L_2-sensitivity realization without limit cycles among these minimum L_2-sensitivity realizations. The controllability and observability Gramians of the selected minimum L_2-sensitivity realization satisfy a sufficient condition for the absence of limit cycles.

II. Preliminaries

A. State-Space Digital Filters

Consider a stable, controllable, and observable Nth-order state-space digital filter described by

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{b}u(n) \tag{1}$$

$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) + du(n) \tag{2}$$

where $\boldsymbol{x}(n) \in \boldsymbol{R}^{N\times 1}$ is a state-vector, $u(n) \in \boldsymbol{R}$ is a scalar input, $y(n) \in \boldsymbol{R}$ is a scalar output, and $\boldsymbol{A} \in \boldsymbol{R}^{N\times N}$, $\boldsymbol{b} \in \boldsymbol{R}^{N\times 1}$, $\boldsymbol{c} \in \boldsymbol{R}^{1\times N}$, $d \in \boldsymbol{R}$ are real constant matrices called coefficient matrices. The block diagram of the state-space digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ is shown in Fig. 1. The transfer function $H(z)$ is described in terms of the coefficient matrices $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ as $H(z) = \boldsymbol{c}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d$.

B. L_2-Sensitivity

The L_2-sensitivity of the filter $H(z)$ with respect to the realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ is defined by

$$S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \left\| \frac{\partial H(z)}{\partial \boldsymbol{A}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{b}} \right\|_2^2 + \left\| \frac{\partial H(z)}{\partial \boldsymbol{c}} \right\|_2^2. \tag{3}$$

Hinamoto *et al.* [4] expressed the L_2-sensitivity in terms of the general Gramians such as

$$S(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \mathrm{tr}(\boldsymbol{W}_0)\mathrm{tr}(\boldsymbol{K}_0) + \mathrm{tr}(\boldsymbol{W}_0) + \mathrm{tr}(\boldsymbol{K}_0) + 2\sum_{i=1}^{\infty} \mathrm{tr}(\boldsymbol{W}_i)\mathrm{tr}(\boldsymbol{K}_i) \tag{4}$$

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where $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ are the controllability and observability Gramians, respectively, $\boldsymbol{K}_i$ and $\boldsymbol{W}_i$ are the general controllability and observability Gramians, respectively. The controllability Gramian $\boldsymbol{K}_0$ and the observability Gramian $\boldsymbol{W}_0$ can be calculated by solving the following Lyapunov equations:

$$\boldsymbol{K}_0 = \boldsymbol{A}\boldsymbol{K}_0\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \tag{5}$$
$$\boldsymbol{W}_0 = \boldsymbol{A}^T\boldsymbol{W}_0\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c}. \tag{6}$$

Our group previously proposed a novel expression of general Gramians as follows [5]:

$$\boldsymbol{K}_i = \frac{1}{2}\left(\boldsymbol{A}^i\boldsymbol{K}_0 + \boldsymbol{K}_0(\boldsymbol{A}^T)^i\right) \tag{7}$$
$$\boldsymbol{W}_i = \frac{1}{2}\left(\boldsymbol{W}_0\boldsymbol{A}^i + (\boldsymbol{A}^T)^i\boldsymbol{W}_0\right). \tag{8}$$

We only need to solve the initial Gramians $\boldsymbol{K}_0$ and $\boldsymbol{W}_0$ by the Lyapunov equations (5) and (6) in order to calculate the general Gramians.

C. Sufficient Condition for the Absence of Limit Cycles

Under zero input condition, the following state-space equations are obtained:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) \tag{9}$$
$$y(n) = \boldsymbol{c}\boldsymbol{x}(n) \tag{10}$$

by letting $u(n) = 0$ in (1) and (2). Equation (9) describes the autonomous behavior of the state-space digital filter. When this digital filter is stable, we have $\lim_{n\to\infty}\boldsymbol{x}(n) = \boldsymbol{0}$ for any initial state $\boldsymbol{x}(0)$. However, the actual digital filters implemented by finite word-length have nonlinearities due to adder overflow and quantization errors. For recursive digital filters, these nonlinearities cause *limit cycles*, which can be classified into *overflow limit cycles* and *granular limit cycles*. Adder overflow causes large-amplitude autonomous oscillations, which is called *overflow limit cycles*. On the other hand, quantization causes small-amplitude autonomous oscillations, which is called *granular limit cycles*.

The state transition of the digital filter considering the overflow is described by

$$\boldsymbol{x}(n+1) = f\left[\boldsymbol{A}\boldsymbol{x}(n)\right] \tag{11}$$

where f is a nonlinear function describing overflow characteristic. The nonlinear function f satisfies

$$|f_i(x_i)| \le |x_i|. \tag{12}$$

Overflow characteristics (two's complement, saturation, and zeroing) satisfy the above inequality. It is known that nonlinearity of the quantization using signed-magnitude truncation after addition is also described by the function f which satisfies the inequality (12).

Under the conditions described by (11) and (12), some sufficient conditions for state-space digital filters to be free of limit cycles have been proposed by Lyapunov approach [1], [6]–[10]. In [1], a sufficient condition for the absence of the limit cycles is given in terms of the controllability and observability Gramians as follows.

The transition matrix $\boldsymbol{A}$ of an Nth-order state-space digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ satisfies

$$\boldsymbol{D} - \boldsymbol{A}^T\boldsymbol{D}\boldsymbol{A} > 0 \tag{13}$$

if the controllability Gramian $\boldsymbol{K}_0$ and observability Gramian $\boldsymbol{K}_0$ has the following relation:

$$\boldsymbol{W}_0 = \xi^2\boldsymbol{D}\boldsymbol{K}_0\boldsymbol{D} \tag{14}$$

for a positive definite diagonal matrix $\boldsymbol{D}$ and a real scalar ξ.[1]

Equation (13) means that the Lyapunov function $\boldsymbol{x}^T(n)\boldsymbol{D}\boldsymbol{x}(n)$ is monotonically decreasing. It is already known that (13) is a sufficient condition for the absence of limit cycles [7]. Therefore, the state-space digital filter satisfying (14) is free of limit cycles [1].

III. L_2-Sensitivity Minimization Problem

In this section, we introduce the formulation of L_2-sensitivity minimization problem and solutions to the L_2-sensitivity minimization problem.

A. Formulation of the L_2-Sensitivity Minimization Problem

Let $\boldsymbol{T}$ be a nonsingular $N \times N$ real matrix. If a coordinate transformation defined by $\bar{\boldsymbol{x}}(n) = \boldsymbol{T}^{-1}\boldsymbol{x}(n)$ is applied to a filter realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$, we obtain a new realization which has the following coefficient matrices:

$$\left[\begin{array}{c|c} \bar{\boldsymbol{A}} & \bar{\boldsymbol{b}} \\ \hline \bar{\boldsymbol{c}} & d \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} & \boldsymbol{T}^{-1}\boldsymbol{b} \\ \hline \boldsymbol{c}\boldsymbol{T} & d \end{array}\right] \tag{15}$$

and the following general Gramians:

$$(\bar{\boldsymbol{K}}_i, \bar{\boldsymbol{W}}_i) = (\boldsymbol{T}^{-1}\boldsymbol{K}_i\boldsymbol{T}^{-T}, \boldsymbol{T}^T\boldsymbol{W}_i\boldsymbol{T}) \tag{16}$$

respectively. It should be noted that the coordinate transformation does not affect the transfer function $H(z)$. It implies that there exist infinite realizations for a given transfer function $H(z)$ since nonsingular $N \times N$ matrices exist infinitely. Therefore, one can synthesize infinite filter realizations by the coordinate transformation while keeping the transfer function invariant. The value of L_2-sensitivity depends on not only the transfer function $H(z)$ but also the coordinate transformation matrix $\boldsymbol{T}$. The L_2-sensitivity of the filter $(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{b}, \boldsymbol{c}\boldsymbol{T}, d)$ can be expressed in terms of the infinite summation of general Gramians as

$$S(\boldsymbol{P}) = \mathrm{tr}(\boldsymbol{W}_0\boldsymbol{P})\mathrm{tr}(\boldsymbol{K}_0\boldsymbol{P}^{-1}) + \mathrm{tr}(\boldsymbol{W}_0\boldsymbol{P}) + \mathrm{tr}(\boldsymbol{K}_0\boldsymbol{P}^{-1}) + 2\sum_{i=1}^{\infty}\mathrm{tr}(\boldsymbol{W}_i\boldsymbol{P})\mathrm{tr}(\boldsymbol{K}_i\boldsymbol{P}^{-1}) \tag{17}$$

[1]In this context, it may be mentioned that, pertaining to saturation overflow arithmetic, some less restrictive conditions than (13) for the elimination of overflow oscillations have been obtained, see, for instance, [8]–[10] and the references cited therein.

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where $\boldsymbol{P}$ is a positive definite symmetric matrix defined by $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$. We call the positive definite symmetric matrix which gives the global minimum of $S(\boldsymbol{P})$ *the optimal positive definite symmetric matrix* $\boldsymbol{P}_{\rm opt}$. The L_2-sensitivity minimization problem is formulated as follows: For an initial digital filter $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ with a given transfer function $H(z)$, minimize the L_2-sensitivity $S(\boldsymbol{P})$ with respect to $\boldsymbol{P}$, where $\boldsymbol{P}$ is an arbitrary positive definite symmetric matrix.

B. Solutions to the L_2-Sensitivity Minimization Problem

Several approaches to solve the L_2-sensitivity minimization problem are proposed [3]–[5]. We have to derive the optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ which satisfies

$$\left.\frac{\partial S(\boldsymbol{P})}{\partial \boldsymbol{P}}\right|_{\boldsymbol{P}=\boldsymbol{P}_{\rm opt}} = \boldsymbol{0}. \tag{18}$$

The optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ minimizes the L_2-sensitivity $S(\boldsymbol{P})$.

For high-order digital filters, we can solve the L_2-sensitivity minimization problem by using iterative calculations [3], [4]. On the other hand, for second-order digital filters, we previously proposed a closed form solution to L_2-sensitivity minimization of second-order state-space digital filters [5]. We first synthesize the balanced realization $(\boldsymbol{A}_{\rm b}, \boldsymbol{b}_{\rm b}, \boldsymbol{c}_{\rm b}, d_{\rm b})$ as an initial digital filter. The controllability Gramian $\boldsymbol{K}_{0\rm b}$ and the observability Gramian $\boldsymbol{W}_{0\rm b}$ of the balanced realization $(\boldsymbol{A}_{\rm b}, \boldsymbol{b}_{\rm b}, \boldsymbol{c}_{\rm b}, d_{\rm b})$ are given by

$$\boldsymbol{K}_{0\rm b} = \boldsymbol{W}_{0\rm b} = \boldsymbol{\Theta}\,(= \mathrm{diag}(\theta_1, \cdots, \theta_N)) \tag{19}$$

where $\theta_1, \cdots, \theta_N$ are the second-order modes. It is proved that the positive definite symmetric matrix $\boldsymbol{P}$ which minimizes the L_2-sensitivity $S(\boldsymbol{P})$ is expressed as

$$\boldsymbol{P} = \begin{bmatrix} \cosh(p) & \sinh(p) \\ \sinh(p) & \cosh(p) \end{bmatrix} \tag{20}$$

where p is a real scalar. Substituting (20) into (17) yields the L_2-sensitivity $S(p)$, a function of the scalar parameter p, as follows:

$$S(\boldsymbol{P}) = S(p) = \sum_{n=-2}^{2} c_n e^{np} \tag{21}$$

which does not contain infinite summations. These coefficients $c_n (n = -2, -1, 0, 1, 2)$ are easily computed directly from the transfer function $H(z)$. We derive the parameter $p_{\rm opt}$ which minimizes $S(p)$ in (21) by solving a fourth-degree equation, and obtain the optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ as follows:

$$\begin{aligned}\boldsymbol{P}_{\rm opt} &= \begin{bmatrix} \cosh(p_{\rm opt}) & \sinh(p_{\rm opt}) \\ \sinh(p_{\rm opt}) & \cosh(p_{\rm opt}) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \beta_{\rm opt} + \beta_{\rm opt}^{-1} & \beta_{\rm opt} - \beta_{\rm opt}^{-1} \\ \beta_{\rm opt} - \beta_{\rm opt}^{-1} & \beta_{\rm opt} + \beta_{\rm opt}^{-1} \end{bmatrix}\end{aligned} \tag{22}$$

where $\beta_{\rm opt} = e^{p_{\rm opt}}$.

C. Synthesis of Minimum L_2-Sensitivity Realizations

For high-order digital filters, we can obtain the minimum L_2-sensitivity realization by successive approximation [3], [4]. On the other hand, for second-order digital filters, we can obtain the minimum L_2-sensitivity realization by closed form solution analytically [5]. We next state how to synthesize the minimum L_2-sensitivity using the optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ as the solution to the L_2-sensitivity minimization problem.

The relation between the optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ and the optimal coordinate transformation matrix $\boldsymbol{T}_{\rm opt}$ is given by

$$\boldsymbol{P}_{\rm opt} = \boldsymbol{T}_{\rm opt}\boldsymbol{T}_{\rm opt}^T. \tag{23}$$

Once the optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ is obtained, the optimal coordinate transformation matrix $\boldsymbol{T}_{\rm opt}$ is constructed as

$$\boldsymbol{T}_{\rm opt} = \boldsymbol{P}_{\rm opt}^{\frac{1}{2}}\boldsymbol{U} \tag{24}$$

where $\boldsymbol{U}$ is an *arbitrary* orthogonal matrix [3], [4]. When we adopt the balanced realization $(\boldsymbol{A}_{\rm b}, \boldsymbol{b}_{\rm b}, \boldsymbol{c}_{\rm b}, d_{\rm b})$ as an initial realization, the minimum L_2-sensitivity realization $(\boldsymbol{A}_{\rm opt}, \boldsymbol{b}_{\rm opt}, \boldsymbol{c}_{\rm opt}, d_{\rm opt})$ is given by

$$\left[\begin{array}{c|c} \boldsymbol{A}_{\rm opt} & \boldsymbol{b}_{\rm opt} \\ \hline \boldsymbol{c}_{\rm opt} & \boldsymbol{d}_{\rm opt} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{T}_{\rm opt}^{-1}\boldsymbol{A}_{\rm b}\boldsymbol{T}_{\rm opt} & \boldsymbol{T}_{\rm opt}^{-1}\boldsymbol{b}_{\rm b} \\ \hline \boldsymbol{c}_{\rm b}\boldsymbol{T}_{\rm opt} & \boldsymbol{d}_{\rm b} \end{array}\right]. \tag{25}$$

We have to note that the optimal coordinate transformation matrix $\boldsymbol{T}_{\rm opt}$ has freedom due to the arbitrariness of the orthogonal matrix. Therefore, the minimum L_2-sensitivity realization is not unique.

IV. Minimum L_2-Sensitivity Realization Without Limit Cycles

This section presents our main results, where we propose the novel method for synthesizing the minimum L_2-sensitivity realization *without limit cycles*.

A. High-Order Digital Filters

For high-order minimum L_2-sensitivity realization obtained by the iterative methods [3], [4], we can construct the minimum L_2-sensitivity realization *without limit cycles*.

We adopt the balanced realization $(\boldsymbol{A}_{\rm b}, \boldsymbol{b}_{\rm b}, \boldsymbol{c}_{\rm b}, d_{\rm b})$, whose controllability and observability Gramians are given by (19), as an initial digital filter. The optimal positive definite symmetric matrix $\boldsymbol{P}_{\rm opt}$ obtained by solving the L_2-sensitivity minimization problem using the successive approximation is decomposed as follows:

$$\boldsymbol{P}_{\rm opt} = \boldsymbol{R}^T\boldsymbol{B}\boldsymbol{R} \tag{26}$$

where $\boldsymbol{R}$ is an orthogonal matrix and $\boldsymbol{B}$ is a positive definite diagonal matrix. The optimal coordinate transformation matrix $\boldsymbol{T}_{\rm opt}$ is given by

$$\begin{aligned}\boldsymbol{T}_{\rm opt} &= \boldsymbol{P}_{\rm opt}^{\frac{1}{2}}\boldsymbol{U} \\ &= \boldsymbol{R}^T\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{U}.\end{aligned} \tag{27}$$

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In the above expression, $\boldsymbol{U}$ is an arbitrary orthogonal matrix. In order to synthesize the limit cycle free realization, we let $\boldsymbol{U} = \boldsymbol{R}^T$, which yields

$$\boldsymbol{T}_{\text{opt}} = \boldsymbol{R}^T \boldsymbol{B}^{\frac{1}{2}}. \tag{28}$$

We can show that the minimum L_2-sensitivity realization obtained by the optimal coordinate transformation matrix $\boldsymbol{T}_{\text{opt}}$ in (28) does not generate limit cycles. The proof is given as follows: under the coordinate transformation by $\boldsymbol{T}_{\text{opt}}$ in (28), the controllability Gramian $\boldsymbol{K}_{0\text{opt}}$ and the observability Gramian $\boldsymbol{W}_{0\text{opt}}$ are expressed as

$$\begin{aligned}\boldsymbol{K}_{0\text{opt}} &= \boldsymbol{T}_{\text{opt}}^{-1}\boldsymbol{K}_{0\text{b}}\boldsymbol{T}_{\text{opt}}^{-T}\\ &= \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T\boldsymbol{B}^{-\frac{1}{2}} \qquad (29)\\ \boldsymbol{W}_{0\text{opt}} &= \boldsymbol{T}_{\text{opt}}^{T}\boldsymbol{W}_{0\text{b}}\boldsymbol{T}_{\text{opt}}\\ &= \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T\boldsymbol{B}^{\frac{1}{2}} \qquad (30)\end{aligned}$$

where $\boldsymbol{\Theta} = \text{diag}(\theta_1, \cdots, \theta_N)$. From (29) and (30), we can derive the relation between the controllability and observability Gramians as follows:

$$\boldsymbol{W}_{0\text{opt}} = \boldsymbol{B}\boldsymbol{K}_{0\text{opt}}\boldsymbol{B}. \tag{31}$$

Equation (31) is equivalent to (14) with $\xi = 1$ and $\boldsymbol{D} = \boldsymbol{B}$. Therefore, we can synthesize the minimum L_2-sensitivity realization without limit cycles by choosing appropriate orthogonal matrix.

B. Second-Order Digital Filters

For second-order minimum L_2-sensitivity realization obtained by the closed form solution [5], we can also construct the minimum L_2-sensitivity realization *without limit cycles*. It is remarkable that the minimum L_2-sensitivity realization without limit cycles is derived in closed form in case of second-order digital filters.

We adopt the balanced realization $(\boldsymbol{A}_\text{b}, \boldsymbol{b}_\text{b}, \boldsymbol{c}_\text{b}, d_\text{b})$, whose controllability and observability Gramians are given by (19), as an initial digital filter. The optimal positive definite symmetric matrix $\boldsymbol{P}_{\text{opt}}$ obtained by solving the L_2-sensitivity minimization problem using the closed form solution is decomposed as follows:

$$\begin{aligned}\boldsymbol{P}_{\text{opt}} &= \frac{1}{2}\begin{bmatrix}\beta_{\text{opt}} + \beta_{\text{opt}}^{-1} & \beta_{\text{opt}} - \beta_{\text{opt}}^{-1}\\ \beta_{\text{opt}} - \beta_{\text{opt}}^{-1} & \beta_{\text{opt}} + \beta_{\text{opt}}^{-1}\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\beta_{\text{opt}} & 0\\ 0 & \beta_{\text{opt}}^{-1}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\\ &\equiv \boldsymbol{R}^T\boldsymbol{B}\boldsymbol{R} \qquad (32)\end{aligned}$$

where $\boldsymbol{R}$ is an orthogonal matrix which rotates coordinate axis $\pi/4$ [rad] and $\boldsymbol{B} = \text{diag}(\beta_{\text{opt}}, \beta_{\text{opt}}^{-1})$ is a positive definite diagonal matrix. The optimal coordinate transformation matrix $\boldsymbol{T}_{\text{opt}}$ is given by

$$\begin{aligned}\boldsymbol{T}_{\text{opt}} &= \boldsymbol{P}_{\text{opt}}^{\frac{1}{2}}\boldsymbol{U}\\ &= \boldsymbol{R}^T\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{U}. \qquad (33)\end{aligned}$$

In the above expression, $\boldsymbol{U}$ is an arbitrary orthogonal matrix. In order to synthesize the limit cycle free realization, we let $\boldsymbol{U} = \boldsymbol{R}^T$, which yields

$$\begin{aligned}\boldsymbol{T}_{\text{opt}} &= \boldsymbol{R}^T\boldsymbol{B}^{\frac{1}{2}}\\ &= \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\beta_{\text{opt}}^{\frac{1}{2}} & 0\\ 0 & \beta_{\text{opt}}^{-\frac{1}{2}}\end{bmatrix}. \qquad (34)\end{aligned}$$

We can show that the minimum L_2-sensitivity realization obtained by the optimal coordinate transformation matrix $\boldsymbol{T}_{\text{opt}}$ in (34) does not generate limit cycles. The proof is given as follows: under the coordinate transformation by $\boldsymbol{T}_{\text{opt}}$ in (34), the controllability Gramian $\boldsymbol{K}_{0\text{opt}}$ and the observability Gramian $\boldsymbol{W}_{0\text{opt}}$ are expressed as

$$\begin{aligned}\boldsymbol{K}_{0\text{opt}} &= \boldsymbol{T}_{\text{opt}}^{-1}\boldsymbol{K}_{0\text{b}}\boldsymbol{T}_{\text{opt}}^{-T}\\ &= \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T\boldsymbol{B}^{-\frac{1}{2}}\\ &= \frac{1}{2}\begin{bmatrix}\beta_{\text{opt}}^{-1}(\theta_1+\theta_2) & -\theta_1+\theta_2\\ -\theta_1+\theta_2 & \beta_{\text{opt}}(\theta_1+\theta_2)\end{bmatrix} \qquad (35)\\ \boldsymbol{W}_{0\text{opt}} &= \boldsymbol{T}_{\text{opt}}^{T}\boldsymbol{W}_{0\text{b}}\boldsymbol{T}_{\text{opt}}\\ &= \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{\Theta}\boldsymbol{R}^T\boldsymbol{B}^{\frac{1}{2}}\\ &= \frac{1}{2}\begin{bmatrix}\beta_{\text{opt}}(\theta_1+\theta_2) & -\theta_1+\theta_2\\ -\theta_1+\theta_2 & \beta_{\text{opt}}^{-1}(\theta_1+\theta_2)\end{bmatrix}. \qquad (36)\end{aligned}$$

From (35) and (36), we can derive the relation between the controllability and observability Gramians as follows:

$$\boldsymbol{W}_{0\text{opt}} = \boldsymbol{B}\boldsymbol{K}_{0\text{opt}}\boldsymbol{B}. \tag{37}$$

Equation (37) is equivalent to (14) with $\xi = 1$ and $\boldsymbol{D} = \boldsymbol{B}$. Therefore, we can synthesize the minimum L_2-sensitivity realization without limit cycles by choosing appropriate orthogonal matrix.

V. Numerical Example

We present a numerical example to demonstrate the validity of the proposed method. Consider a second-order narrowband bandpass digital filter $H(z)$ given by

$$H(z) = \frac{0.0040 + 0.0078z^{-1} + 0.0040z^{-2}}{1 - 1.8546z^{-1} + 0.9506z^{-2}}. \tag{38}$$

The poles of the transfer function (38) are $0.975\exp(\pm j0.1\pi)$, which are very close to the unit circle. The frequency amplitude response of the digital filter (38) is shown in Fig. 2. The minimum L_2-sensitivity realization which is free of limit cycle is given by

$$\left[\begin{array}{c|c}\boldsymbol{A}_{\text{opt}} & \boldsymbol{b}_{\text{opt}}\\ \hline \boldsymbol{c}_{\text{opt}} & d_{\text{opt}}\end{array}\right] = \left[\begin{array}{cc|c}0.9273 & 0.3015 & 0.0345\\ -0.3011 & 0.9273 & 0.2202\\ \hline 0.2202 & 0.0345 & 0.0040\end{array}\right]. \tag{39}$$

$\boldsymbol{K}_{0\text{opt}}$ and $\boldsymbol{W}_{0\text{opt}}$ are given as follows:

$$\boldsymbol{K}_{0\text{opt}} = \begin{bmatrix}0.5006 & 0.0410\\ 0.0410 & 0.5066\end{bmatrix} \tag{40}$$

$$\boldsymbol{W}_{0\text{opt}} = \begin{bmatrix}0.5066 & 0.0410\\ 0.0410 & 0.5006\end{bmatrix}. \tag{41}$$

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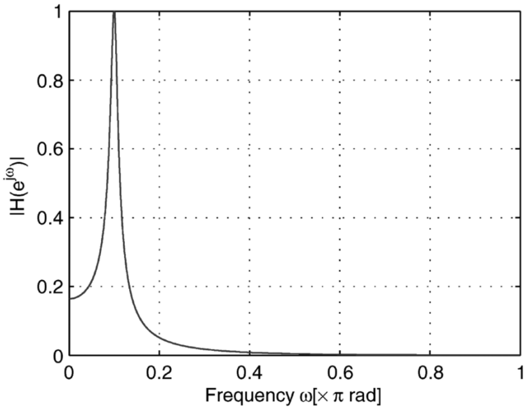

Fig. 2. Frequency response of digital filter (38).

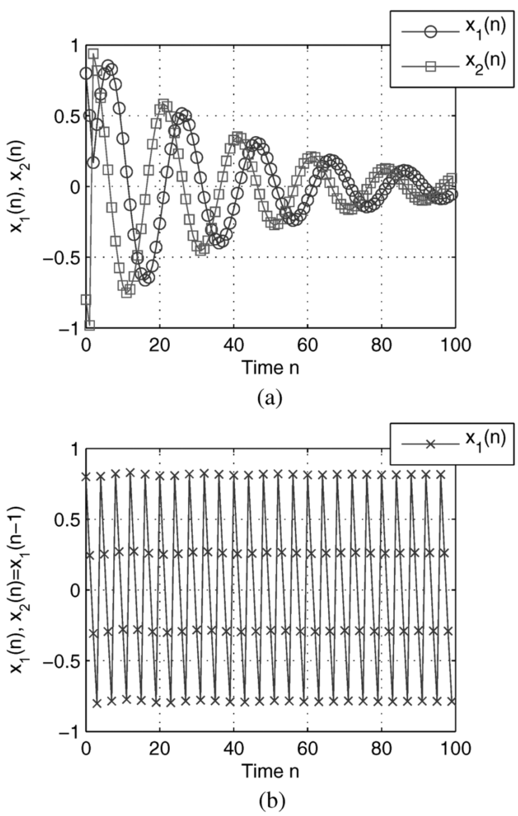

Fig. 3. Zero-input responses. $x_1(n)$ and $x_2(n)$ are state variables denoted by $\boldsymbol{x}(n) = [x_1(n)\ x_2(n)]^T$. (a) Minimum L_2-sensitivity realization. (b) Direct Form II.

We have to note that the controllability Gramian $\boldsymbol{K}_{0\mathrm{opt}}$ and the observability Gramian $\boldsymbol{W}_{0\mathrm{opt}}$ satisfy the sufficient condition of the absence of limit cycles given in (37) with

$$\boldsymbol{B} = \mathrm{diag}(1.0060, 0.9941). \tag{42}$$

Therefore, $(\boldsymbol{A}_{\mathrm{opt}}, \boldsymbol{b}_{\mathrm{opt}}, \boldsymbol{c}_{\mathrm{opt}}, d_{\mathrm{opt}})$ is the minimum L_2-sensitivity realization without limit cycles.

We demonstrate the absence of limit cycles in the minimum L_2-sensitivity realization by observing its zero-input response. We calculate the zero-input responses of the minimum L_2-sensitivity realization and the Direct Form II, setting the initial state as $\boldsymbol{x}(0) = [0.8 - 0.8]^T$. We let the dynamic range of signals to be $[-1, 1)$ and adopt two's complement as the overflow characteristic. The zero-input responses are shown in Fig. 3(a) and (b). In this numerical example, the overflow of the state variables occurs in both cases. It is desirable that the effect of the overflow is decreasing since the digital filter (38) is stable. For the minimum L_2-sensitivity realization synthesized by our proposed method, the state variables $x_1(n)$ and $x_2(n)$ converge to zero after the overflow, as shown in Fig. 3(a). Therefore, there are no limit cycles. On the other hand, for the Direct Form II, a large-amplitude autonomous oscillation is observed as shown in Fig. 3(a). The state variable $x_2(n)$ has the same behavior as $x_1(n)$ since $x_2(n) = x_1(n-1)$ in the Direct Form II. Therefore, the Direct Form II generates the limit cycles.

VI. Conclusion

This brief has discussed the synthesis of limit cycle free state-space digital filters with minimum L_2-sensitivity. We have shown that the controllability Gramian and observability Gramian of the minimum L_2-sensitivity realization satisfy a sufficient condition of the absence of limit cycles when we select the appropriate orthogonal matrix.

In this brief, we have discussed the absence of limit cycles of the minimum L_2-sensitivity realization *without L_2-scaling constraints*. However, it has been known that the use of L_2-scaling constraints can be beneficial in order to suppress the overflow of the internal state variables. Since the overflow of the internal state variables is serious and undesirable effects, it is better to consider the L_2-scaling constraints for further progress of our research. Our future work is thus to give theoretical proof of the absence of limit cycles of the minimum L_2-sensitivity realization *subject to L_2-scaling constraints*.

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