Nonequilibrium stationary coupling of solitons

doi: 10.1063/1.321554

Nonequilibrium stationary coupling of solitons

K. Nakajima, Y. Sawada, and Y. Onodera

Research Institute of Electrical Communication, Tohoku University, Sendai, Japan
(Received 20 December 1974; in final form 31 July 1975)

By computer simulation, interaction of two solitons with the same screw sense of the sine-Gordon equation, which retain their shapes and velocities upon collision with other solitons even in the presence of bias and loss terms, are examined. It is confirmed that they can couple when the bias is greater than a critical value. The conditions and mechanism of coupling are examined in detail. They are explained in terms of the energy of interaction between the ripple structures trailed by energy dissipative moving solitons. The distance D between coupled solitons can be expressed as $D=(n-1/2-\delta)\lambda$ (n is an integer), where λ is a wavelength of the ripple structure and $\delta \ll 1$.

PACS numbers: 84.40.M, 85.25.

I. INTRODUCTION

Research in recent years exhibits an increase of interest in the nonlinear partial differential equation, the sine-Gordon equation.[1] Mathematical properties and physical applications of the pulselike solitary wave as the solution of this nonlinear differential equation has been the subject for many investigators.[1–25] In particular, the sine-Gordon equation with dimensionless bias and loss terms which has been known to describe an active Josephson-junction transmission line has been studied by Scott,[10] Johnson,[13] Scott *et al.*,[14] and Barone.[15] They investigated the velocity, the stability, and various properties of propagating flux quanta which are described as solitons.[4,12] They showed that an arbitrary number of flux quanta propagate with permanent profiles, but that this state might exhibit a mode of instability of which individual fluxons detatch themselves from the trailing edge.[12] We have presented in our previous papers a mechanical analogue[18] and the numerical analysis[19] of the sine-Gordon equation with dimensionless bias and loss terms. In these papers, we reported that the soliton in a dissipative system is characterized by its final velocity $u(\gamma,\Gamma)$.[19] For a system with $\gamma=\Gamma=0$, the velocity of a soliton can be arbitrarily chosen, while in a lossy system with a bias, the final velocity is uniquely determined as a result of energy balance between the dissipative term and the bias term. This characteristic velocity $u(\gamma,\Gamma)$ is always recovered after a certain duration of time, so as long as solitary waves having opposite screw senses do not disappear upon collision.[19] In the preceding paper[19] we also reported that for a value of bias larger than a critical value, coupled solitons with the same screw sense were in stable states.

The purpose of the present paper is to make clear the coupling mechanism of two solitons with the same screw sense for a relatively large value of bias by using computer simulations. In this paper we discuss a γ-dependent periodic potential which is formed in the tail of solitons stably propagating on a line with bias and dynamic losses. It was found that for a relatively large value of γ, two propagating solitons with the same screw sense are coupling by attractive force caused by the periodic potentials created by the leading and following solitons in spite of the existence of a short-range repulsive force[1] acting between the two solitons. It was found that the distance D between the centers of two solitons can be described in the following form: $D=(n-\frac{1}{2}-\delta)\lambda$, where n is an integer which depends on the initial condition and δ is a small number compared to unity. In Sec. III we present numerical data on the periodic potential trailed by energy-absorbing solitons and the data on the coupled solitons with the same screw sense propagating with a constant velocity. In Sec. IV we discuss the nature of the trailed ripple structure created in tails of solitons, the short-range repulsive force acting between two solitons propagating with a constant velocity, the physical meaning of the entire potential for a following soliton, and the distance between coupled solitons.

II. COMPUTER SIMULATION

We study the behaviors of two solitons obeying the following nonlinear partial differential equation[18,19]:

$$\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial t^2}-\Gamma\frac{\partial\phi}{\partial t}=\sin\phi-\gamma. \tag{1}$$

Equation (1) describes a Josephson-junction transmission line which includes the effect of losses associated with normal electrons tunnelling through the junction and a distributed bias current source.[10,13] Equation (1) also describes approximately a mechanical transmission line which we have constructed and reported.[18] The solitons given by Eq. (1) correspond to the flux quanta in a Josephson-junction transmission line and the kinks on a mechanical line.[18] The following finite difference equation (2)[19] is used to calculate Eq. (1) using a NEAC 2200/700 digital computer in the computer center of Tohoku University:

$$\frac{\phi_{i+1,n}-2\phi_{i,n}+\phi_{i-1,n}}{\Delta x^2}-\frac{\phi_{i,n+1}-2\phi_{i,n}+\phi_{i,n-1}}{\Delta t^2}-\Gamma\frac{\phi_{i,n}-\phi_{i,n-1}}{\Delta t}=\sin\phi_{i,n}-\gamma. \tag{2}$$

We chose Eqs. (3) and (4)[1] as initial conditions for the calculation of Eq. (2).

$$\phi_{i,0}=4\tan^{-1}\exp[(-x_i+q)(1-u'^2)^{-1/2}]+4\tan^{-1}\exp[(-x_i+q')(1-u'^2)^{-1/2}]+\sin^{-1}\gamma; \tag{3}$$

$$\phi_{i,1}=4\tan^{-1}\exp[(-x_i+u'\Delta t+q)(1-u'^2)^{-1/2}]+4\tan^{-1}\exp[(-x_i+u'\Delta t+q')(1-u'^2)^{-1/2}]+\sin^{-1}\gamma. \tag{4}$$

Equation (3) would correspond to a superposition of two free solitons of a velocity u' located at $x=q$ and q' for $t=0$, which are biased with a trivial solution $\phi=\sin^{-1}\gamma$. A cyclic boundary condition is used such that the end

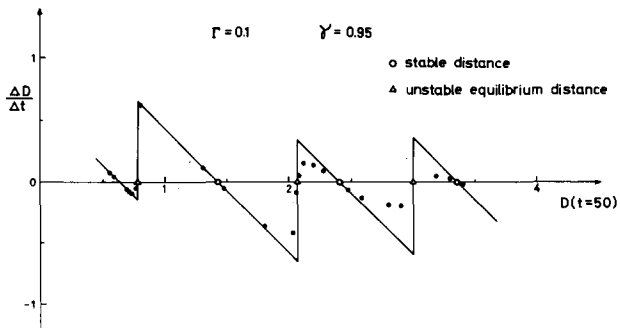

FIG. 1. Small change in the distance between two propagating solitons for $\gamma = 0.95$ from $t = 50$ to $t = 300$ as a function of the distance measured at $t = 50$, $\Delta D/\Delta t = [D(t=300) - D(t=50)]/250$. (Computer-simulated results, $\Delta x = 0.1$, $\Delta t = 0.05$).

effect should not disturb the results of calculation. The dimensionless small increment of space Δx and time Δt used in most cases are 0.1 and 0.05, respectively. The number of iterations in our calculations varies from 4000 to 24 000.

III. RESULTS

In the case of $\gamma = \Gamma = 0$, from Eqs. (2)—(4) it is shown that the two solitons with the same screw sense begin to propagate down with initial velocity u', and that the distance between the two solitons becomes greater with increasing time, owing to the repulsive force[1] acting between the two solitons. This result agrees with that given by Scott[12] and Hirota.[24] Therefore the total waveform of a two-soliton system changes with time, and thus the permanent profile of a two-soliton system does not exist.

In our numerical calculations under the conditions of $1 > \gamma > 0$ and $\Gamma = 0.1$, the distance D between the centers of two solitons with the same screw sense increased with time for the value of γ smaller than a critical value ($\gamma_c \sim 0.4$). For a value of γ greater than γ_c the distance between the two solitons tends to a constant value, and they apparently form a stable coupled state. We study in detail the propagation of coupled solitons with the same screw sense in the case of $\gamma = 0.95$, $\gamma = 0.7$, and $\Gamma = 0.1$. To see the time dependence of the distance D between two propagating solitons with the same screw sense, we calculated the positions of two solitons for

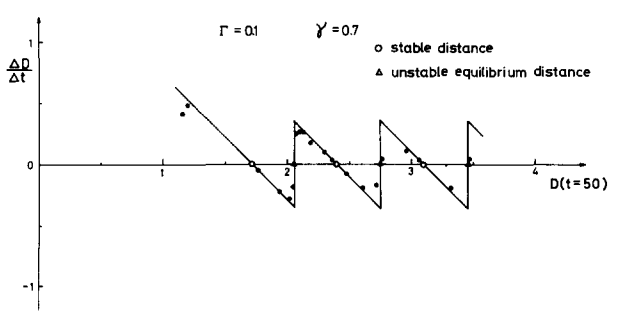

FIG. 2. Small change in the distance between two propagating solitons for $\gamma = 0.7$ from $t = 50$ to $t = 300$ as a function of the distance measured at $t = 50$, $\Delta D/\Delta t = [D(t=300) - D(t=50)]/250$. (Computer-simulated results, $\Delta x = 0.1$, $\Delta t = 0.05$).

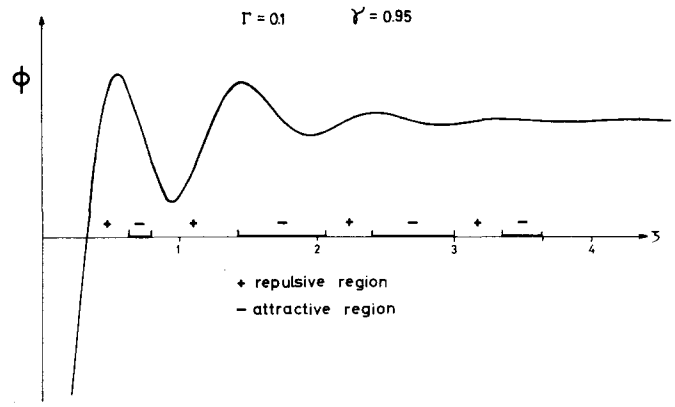

FIG. 3. Waveform of the tail of propagating soliton for $\gamma = 0.95$, $\Gamma = 0.1$, $\Delta x = 0.1$, and $\Delta t = 0.05$ as a function of dimensionless distance ξ measured from the center of the soliton. Attractive (−) and repulsive (+) regions, which are obtained from the analyses of Fig. 1, are also marked.

various values of initial distances D_i. The distance between two solitons become almost stable after $t = 50$, but still they adjust themselves slightly to settle down at the most favored distance taking a long time. The small change of the distance from $t = 50$ to $t = 300$ as a function of the distances measured at $t = 50$ are shown in Figs. 1 and 2 for $\gamma = 0.95$ and $\gamma = 0.7$, respectively. Figures 1 and 2 show the existence of attractive and repulsive regions for the force acting between two solitons depending upon their distance D. The region which has positive value of $\Delta D/\Delta t$ corresponds to a repulsive region, the region for negative $\Delta D/\Delta t$ corresponds to an attractive region, and stable and unstable equilibrium points are alternatively situated at the boundaries between repulsive and attractive regions. In order to understand these attractive and repulsive regions, it is necessary to examine carefully the waveforms of solitons obeying Eq. (2) in the presence of γ. Figures 3 and 4 show the waveform of the tail of propagating solitons for $\gamma = 0.95$ and $\gamma = 0.7$, respectively. From Figs. 1 and 2, the regions of attractive and repulsive forces acting on the following soliton are marked by + and − regions in Figs. 3 and 4. In Fig. 4 we try to separate

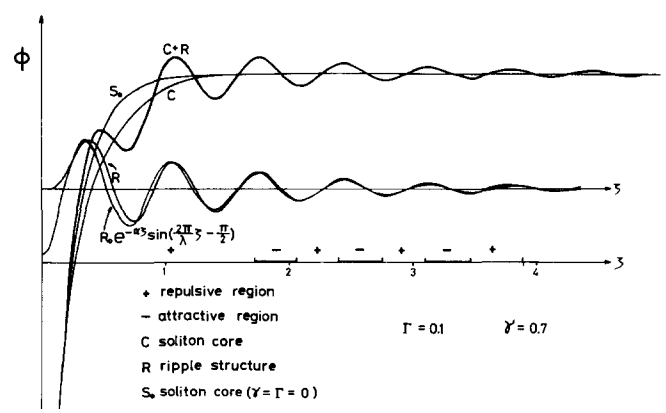

FIG. 4. Waveform of the tail of propagating soliton for $\gamma = 0.7$, $\Gamma = 0.1$, $\Delta x = 0.1$, and $\Delta t = 0.05$ marked by $(C+R)$ are separated into soliton core C and ripple structure R. Soliton S_0 for $\gamma = \Gamma = 0$ and, for reference, the function of the ripple structure are also drawn. Attractive and repulsive regions analyzed from Fig. 2 are marked.

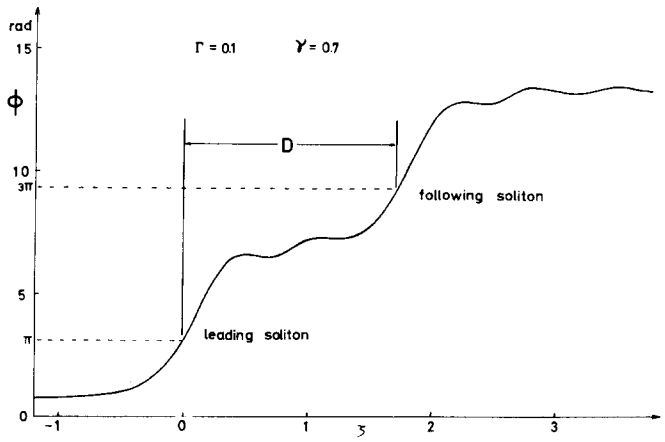

FIG. 5. Waveform of two propagating solitons obtained numerically from Eq. (2) with $\gamma=0.7$, $\Gamma=0.1$, $\Delta x=0.1$, and $\Delta t=0.05$ as a function of dimensionless distance ζ.

the complex structure of propagating soliton into "soliton core" C and "ripple structure" R. It can be seen from Figs. 3 and 4 that the attractive and repulsive regions are approximately periodic. This periodicity corresponds well to the periodicity of the oscillation in the tail of the soliton. This trailed ripple structure of the soliton is not included in the initial waveforms. As an example, in Fig. 5 a waveform of two propagating solitons is shown which are obtained numerically from Eq. (2) with $\gamma=0.7$ and $\Gamma=0.1$. It can be seen from Figs. 4 and 5 that the center of the following soliton is situated at the boundary between the attractive and repulsive regions. From these results it seems reasonable to assume that the trailed ripple structure of the soliton generates the attractive and repulsive regions for the following soliton. Figure 6 shows the envelope lines of waves of ripple structures as a function of dimensionless distance ζ measured from the center of the soliton for various values of γ with $\Gamma=0.1$, $\Delta x=0.1$, and $\Delta t=0.05$. It can be seen from Fig. 6 that the amplitude of these oscillations decays with ζ and an envelope is approximated as $R_0\exp(-\alpha\zeta)$. Figure 7 shows the wavelength λ, the attenuation constant α, and the wave height R_0 of this oscillation of ripple structure as a function of the dimensionless bias γ for $\Gamma=0.1$. The wavelength λ, the attenuation constant α, and the wave height R_0 increase with increasing γ. The value of R_0 is nearly zero at $\gamma=0.4$ for this calculation. This result is consistent with our calculation that the coupling of two solitons cannot occur at the value of γ smaller than 0.4. It can be seen from Figs. 1, 2, and 7 that the stable distance D between two propagating solitons is described as $D=(n-\frac{1}{2}-\delta)\lambda$ (n is an integer), where δ is a small value compared to unity.

IV. DISCUSSION

For two solitons with the same screw sense which are in static state, it has been reported that each soliton is affected by the repulsive force from the other soliton.[1,2] But in the case of propagating solitons, each soliton is affected not only by the short-range repulsive force from the other soliton but also by a disordered field brought about by the motion of the other soliton. In the present work it is found that two solitons propagating nearby satisfying Eq. (2) can coupled in the presence of the bias γ. The propagation of coupled solitons under the existence of γ suggests that the stationary ripple structures created by the motions of leading and following

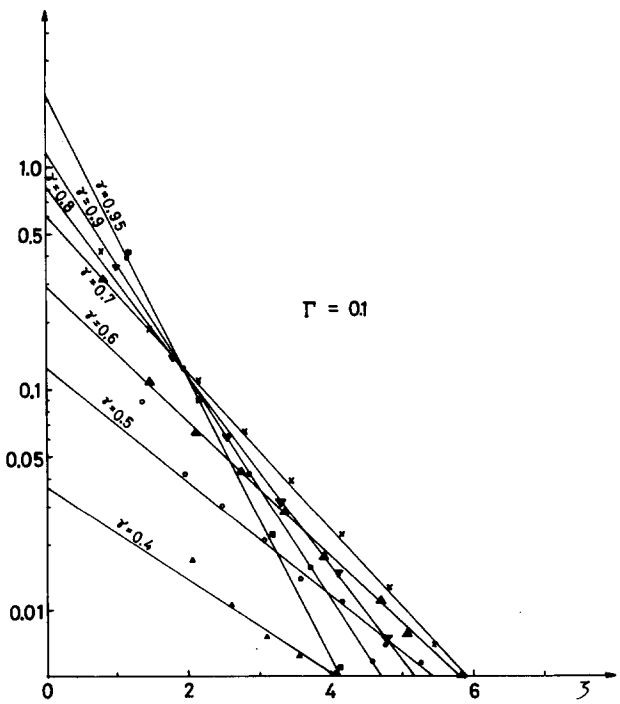

FIG. 6. Envelope lines of waves of ripple structures as a function of dimensionless distance ζ measured from the center of the soliton for various values of γ with $\Gamma=0.1$, $\Delta x=0.1$, and $\Delta t=0.05$.

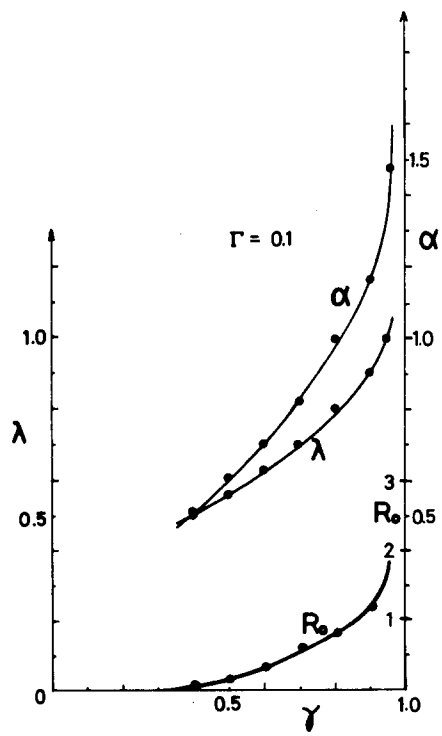

FIG. 7. Wavelength λ, attenuation constant α, and wave height R_0 of the ripple structure as a function of dimensionless bias γ for $\Gamma=0.1$, $\Delta x=0.1$, and $\Delta t=0.05$.

solitons bring about an attractive force between two solitons which overcomes the short-range repulsive force. The trailed ripple structure R is empirically described as $R \approx R_0 \exp(-\alpha\zeta)\sin(2\pi\zeta/\lambda - \frac{1}{2}\pi)$. The waveform of the propagating soliton of Eq. (2) in the presence of γ and Γ is found to be approximated as a superposition of a ripple structure and a bare soliton (soliton core C). We discuss the whole interaction between two moving solitons as a sum of core-core interaction, core-ripple interaction, and ripple-ripple interaction. These three parts of forces are investigated in detail using a computer.

A. Waveform of soliton in presence of bias and loss

A propagating soliton in Eq. (1) with $\gamma = \Gamma = 0$ is described as $\phi = 4\tan^{-1}\exp[\pm(x-ut)(1-u^2)^{-1/2}]$ $(\equiv S_0)$.[1,10] But in the presence of the relatively large value of γ and Γ, it is obvious that S_0 no longer represents a propagating soliton. It can be seen from Figs. 3—5 that a propagating soliton obtained from Eq. (2) in a system of finite γ and Γ trails a ripple structure R. To understand the coupling mechanism, we assume that the waveform of the propagating soliton (S) in the presence of γ and Γ is a superposition of the soliton core (C) and the ripple structure (R), $S = C + R$. In Fig. 4 we tried to separate the propagating soliton into C and R. C is approximately equal to S_0, except for the most curving part. R is drawn in the same figure and is numerically approximated as

$$R(\zeta) \approx R_0 \exp(-\alpha\zeta)\sin(2\pi\zeta/\lambda - \tfrac{1}{2}\pi), \quad \zeta > 0 \tag{5}$$

where ζ is the dimensionless distance measured from the center of propagating soliton in the direction opposite to the propagation. The ripple structure parameters λ, α, and R_0 in Eq. (5) depend on γ as shown in Fig. 7. And they also depend slightly on the mesh interval of computer calculation for space and time, Δx and Δt. Thus we consider that the shape of this oscillatory wave might have some relevance to those which appear after the shock-wave front[26] and after the moving dislocation.[27] The ripple structure treated in the present paper is fundamentally different from the so-called small-signal oscillation given by the dispersion relation $-k^2 + \omega^2 = 1$. The ripple structure is trailed stationary by the soliton core, while small-signal oscillation propagates with a velocity which is greater than the limiting velocity of the soliton core. The physical origin of the ripple structure is not completely understood at the present time and is the subject for future study.

B. Core-core interaction (C-C interaction)

Although we are interested in this section in the repulsive force acting between two soliton cores C which are obtained from the waveform of propagating soliton drawn in Fig. 4 in the presence of γ and Γ, we assume that the repulsive force is not very different from the force acting between two soliton cores $S_0(u)$ which are the analytical waveform of the propagating soliton in absence of γ and Γ, $(C \approx S_0)$. The repulsive force acting between two standing solitons $[S_0(u=0) - S_0(u=0)$ interaction] was investigated by Rubinstein,[1] Seeger *et al.*,[2] and Perring *et al.*[22] But the repulsive force acting between two solitons having a certain velocity has not been investigated. There is no analytical form for two solitons propagating in the same direction. However, it is known[22] that exact analytical form for two-soliton systems is given by $\phi_1 = 4\tan^{-1}u\{\exp[x(1-u^2)^{-1/2}] - \exp[-x(1-u^2)^{-1/2}]\}/\{\exp[ut/(1-u^2)^{-1/2}] + \exp[-ut(1-u^2)^{-1/2}]\}$, where u is an arbitrary value smaller than unity, which represents two solitons propagating with equal but opposite velocities in static coordinate. Thus we first try to understand the empirical repulsive force between two solitons given by ϕ_2, carrying out subsequent calculations. We obtain numerically the distance $2r$ between two solitons as a function of time. And by using the obtained $2r(t)$, we numerically calculate the repulsive force $F = \partial[m_0\dot{r}(1-\dot{r}^2)^{-1/2}]/\partial t$ where m_0 is a rest mass of the soliton.[1] Figure 8 shows the repulsive force F acting between two propagating solitons with the same screw sense and oppositely propagating directions as a function of their distance $2r$ for two values of u. It can be seen from Fig. 8 that as the distance $2r$ increases, F increases first and reaches its maximum value, and decreases exponentially. It is interesting to notice that from Fig. 8 for $2r \gg 1$, the repulsive force can be written in the form

$$F = F_0(u)\exp[-2r(1-u^2)^{-1/2}], \quad 2r \gg 1. \tag{6}$$

The abnormal behavior of F at $2r < 1$ in Fig. 8 seems to have originated from an incorrect definition of r when two solitons are too close. The factor $(1-u^2)^{-1/2}$ represents the effect of a Lorentz contraction. By an analytical treatment in the case of two solitons running either in opposite directions or in the same direction (Appendix A), F can be written

$$F_0(u) \approx \frac{8(1+u^2)}{1-u^2}\exp\left(-\frac{2r}{(1-u^2)^{1/2}}\right) \quad \text{for } 2r \gg (1-u^2)^{1/2}. \tag{7}$$

If we set $u = 0$, formula (7) agrees with the repulsive force between two static solitons treated by Rubinstein[1] except by a factor of $\frac{1}{4}$. The repulsive force decreases sharply with increasing the distance $2r$ of two solitons

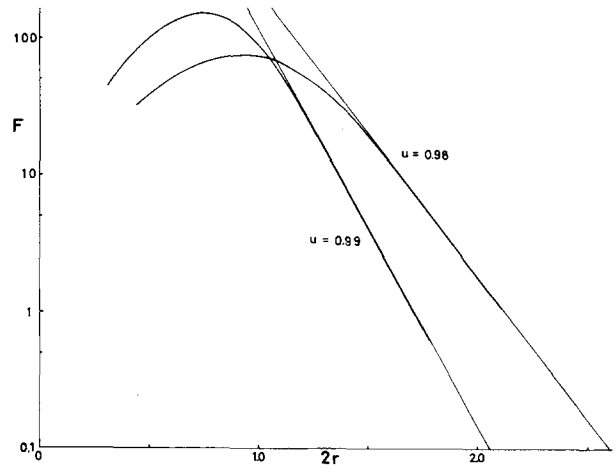

FIG. 8. Repulsive force F acting between two solitons propagating with equal but opposite velocities in static coordinate. which is obtained from an analysis of $\phi_1 = 4\tan^{-1}u\{\exp[x(1-u^2)^{-1/2}] - \exp[-x(1-u^2)^{-1/2}]\}/\{\exp[ut(1-u^2)^{-1/2}] + \exp[-ut(1-u^2)^{-1/2}]\}$, as a function of their distance $2r$ for two values of u.

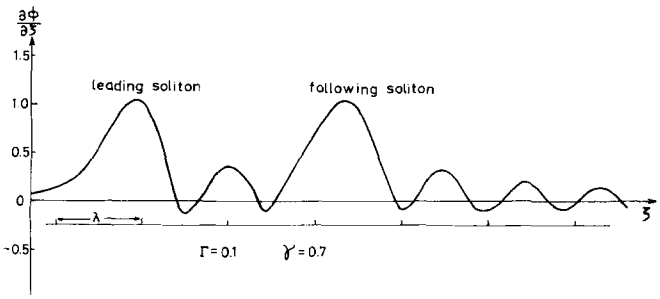

FIG. 9. First derivative $\partial\phi/\partial\zeta$ for a propagating coupled pair of solitons as a function of dimensionless distance ζ for $\gamma=0.7$, $\Gamma=0.1$, $\Delta x=0.1$, and $\Delta t=0.05$.

which are separated from each other by a distance of more than unity.

C. Ripple-core interaction (R-C interaction) and ripple-ripple interaction (R-R interaction)

As shown in Figs. 3 and 4 that the attractive and repulsive regions for the following soliton are formed behind the leading soliton in the presence of γ. The attractive and repulsive regions are alternate, and the periodicity coincides with the periodicity of R. The attractive regions correspond to the negative value of $\partial\phi/\partial\zeta$ $(=\partial R/\partial\zeta)$, where $\zeta=-x+ut$, and the repulsive regions correspond to the positive value of $\partial R/\partial\zeta$. By way of example, $\partial\phi/\partial\zeta$ for a propagating coupled pair of solitons is shown in Fig. 9. The following soliton core is at the boundary of the positive and negative regions of $\partial R/\partial\zeta$ brought about by the leading soliton. The $\min(\partial R/\partial\zeta)$ of the following soliton are found to be approximately at the point which corresponds to $\max(\partial R/\partial\zeta)$ of the leading soliton. As discussed in Sec. IV B, the repulsive force F between two soliton cores is a short-range force. Therefore it is considered that the following soliton is hardly affected by the short-range repulsive force F except in the vicinity of the leading soliton core. In order to investigate the interaction between the ripple structure R and the soliton core C, we have numerically calculated the interaction energy defined by the following expression:

$$H_{RC}=\int\frac{\partial R}{\partial\zeta}\frac{\partial C}{\partial\zeta}\,d\zeta,\quad \zeta=-x+ut. \tag{8}$$

This expression is an approximation derived from the energy density for the sine-Gordon equation (Appendices A and B). H_{RC} as a function of distance D between two solitons is shown in Fig. 10. This figure also shows the numerical results of the interaction energy between two ripple structures $[H_{RR}=\int(\partial R/\partial\zeta)(\partial R/\partial\zeta)\,d\zeta]$ and the sum of R-C and R-R interaction energies. In this figure the empirical positions D_n of the following soliton which are obtained from Eq. (2) are also marked by arrows. It can be seen from Fig. 10 that considering R-C interaction only, the point of minimum energy should be situated at a distance greater than D_n, while considering R-R interaction only, the point of minimum energy should be located at a distance slightly smaller than D_n. As one can see from Fig. 10, the minimum points of R-$R+R$-C agree well with the empirically obtained distances between the two running solitons. The interaction energy obtained by Eq. (8) includes only terms $(\partial\phi/\partial x)^2$ and $(\partial\phi/\partial t)^2$. It seems reasonable to conclude that the term $(1-\cos\phi)$ in the energy density is not important in understanding the interaction of two solitons in this case.

It is interesting to point out that the ripple-ripple interaction plays a more important role than the ripple-core interaction, contrary to our initial expectation that the ripple-core interaction should dominate. We believe this result originates from the finite size of the soliton core. In Appendix B it is verified analytically that R-C interaction energy is greater than R-R interaction energy, if the core is localized to a point.

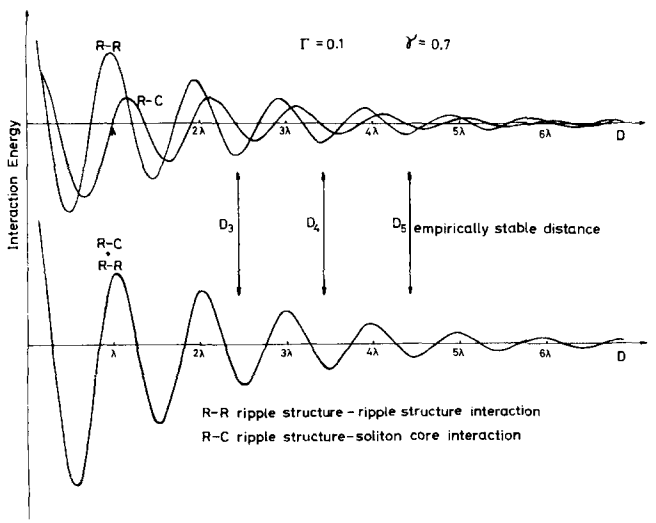

FIG. 10. Interaction energies between two ripple structures marked by R-R are shown as a function of their distance D. Interaction energies between the ripple structure and the core of a following soliton R-C are also drawn. In the lower part, a summation of these two interaction energies are shown as a function of D. Arrows correspond to the distances at which two solitons are empirically found.

D. Total interaction between two solitons with ripple structure

It is considered that as a whole a following soliton is repelled by the short-range repulsive force of the leading soliton which is shown in Fig. 8, and is simulta-

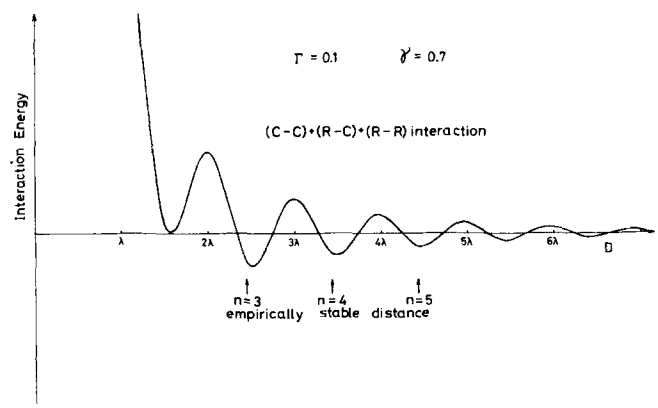

FIG. 11. Total interaction energy between two solitons with ripple structures as a function of their distance D.

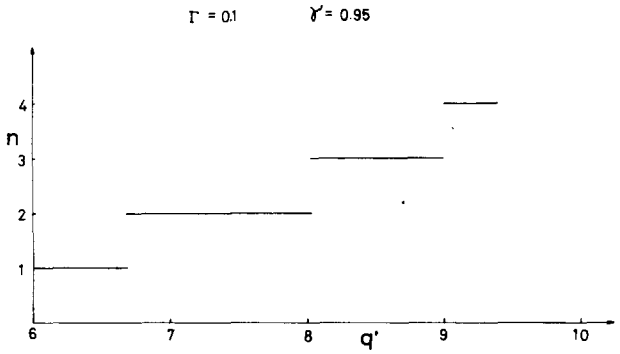

FIG. 12. Empirically obtained values of n for $\gamma = 0.95$, $u' = v = 0.99$, and $q = 6.0$ as a function of q' where q, q', u', and v are the parameters in the initial waveform $\phi_{i,1} = 4\tan^{-1}\exp[-(x_i - u'\Delta t - q)(1-v^2)^{-1/2}] + 4\tan^{-1}\exp[-(x_i - u'\Delta t - q')(1-v^2)^{-1/2}] + \sin^{-1}\gamma$. (Computer-simulated results $\Delta x = 0.1$ and $\Delta t = 0.05$).

neously affected by the potential created by the ripple structures brought about by running solitons. It can be seen from the discussion in Sec. IV C that the term $(1 - \cos\phi)$ in energy density is considered to contribute negligibly to the total interaction. In Sec. IV C we calculated R-C and R-R interactions for a system of two solitons with ripple structures obtained independently from Eq. (2). In this section we calculate numerically the total interaction energy between the two solitons obtained directly from Eq. (2). Figure 11 shows the obtained results as a function of the distance D using the following equation: $H = \int[\partial S(\zeta)/\partial\zeta][\partial S(\zeta, D)/\partial\zeta]\,d\zeta$, where S is obtained from Eq. (2) for $\gamma = 0.7$ and $\Gamma = 0.1$. It can be seen from Fig. 11 that the point of minimum energy is located at $D = (n - \frac{1}{2} - \delta)\lambda$, n is an integer. The reason that the point corresponding to $n = 1$ is not a minimum energy point in this figure is that the potential created by the ripple structure is overcome by the short-range repulsive force F. But if $\gamma = 0.95$ and $\Gamma = 0.1$, it is supposed from Fig. 3 that the point for $n = 1$ corresponds to a minimum energy. The larger γ becomes the nearer the two solitons are allowed to come, because R_0 increases with increasing γ. Although the point for $n = 2$ in Fig. 11 corresponds to a minimum energy, no soliton can be found there as one can see from Figs. 2 and 4. The reason for this is considered to be an effect of initial conditions, since it is supposed that the formation of ripple structure takes a relatively long time but the repulsive force acts between two solitons from the beginning.

E. Dependence of the distance of coupled solitons on the initial condition

The position of the following soliton coupled to the leading soliton is located at one of the minimum energy points. If two solitons separate enough not to be affected by F, the position of coupled solitons is determined by the potential created by the ripple structures. In this case the spacing of two solitons is described as $(n - \frac{1}{2} - \delta)\lambda$. n is decided by initial conditions. Figure 12 shows the value of n for $\gamma = 0.95$ as a function of q' and Fig. 13 shows the value of n for $\gamma = 0.7$ as a function of u' and v where q',[28] u',[29] and v[30] are the parameters in the initial value $\phi_{i,1} = 4\tan^{-1}\exp[-(x_i - u'\Delta t - q) \times (1-v^2)^{-1/2}] + 4\tan^{-1}\exp[-(x_i - u'\Delta t - q')(1-v^2)^{-1/2}] + \sin^{-1}\gamma$. As discussed before, λ is a function of γ as shown in Fig. 7. δ depends on the waveforms and distortions of soliton core and ripple structure, but in general it is negligibly small compared with unity. From these figures we observe that the equilibrium distance between two solitons are influenced by the initial conditions, such as initial waveform, initial time derivative, and initial distance, while the equilibrium velocity are uniquely determined by the bias γ and the loss Γ.

V. CONCLUSION

By computer simulation, the propagating two solitons with the same screw sense of the sine-Gordon equation with dimensionless loss Γ and bias γ terms can be stationary coupled, if γ is greater than a critical value. Physical bases for the coupling mechanism are examined in detail. In spite of the short-range repulsive force acting between two solitons with the same screw sense, two propagating solitons can be coupled through the periodic potentials which are produced by the ripple structures created by a leading and a following soliton. This ripple structure can, in general, be described as

$$R \approx R_0(\gamma)\exp[-\alpha(\gamma)\zeta]\sin[2\pi\zeta/\lambda(\gamma) - \tfrac{1}{2}\pi].$$

The interaction energy is dominanted by ripple-ripple interaction, not by ripple-core interaction. The repulsive force acting between two propagating solitons with the same screw sense have been calculated also. The spacing of coupled solitons are found to be expressed as

$$D = (n - \tfrac{1}{2} - \delta)\lambda, \quad n \text{ is an integer},$$

where λ is a wavelength of the ripple structure, n is determined by an initial value of velocities, positions, and waveforms. A small number δ depends upon the wave distortion of a propagating soliton. It is interesting to note that this type of coupling is an example of a formation of macroscopic ordering structure in an energy dissipating system.

ACKNOWLEDGMENTS

The authors are grateful to Professor T. Anayama and Professor R. Sato for their constant interest and

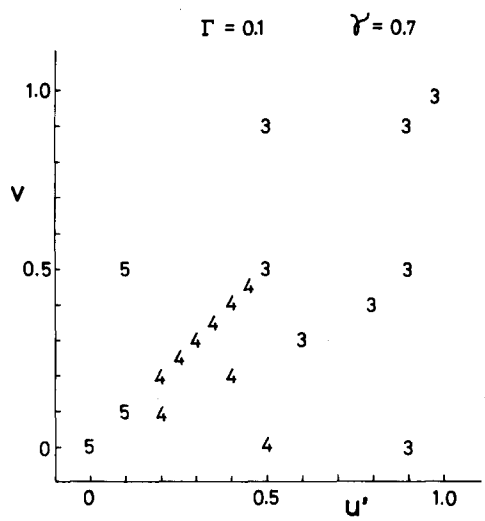

FIG. 13. Empirically obtained values of n for $\gamma = 0.7$ and $q = q' = 6.0$ as a function of u' and v where q, q', u', and v are the parameters in the initial waveform $\phi_{i,1} = 4\tan^{-1}\exp[-(x_i - u'\Delta t - q)(1-v^2)^{-1/2}] + 4\tan^{-1}\exp[-(x_i - u'\Delta t - q')(1-v^2)^{-1/2}] + \sin^{-1}\gamma$. (Computer-simulated results $\Delta x = 0.1$ and $\Delta t = 0.05$).

encouragement in this problem. They are deeply indebted for stimulating discussions to Dr. T. Nakamura, Professor Y. Ebina, Professor S. Kawakami, Dr. M. Miyagi, Dr. Y. Ogawa, Dr. G. Oya, H. Tsuge, and T. Hoshimiya.

APPENDIX A: ANALYTICAL CALCULATION AND PHYSICAL ORIGIN OF REPULSIVE FORCE BETWEEN TWO PROPAGATING SOLITONS

In order to understand Eq. (6) analytically, we carried out the following calculation. The energy density for the sine-Gordon equation can be written in the form[1]

$$H_D = \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_t^2 + 1 - \cos\phi. \tag{A1}$$

The force F acting between two solitons has been written for static solitons in the following form by Seeger *et al.*[2]:

$$F \sim -\frac{\partial H}{\partial(2r)}, \quad H = \int_{-\infty}^{\infty} H_D\, dx, \tag{A2}$$

where $2r$ is the distance between the two solitons. Therefore we apply this formula for the moving solitons. In our case,

$$\begin{aligned} F &= -\frac{\partial}{\partial(2r)} \int_{-\infty}^{\infty} H_D\, dx \\ &= -\int_{-\infty}^{\infty} \frac{\partial H_D}{(2r)}\, dx \\ &= -\int_{-\infty}^{\infty} \left(\phi_x \frac{\partial \phi_x}{\partial(2r)} + \phi_t \frac{\partial \phi_t}{\partial(2r)} + \sin\phi \frac{\partial \phi}{\partial(2r)}\right) dx. \end{aligned} \tag{A3}$$

If $1-u \ll 1$, the analytical solution $\phi = \phi_1$ with $1-u \ll 1$ can be approximated as $\phi \approx 4\tan^{-1}\exp[(x-ut)(1-u^2)^{-1/2}] - 4\tan^{-1}\exp[(-x-ut)(1-u^2)^{-1/2}]$. Since the centers of the two solitons are located at $x = ut$ and $x = -ut$ respectively, $\partial/\partial(2r) \approx \partial/\partial(2ut)$. Therefore

$$\begin{aligned} F \approx{}& \frac{-8(1+u^2)}{(1-u^2)(1-u^2)^{1/2}} \int_{-\infty}^{\infty} \left(\frac{1}{e^{-p}+e^{p}} + \frac{1}{e^{-Q}+e^{Q}}\right)\left(\frac{-e^{-p}+e^{p}}{(e^{-p}+e^{p})^2} \right. \\ &\left. + \frac{-e^{-Q}+e^{Q}}{(e^{-Q}+e^{Q})^2}\right) dx + \frac{8}{(1-u^2)^{1/2}} \int_{-\infty}^{\infty} \left(\frac{1}{e^{-p}+e^{p}} - \frac{1}{e^{-Q}+e^{Q}}\right) \\ &\times \left(\frac{(e^{-p}-e^{p})(e^{-2Q}+e^{2Q}-6)}{(e^{-p}+e^{p})(e^{-Q}+e^{Q})^2} + \frac{(e^{-2p}+e^{2p}-6)(e^{-Q}-e^{Q})}{(e^{-p}+e^{p})^2(e^{-Q}+e^{Q})^2}\right) dx, \end{aligned} \tag{A4}$$

where $p = (x-ut)(1-u^2)^{-1/2}$ and $Q = (-x-ut)(1-u^2)^{-1/2}$. Since $1-u \ll 1$, sech(p) and sech(Q) may be replaced in zeroth approximation by $\pi(1-u^2)^{1/2}\delta(x-ut)$ and $\pi(1-u^2)^{1/2}\delta(-x-ut)$, respectively. Therefore the repulsive force acting between two oppositely running solitons with the same screw sense is approximated as

$$F \approx \frac{8\pi(1+u^2)}{1-u^2} \exp\left(-\frac{2r}{(1-u^2)^{1/2}}\right), \quad 2r \gg (1-u^2)^{1/2}, \tag{A5}$$

where $2r$ is the distance between two solitons. It can also be verified that the exact result is obtained if one uses superposition of two solitons with the same screw sense running in the same direction, although it is not an exact solution of the sine-Gordon equation. Let us consider the physical origin of F in the S_0-S_0 interaction. It is considered that F consists of F_1, F_2, and F_3 where F_1, F_2, and F_3 are derived from terms $\frac{1}{2}\phi_x^2$, $\frac{1}{2}\phi_t^2$, and $1-\cos\phi$ in the energy density of Eq. (A1), respectively. And it can be shown from Eqs. (A3) and (A4) that F_1, F_2, and F_3 are written

$$F_1 = \frac{8\pi}{1-u^2} \exp\left(-\frac{2r}{(1-u^2)^{1/2}}\right), \tag{A6}$$

$$F_2 = \frac{8\pi u^2}{1-u^2} \exp\left(-\frac{2r}{(1-u^2)^{1/2}}\right), \tag{A7}$$

and

$$F_3 = 0. \tag{A8}$$

Therefore the term $1-\cos\phi$ in the energy density does not contribute to the S_0-S_0 interaction in the zeroth approximation. In a Josephson-junction transmission line F_1 and F_2 are related to magnetic and electric field potentials, because $\phi_{x'}$ is equal to $(2e/h)\, d\mu H_{y'}$ and $\phi_{t'}$ is equal to $(2e/h)V$, according to the Josephson equations where $H_{y'}$ is a magnetic field parallel to the layer of insulator, V is a transverse voltage, d is an effective thickness of the layer of insulator, μ is the permeability, x' is the distance measured in meters, and t' is the time measured in seconds. From the above discussion it is considered that the S_0-S_0 interaction is mainly acted through magnetic and electric fields in the case of the Josephson line. For a standing soliton F_2 vanishes because $u=0$, therefore it is considered that the interaction between two standing solitons is acted through a magnetic field only.

APPENDIX B: ANALYTICAL CALCULATIONS OF RIPPLE-CORE AND RIPPLE-RIPPLE INTERACTIONS

Let us carry out approximate analytical calculations of R-C and R-R interactions. We assume that the soliton core and the ripple structure are expressed as

$$C \approx 4\tan^{-1}\exp[(\xi - D)(1-u^2)^{-1/2}], \quad D > 0, \tag{B1}$$

and

$$R \approx -R_0 \exp(-\alpha\xi)\cos(2\pi\xi/\lambda), \quad \xi > 0, \tag{B2}$$

respectively. The R-C interaction energy H_{RC} can be written

$$\begin{aligned} H_{RC} = \int_0^{\infty} C_\xi R_\xi\, d\xi = \frac{2(1+u^2)R_0}{(1-u^2)^{1/2}} \int_0^{\infty} \operatorname{sech}\left(\frac{\xi - D}{(1-u^2)^{1/2}}\right) \\ \times \left(\alpha\cos\frac{2\pi}{\lambda}\xi + \frac{2\pi}{\lambda}\sin\frac{2\pi}{\lambda}\xi\right)\exp(-\alpha\xi)\, d\xi. \end{aligned} \tag{B3}$$

In the zeroth approximation in which we assume that the core is localized to a point, H_{RC} may be written as

$$\begin{aligned} H_{RC} \approx{}& 2\pi(1+u^2)R_0\left(\alpha^2 + \frac{4\pi^2}{\lambda^2}\right)^{1/2} \exp(-\alpha D) \\ &\times \left(\sin\frac{2\pi}{\lambda}D + \tan^{-1}\frac{\alpha\lambda}{2\pi}\right). \end{aligned} \tag{B4}$$

On the other hand the R-R interaction energy H_{RR} can be written

$$\begin{aligned} H_{RR} = \int_D^{\infty} \frac{\partial}{\partial\xi}\left(-R_0\exp(-\alpha\xi)\cos\frac{2\pi}{\lambda}\xi\right)\frac{\partial}{\partial\xi} \\ \times \left(-R_0\exp[-\alpha(\xi - D)]\cos\frac{2\pi}{\lambda}(\xi - D)\right) d\xi \end{aligned} \tag{B5}$$

$$= R_0^2 \exp(-\alpha D)\left[\frac{2\pi\lambda^2\alpha^2 + 8\pi^2}{4\alpha^2\lambda^3 + 16\pi^2\lambda} \sin\frac{2\pi}{\lambda}D + \left(\frac{4\pi^2\alpha + \alpha^3\lambda^2}{4\alpha^2\lambda^2 + 16\pi^2} + \frac{\alpha}{4} + \frac{\pi^2}{\alpha\lambda^2}\right)\cos\frac{2\pi}{\lambda}D\right].$$

Using the parameters $\lambda = 0.7$, $\alpha = 0.82$, $R_0 = 0.6$, and $u = 0.98$ of ripple structure numerically obtained for $\gamma = 0.7$,

$$H_{RC} \approx 67 \exp(-\alpha D)\sin[(2\pi/\lambda)D + 0.09], \quad \text{(B6)}$$

$$H_{RR} \approx 9 \exp(-\alpha D)\sin[(2\pi/\lambda)D + 1.48]. \quad \text{(B7)}$$

These forms of energies coincide very well with the real interaction energy shown in Fig. 10. However the ratio H_{RC} to H_{RR} is greater than unity in this calculation, contrary to the results shown in Fig. 10. Further from Eqs. (B6) and (B7) we note that the phase obtained in this model calculation is always a little greater than the real value. This small discrepancy we attribute to the deformation of the interacting cores and ripple structures of running solitons from Eqs. (B1) and (B2).

[1]J. Rubinstein, J. Math. Phys. **11**, 258 (1970).
[2]A. Seeger and A. Kochendorfer, Z. Phys. **130**, 321 (1951).
[3]A. Seeger, H. Donth, and A. Kochendorfer, Z. Phys. **134**, 173 (1952).
[4]A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE **61**, 1443 (1973).
[5]B. D. Josephson, Adv. Phys. **14**, 419 (1965).
[6]I. O. Kulik, Sov. Phys.-JETP. **24**, 1307 (1967).
[7]P. Lebwohl and M. J. Stephen, Phys. Rev. **163**, 376 (1967).
[8]A. C. Scott, Am. J. Phys. **37**, 52 (1969).
[9]J. Matisoo, IEEE Trans. Magn. MAG-**5**, 848 (1969).
[10]A. C. Scott, *Active and Nonlinear Wave-Propagation Electronics* (Interscience, New York, 1970).
[11]A. Barone, F. Esposito, C. J. Magee, and A. C. Scott, Riv. Nuovo Cimento **1**, 227 (1971).
[12]A. C. Scott, Nuovo Cimento B **69**, 241 (1970).
[13]W. J. Johnson, Ph. D. thesis (University of Wisconsin, 1968) (unpublished).
[14]A. C. Scott and W. J. Johnson, Appl. Phys. Lett. **14**, 316 (1969).
[15]A. Barone, J. Appl. Phys. **42**, 2747 (1971).
[16]T. A. Fulton, R. C. Dynes, and P. W. Anderson, Proc. IEEE **61**, 28 (1973).
[17]T. A. Fulton and R. C. Dynes, Solid State Commun. **12**, 57 (1973).
[18]K. Nakajima, T. Yamashita, and Y. Onodera, J. Appl. Phys. **45**, 3141 (1974).
[19]K. Nakajima, Y. Onodera, T. Nakamura, and R. Sato, J. Appl. Phys. **45**, 4095 (1974).
[20]G. L. Lamb, Rev. Mod. Phys. **43**, 99 (1971).
[21]U. Enz, Helv. Phys. Acta **37**, 245 (1964).
[22]J. K. Perring and T. H. R. Skyrme, Nucl. Phys. **31**, 550 (1962).
[23]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **30**, 1262 (1973).
[24]R. Hirota, J. Phys. Soc. Jpn. **33**, 1459 (1972).
[25]A. J. Callegari, and E. L. Reiss, J. Math. Phys. **14**, 267 (1973).
[26]R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-Value Problems*, 2nd ed. (Interscience, New York, 1967).
[27]N. Flytzanis, V. Celli, and A. Nobile, J. Appl. Phys. **45**, 5176 (1974).

[28]$q' - q$ is not the exact distance between two solitons, because the positions of the solitons are defined as the position for which $\phi = (2n+1)\pi$.
[29]u' is a velocity of the soliton in the initial state.
[30]v is a parameter to determine the waveform of the initial state, and $(1 - v^2)^{1/2}$ roughly corresponds to a length of the soliton.