Geometry of Gears

著者	MAEDA Kazuhiko
journal or publication title	Science reports of the Research Institutes, Tohoku University. Ser. A, Physics, chemistry and metallurgy
volume	3
page range	364-382
year	1951
URL	http://hdl.handle.net/10097/26445

Geometry of Gears*

Kazuhiko MAEDA

The Research Institute for Scientific Measurements

(Received March 31, 1951)

Chapter I. Theory of Spur Gears

1. In order to deal with tooth profiles from the general point of view, we shall use complex numbers and dual complex numbers and solve problems by the method of the natural geometry. The courses are similar for plane, spherical and spacial cases.

2. Analytic representation of plane curves

We shall fix an orthogonal frame O-xy of right-hand system. Let (M) be an oriented plane curve, A be a fixed point on (M), s be the arc length measured from A to a movable point M on (M) and let $x(s)$, $y(s)$ be coordinates of the point M. If we put

$$z = x + iy \qquad (i^2 = -1) ,$$

then the equation of the curve (M) can be represented as follows:

$$z = z(s) .$$

Let us denote the curvature of the oriented curve (M) by κ. It may be $\kappa \gtreqless 0$ (cf. Fig. 1).

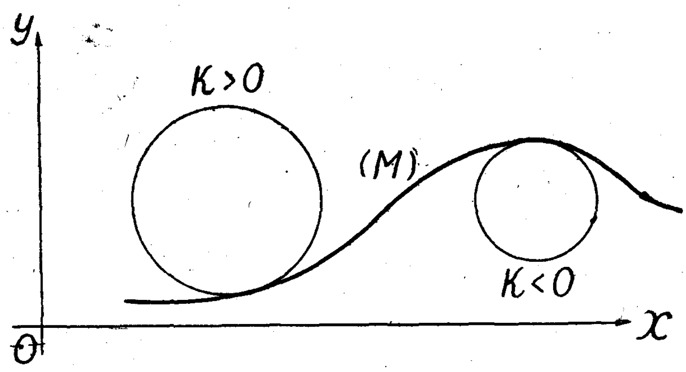

Fig. 1.

We can easily verify that the following relation holds good:

$$z'' = i\kappa z' \qquad \left(z' = \frac{dz}{ds}\right) . \tag{2.1}$$

---

* The author, Kazuhiko Maeda, Doctor of Science, was a mathematician, who had made a special study of geometry with respect to gears until he died in the spring of 1948 at the age of 38. During the Second World War, also in Japan there were many studies done on engineering by mathematicians. Dr. Maeda was one of them, who formulated a plain but systematically beautiful theory on the meshing of gears by means of natural geometry.

In Chapter I, he dealt with the meshing of the spur gears in general, in Chapter II, with the meshing of the bevel gears in general, and in Chapter III, with the meshing of the skew gears toothed with skew ruled surfaces by utilizing Study's coordinate of a straight line.

In translating his thesis into English, we owe much to Dr. Shigeo Sasaki, professor at the Department of Science, the Tôhoku University. In his mimeographed script, we found what seemed to be clear errors, which we corrected after consulting with Dr. Kaneo Yamada, professor at the Department of Technology, the Niigata University and with a few others.

Tagaichi Matsuyama
The Research Institute for
Scientific Measurements,
Tôhoku University

When $\kappa=\kappa(s)$ is given, an oriented curve whose curvature is the given function $\kappa(s)$ is determined uniquely except its location. If we reflect a curvew ith respect to a straight line, then the curvature of the image differs only in sign from that of the original curve. If we change the orientation of the curve, the curvature changes in the same manner.

3. Moving frames and relative coordinates

Let ζ be a given point. If we put

$$\zeta = z + z'Z \qquad \left(z' = \frac{dz}{ds}\right), \tag{3.1}$$

then Z is an in variant for any displacement of the curve (M) and the point ζ as a whole. We shall call Z the relative coordinate of the point ζ with respect to the point z on the curve (M).

Let MT be the oriented tangent to (M) at z and MN be the oriented normal such that MT and MN make a right-hand system. The frame M-TN is called the moving frame at a point z of (M). Z is the coordinate of the point ζ with respect to the moving frame at z.

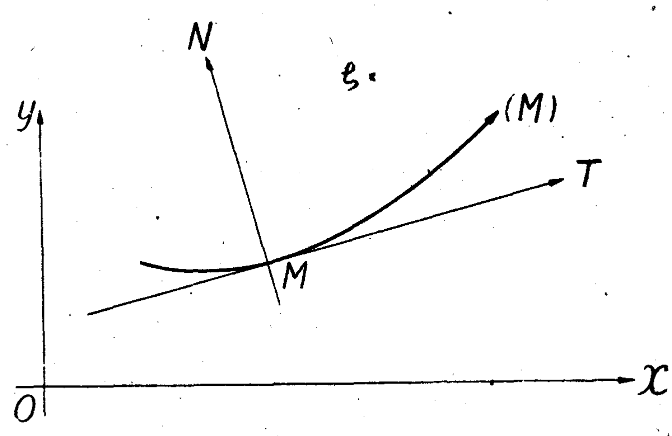

Fig. 2.

4. Adjoint curve and the condition of immovability of a point

If the relative coordinate Z is given as a function of s, ζ describes a curve generally. We shall call the latter curve an adjoint curve of the original curve (M). Differentiating (3.1), we obtain

$$\frac{d\zeta}{ds} = z'(Z' + i\kappa Z + 1) . \tag{4.1}$$

Accordingly, Z is the relative coordinate of a fixed point, if and only if Z is a solution of the following differential equation:

$$\frac{dZ}{ds} + i\kappa Z + 1 = 0 . \tag{4.2}$$

5. The expansion formula for the adjoint crnve

We can give an adjoint curve by $Z=Z(s)$. Let P_0 and P be points on the adjoint curve corresponding to the points s and $s+\delta s$ respectively on the curve (M). The relative coordinate of the point P_0 with respect to $z(s)$ is $Z(s)$, and that of the point P with respect to $z(s+\delta s)$ is $Z(s+\delta s)$. Let W be the relative coordinate of P with respect to $z(s)$. If we fix s, then W is a function of δs. Hence we can write

$$W = W(\delta s) . \tag{5.1}$$

(5.1) is the equation of the adjoint curve with respect to the moving frame at $z(s)$, the parameter being δs. If we put

$$\left(\frac{d^n W}{dh^n}\right)_{h=0} = \frac{D^n Z}{ds^n} \qquad (h = \delta s) \quad (n = 1, 2, 3, \cdots\cdots) , \tag{5.2}$$

then the expansion formula for the adjoint curve is given by

$$W = Z + \frac{\delta s}{1!}\frac{DZ}{ds} + \frac{(\delta s)^2}{2!}\frac{D^2 Z}{ds^2} + \cdots\cdots . \tag{5.3}$$

For the computation of $\frac{D^n Z}{ds^n}$ the following recurrent formulas are useful

$$\left.\begin{aligned} &A_{n+1} = \frac{dA_n}{ds} + i\kappa A_n + \delta_{0n} , \\ &A_n = \frac{D^n Z}{ds^n} , \qquad (n = 0, 1, 2, \cdots\cdots) \\ &A_0 = Z , \qquad \delta_{0n} = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) . \end{cases} \end{aligned}\right\} \tag{5.4}$$

From the last equations we see that

$$\frac{DZ}{ds} = \frac{dZ}{ds} + i\kappa Z + 1 , \tag{5.5}$$

$$\frac{d\zeta}{ds} = \frac{dz}{ds}\frac{DZ}{ds} . \tag{5.6}$$

In particular, in order to obtain the expansion formula for the original curve (M), it is sufficient to put $Z = 0$ in (5.4). Then, we can see easily that

$$A_0 = 0 , \quad A_1 = 1 , \quad A_2 = i\kappa , \quad A_3 = i\kappa' - \kappa^2 , \cdots\cdots\cdots\cdots\cdots$$

Hence we get the following expansion formula for (M):

$$Z = s + \frac{i\kappa}{2!} s^2 + \frac{i\kappa' - \kappa^2}{3!} s^3 + \cdots\cdots . \tag{5.7}$$

In the last formula, we have replaced W by Z and δs by s. The notations κ, κ', $\cdots\cdots$ mean $\kappa(0)$, $\kappa'(0)$, $\cdots\cdots$ respectively.

6. The problem on roulettes

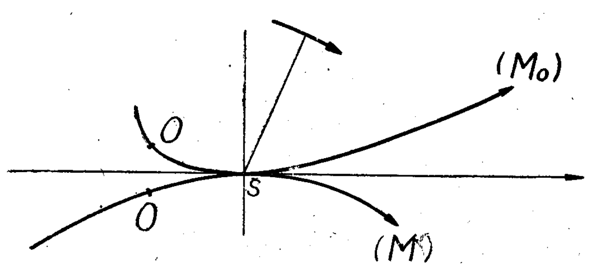

Fig. 3.

Assuming that a curve (M_0) rolls on another curve (M), we shall consider the locus of a fixed point relative to (M_0). From the assumption, we can represent corresponding points of (M) and (M_0) by the same parameter s. Let Z be the relative coordinate of a fixed point relative to (M_0). Then the relative coordinate of the same point with respect to (M) is also Z.

From the condition of immovability relative to (M_0), we get

$$\frac{dZ}{ds} + i\kappa_0 Z + 1 = 0 , \tag{6.1}$$

where κ_0 denotes the curvature of (M_0). Concerning (M) we get

$$\frac{DZ}{ds} = \frac{dZ}{ds} + i\kappa Z + 1 . \tag{6.2}$$

Subtracting (6.1) from (6.2) we obtain

$$\frac{DZ}{ds} = i(\kappa - \kappa_0)Z . \tag{6.3}$$

From the last equation we see that the normal to the roulette at $Z(s)$ passes through the point $z(s)$, a well known fact. Various properties concerning on roulettes can be obtained from (6.3).

7. A necessary and sufficient condition for gearing

Suppose that the curves (M) and (M_0) are rolling each other upon another. We shall now seek for a necessary and sufficient condition in order that two curves (C) and (C_0), which lie at certain fixed positions relative to (M) and to (M_0) respebtively keep contact with each other for every value of s. We will solve this problem.

As the contact point of (C) and (C_0) has the same relative coordinate with respect to (M) and (M_0), we shall denote it by Z. We may consider that the curves (C) and (C_0) are adjoint curves with respect to (M) and (M_0) respectively. They are given by a same equation

$$Z = Z(s) . \tag{7.1}$$

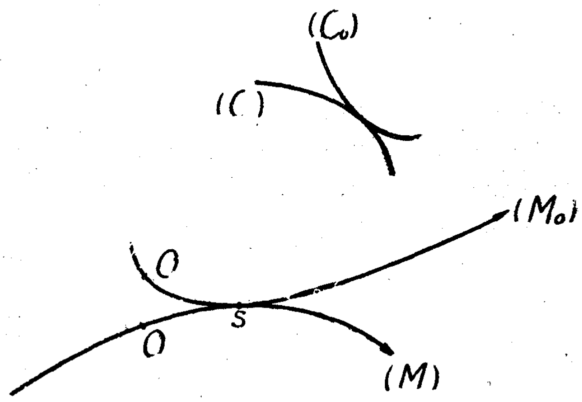

Fig. 4.

For the set of curves (M) and (C), we get

$$\frac{DZ}{ds} = \frac{dZ}{ds} + i\kappa Z + 1 , \tag{7.2}$$

and similarly for the set of curves (M_0) and (C_0) we get

$$\frac{D_0 Z}{ds} = \frac{dZ}{ds} + i\kappa_0 Z + 1 , \tag{7.3}$$

where κ and κ_0 denote curvatures of (M) and (M_0) respectively.

The required condition is that $\frac{DZ}{ds} \Big/ \frac{D_0 Z}{ds}$ is real for any value of s, namely

$$\frac{DZ}{ds} \Big/ \left(\frac{DZ}{ds} - \frac{D_0 Z}{ds}\right) = \mu ,$$

where μ is a real function of s. Now, by virtue of (7.2) and (7.3), the equation

$$\frac{DZ}{ds} - \frac{D_0Z}{ds} = i(\kappa - \kappa_0)Z \tag{7.4}$$

holds good. Consequently, the required condition is that the following differential equation is satisfied:

$$\frac{DZ}{ds} = i\mu(\kappa - \kappa_0)Z \,. \tag{7.5}$$

From the last equation we can easily see a well known property that the common normal to (C) and (C_0) at $Z(s)$ passes through the pitch point.

From (7.4) and (7.5) we obtain also

$$\frac{D_0Z}{ds} = i(\mu - 1)(\kappa - \kappa_0)Z \,, \tag{7.6}$$

which can be written as

$$\frac{D_0Z}{ds} = i\mu_0(\kappa_0 - \kappa)Z \,, \tag{7.7}$$

provided that μ_0 is a real function of s defined by

$$\mu + \mu_0 = 1 \,. \tag{7.8}$$

If $\kappa \neq \kappa_0$, the relation $\frac{DZ}{ds} = \frac{D_0Z}{ds}$ holds good when and only when $Z = 0$. Therefore only the pair of curves (M) and (M_0) keeps the rolling contact and any other pair of curves (C) and (C_0) can not keep the rolling contact. There arises necessarily sliding.

Let σ and σ_0 be specific slidings on the curves (C) and (C_0) respectively, they are given by

$$\sigma = \left(\frac{DZ}{ds} - \frac{D_0Z}{ds}\right) \Big/ \frac{DZ}{ds} = \frac{1}{\mu} \,, \tag{7.9}$$

$$\sigma_0 = \left(\frac{D_0Z}{ds} - \frac{DZ}{ds}\right) \Big/ \frac{D_0Z}{ds} = \frac{1}{\mu_0} \,. \tag{7.10}$$

It is evident that

$$\frac{1}{\sigma} + \frac{1}{\sigma_0} = 1 \,. \tag{7.11}$$

The formulas (7.9) and (7.10) give the meaning of μ and μ_0. The fundamental equation of gearing (7.5) can be written also as

$$\frac{dZ}{ds} + i\kappa_h Z + 1 = 0 \,, \tag{7.12}$$

where

$$\kappa_h = \kappa - \mu(\kappa - \kappa_0) = \mu_0\kappa + \mu\kappa_0 \,. \tag{7.13}$$

From (7.9), (7.10) and (7.12) we can easily see that the following relations exist:

$$\kappa_h = \frac{\kappa}{\sigma_0} + \frac{\kappa_0}{\sigma}, \tag{7.14}$$

$$\sigma = \frac{\kappa - \kappa_0}{\kappa - \kappa_h}, \qquad \sigma_0 = \frac{\kappa_0 - \kappa}{\kappa_0 - \kappa_h}. \tag{7.15}$$

8. Tooth profiles with given specific sliding

When the specific sliding is given, we know $\mu = \mu(s)$ and hence $\kappa_h = \kappa_h(s)$. Consider s as arc length and describe a curve (H) having the curvature $\kappa_h(s)$. Let us carry (H) so that the point $s = 0$ of (H) coincides with the point $s = 0$ of (M) and roll (H) over (M). Then as (7.12) shows us, one of the tooth profile (C) is given as the locus of a fixed point P relative to (H). In the same way, if we carry (H) so that the point $s = 0$ of (H) coincides with the point $s = 0$ of (M) and roll (H) over (M_0), then we obtain another tooth profile (C_0) as the locus of the same fixed point P as above.

Especially the path of contact of (C) and (C_0) passes throuh the pitch point when and only when there exists a value s such that $Z = 0$, then the fixed point P lies on the auxiliary curve (H). Analytically any solution $Z = Z(s)$ of (7.12) gives us the path of contact and at the same time it gives us tooth profiles (C) and (C_0) as adjoint curves of (M) and (M_0) respectively.

9. Tooth profiles with constant specific sliding

Especially, in the case of common gears, that is, in the case when (M) and (M_0) are circles or a circle and a straight line, we shall study tooth profiles with a constant specific sliding. Then, as κ, κ_0 and μ are constants, κ_h is also a constant and the auxiliary curve (H) is a circle or a straight line. Hence tooth profiles under consideration are epicyclics, i. e., they are epitrochoid, hypocycloid, cycloid or involute etc., as the case may be. In particular when the path of contact passes through a pitch point, curves of trochoid type must be omitted.

10. Determination of tooth profiles with a given path of contact

By hypothesis $Z = Z(t)$ is a given function of t. From the fundamental equation (7.12) we have

$$\frac{dZ}{dt}\frac{dt}{ds} + 1 + i\kappa_h Z(t) = 0. \tag{10.1}$$

Accordingly, we get

$$\Re\left[\frac{\dfrac{dZ}{dt}\dfrac{dt}{ds} + 1}{Z(t)}\right] = 0, \tag{10.2}$$

where $\Re w$ means the real part of w. Solving the differential equation (10.2) with respect to t, we get $t = t(s)$. Then $Z = Z(t) = Z(t(s))$, is the parametric equation of the required tooth profiles (C) and (C_0) regarded as adjoint curves of (M) and (M_0) respectively.

11. Determination of the path of contact and the mating tooth profile (C_0) when a tooth profile (C) is given

Let two curves

$$(M) \qquad z = z(s) \tag{11.1}$$

and

$$(C) \qquad \zeta = \zeta(t) \tag{11.2}$$

be given on the one hand and a curve

$$(M_0) \qquad z_0 = z_0(s) \tag{11.3}$$

be given on the other. From (3.1) and (5.6) we get

$$\zeta = z + \frac{dz}{ds} Z \tag{11.4}$$

$$\frac{d\zeta}{ds} = \frac{dz}{ds} \frac{DZ}{ds} . \tag{11.5}$$

If we put (7.5) into the last equation we get

$$\frac{d\zeta}{ds} = i\mu(\kappa - \kappa_0) \frac{dz}{ds} Z , \tag{11.6}$$

that is

$$\frac{d\zeta}{dt} \frac{dt}{ds} = i\mu(\kappa - \kappa_0)(\zeta - Z) . \tag{11.7}$$

Hence we get

$$\Re \left[\frac{\frac{d}{dt} \zeta(t)}{\zeta(t) - Z(s)} \right] = 0 . \tag{11.8}$$

(11.8) is an equation of the form $f(s, t) = 0$. Solving it, we have $t = t(s)$. Then, by virtue of (11.4), we get $Z = Z(s)$, which gives the required tooth profile C_0 as an adjoing curve of (M_0).

The equation of the curve (C_0) is given by

$$\zeta_0 = Z_0(s) + \frac{dz_0}{ds} Z(s) , \tag{11.9}$$

and that of the path of contact is given also by $Z = Z(s)$.

12. Determination of all pairs of tooth profiles (C), (C_0)

Any pair of tooth profiles (C) and (C_0) is given by

$$(C) \qquad \zeta = z(s) + \frac{dz}{ds} Z , \tag{12.1}$$

$$(C_0) \qquad \zeta_0 = z_0(s) + \frac{dz_0}{ds} Z , \tag{12.2}$$

where Z is a solution of (7.12). As (7.12) is a linear differential equation of the first order, the general solution can be given explicitly as follows:

$$Z = A\exp\left(-i\int \kappa_h ds\right) - \exp\left(-i\int \kappa_h ds\right)\int \exp\left(i\int \kappa_h ds\right)ds\ , \qquad (12.3)$$

where A is the (complex) constant of integration and κ_h is an arbitrary real function of s.

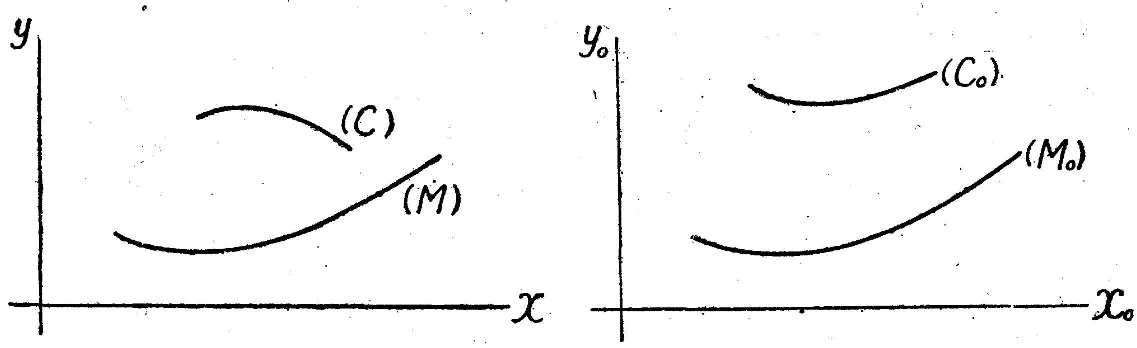

Fig. 5.

13. Another form of the fundamental equation

If we put

$$Z = Re^{i\Theta} \qquad (13.1)$$

then (R, Θ) is the polar coordinates of the point of contact of (C) and (C_0) with respect to the moving frame at $z(s)$. If we substitue it into the fundamental equation (7.12) and separate the real and the imaginary parts, we get

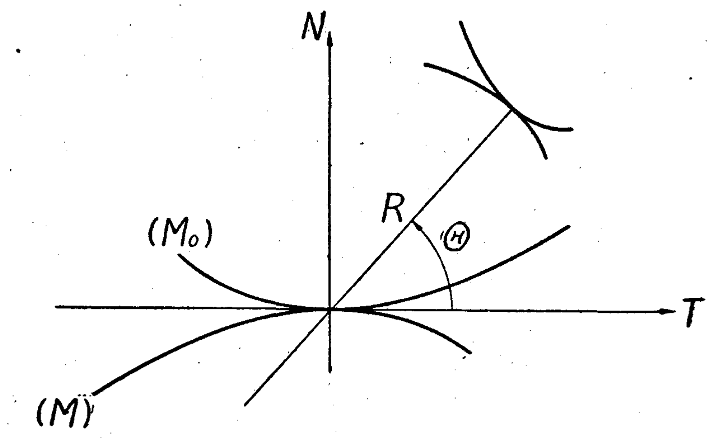

Fig. 6.

$$\frac{dR}{ds} + \cos\Theta = 0\ , \qquad (13.2)$$

$$R\left(\frac{d\Theta}{ds} + \kappa_h\right) = \sin\Theta \qquad (13.3)$$

The last equations are another form of the fundamental equation of gearing (7.12).

14. Deduction of the formula for specific sliding

If we eliminate the term ds from (13.2) and (13.3), we get

$$\kappa_h R = \frac{d}{dR}(R\sin\Theta)\ . \qquad (14.1)$$

Hence, by virtue of (7.15), we have the following formulas for specific sliding:

$$\sigma = \frac{(\kappa_0 - \kappa)\dfrac{dR}{d\Theta}}{\cos\Theta + \left(\dfrac{\sin\Theta}{R} - \kappa\right)\dfrac{dR}{d\Theta}}\ , \qquad (14.2)$$

$$\sigma_0 = \frac{(\kappa - \kappa_0)\dfrac{dR}{d\Theta}}{\cos\Theta + \left(\dfrac{\sin\Theta}{R} - \kappa_0\right)\dfrac{dR}{d\Theta}} . \qquad (14.3)$$

The case of the external gearing. In this case, if we set

$$\kappa = -\frac{1}{R_1} \qquad (R_1 > 0) ,$$

$$\kappa_0 = \frac{1}{R_2} \qquad (R_2 > 0) ,$$

in (14.2) and (14.3), then we have the desired formulas for specific sliding.

The case of the internal gearing. In this case, if we set

$$\kappa = \frac{1}{R_1} \qquad (R_1 > 0) ,$$

$$\kappa_0 = \frac{1}{R_2} \qquad (R_2 > 0)$$

in (14.2) and (14.3), then we have also the desired formulas for specific sliding.

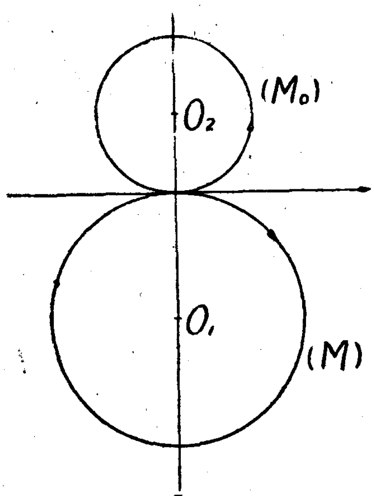

Fig. 7.

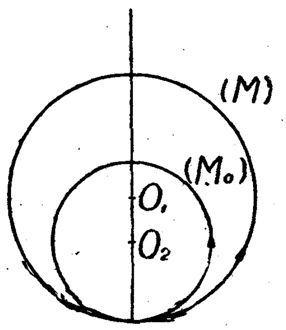

Fig. 8.

15. Formulas of relative coordinates for the cases of common gears

The case where both (M) and (M_0) are circles. In this case we get the following formulas:

$$\text{(M)} \quad z = \frac{1}{|\kappa|} e^{i\kappa s} , \qquad (15.1)$$

$$\zeta = \frac{1}{|\kappa|} e^{i\kappa s} (1 + i\kappa Z) , \qquad (15.2)$$

$$(\text{M}_0) \quad z_0 = \frac{1}{|\kappa_0|} e^{i\kappa_0 s} , \qquad (15.3)$$

$$\zeta_0 = \frac{1}{|\kappa_0|} e^{i\kappa_0 s} (1 + i\kappa_0 Z) . \qquad (15.4)$$

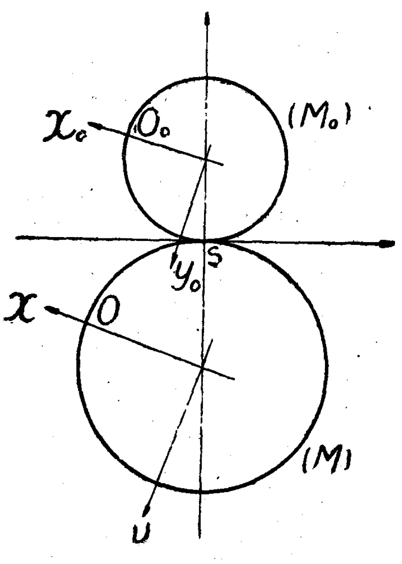

Fig. 9.

The case where (M_0) is a straight line and (M) is a circle. In this case we get the following formulas:

$$\text{(M)} \quad z = \frac{1}{|\kappa|} e^{i\kappa s} , \qquad (15.5)$$

$$\zeta = \frac{1}{|\kappa|} e^{i\kappa s}(1 + i\kappa Z) , \tag{15.6}$$

$$(M_0) \quad z_0 = s , \tag{15.7}$$

$$\zeta_0 = s + Z . \tag{15.8}$$

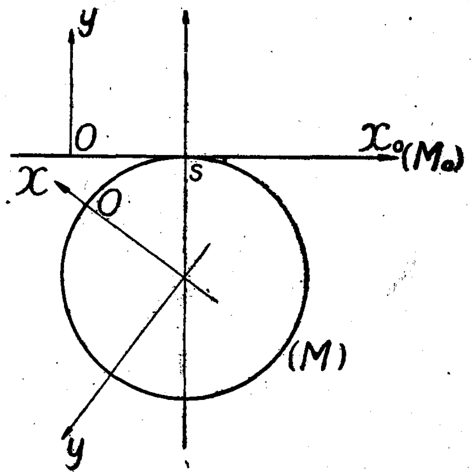

Fig. 10.

16. A remark on the kinematics on a plane

In various cases for the developements of the kinematics on a plane, it is convenient to use complex numbers. However we shall omit the details here.

Chapter II. Theory of bevel gears

17. Coordinates of points on a sphere

Let a be the radius of a sphere. We shall consider oriented great circles NAS, NBS, which cut orthogonally each other at N and S. Let us assume that when we rotate the oriented great circle NAS just $\pi/2$ about the axis $\overrightarrow{SN}$, it coincides with the oriented great circle NBS. We shall take axes of rectangular coordinates x_1, x_2, x_3 as are indicated in the figure 11. We shall project a point P on the sphere onto a point in x_1 x_2-plane stereographically from the point S(0, 0, $-a$). Let Q be the image of P by this projection, and $(x_1, x_2, 0)$ be the coordinates of Q.

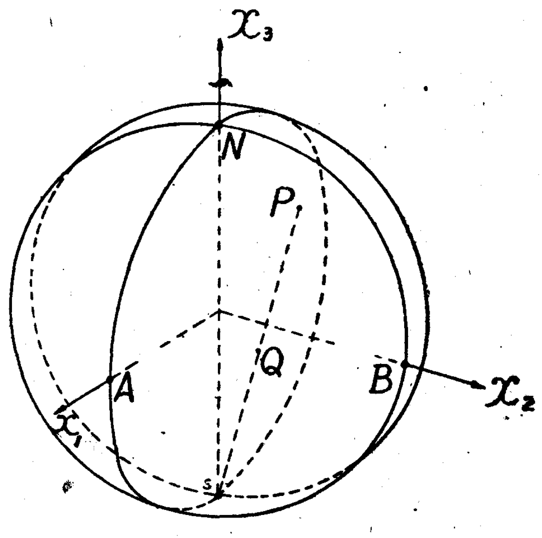

Fig. 11.

Setting

$$z = x_1 + ix_2 \qquad \left(i = \sqrt{-1}\right), \tag{17.1}$$

we shall consider the complex number z as the coordinate of the point P with respect to the coordinate system in consideration.

A rotation of angle φ on the sphere about a point x is given by the equation

$$\frac{z^* - x}{\bar{x}z^* + a^2} = e^{i\varphi} \frac{z - x}{\bar{x}z + a^2} . \tag{17.2}$$

This is an equation of the form

$$z^* = \frac{\alpha z - a^2\bar{\beta}}{\beta z + \bar{\alpha}} \tag{17.3}$$

where $\bar{x}$ is the conjugate complex number of x.

18. Analytic representation of spherical curves

Let s be the arc-length of a spherical curve, Taking s as a parameter, we can put the equation of this curve as follows:

$$z = z(s) . \tag{18.1}$$

Since s is the arc-length, we see that the following relation holds good:

$$4a^4 \frac{z'\bar{z}'}{(a^2+z\bar{z})^2} = 1 . \tag{18.2}$$

Consider an oriented circle on the sphere, and let r be its spherical radius $(0<r<a\pi)$. We call $\kappa(\gtreqless 0)$ defined by

$$\kappa = \frac{1}{a \tan \frac{r}{a}} \tag{18.3}$$

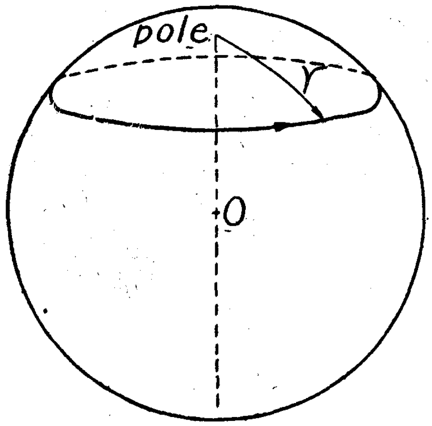

Fig. 12.

the geodesic curvature of the oriented circle. For a great circle, we have $\kappa=0$. If we reverse the orientation of the oriented circle, then the geodesic curvature changes only its sign.

The geodesic curvature of the osculating oriented circle of an oriented curve is nothing but the geodesic curvature of the oriented curve under consideration. The geodesic curvature κ is given by the following formula:

$$i\kappa = \frac{z''}{z'} - \frac{2z'\bar{z}}{a^2+z\bar{z}} \qquad \left(z' = \frac{dz}{ds}\right) . \tag{18.4}$$

If $\kappa = \kappa(s)$ is given, an oriented curve whose geodesic curvature is the given function $\kappa(s)$ is uniquely determined except rotation.

19. Relative coordinates

For a point ζ, we set

$$a^2 \frac{\sqrt{z'\bar{z}'}}{z'} \frac{\zeta - z}{\bar{z}\zeta + a^2} = Z , \tag{19.1}$$

or, equivalent to it, we put

$$\frac{(\zeta - z)(z\bar{z}+a^2)}{2z'(\bar{z}\zeta+a^2)} = Z . \tag{19.2}$$

Then Z is an invariant for any rotation of the curve z and the point ζ as a whole on the sphere. We shall call Z the relative coordinate of the point ζ with respect to the frame which consists of the oriented tangent great circle and the oriented normal great circle at a point z of the curve. By (19.2), ζ-plane is mapped onto

Z-plane. This mapping is a rotation on the sphere. The point z on the curve is mapped onto the origin in Z-plane. The tangent great circle and the normal great circle are mapped onto the real and imaginary axes respectively.

20. The condition of immovability of a point

From (19.2), we get by differentiation

$$\frac{(1-UZ)^2}{2z'}\zeta' = Z' + \frac{1}{2} + i\kappa Z + \frac{1}{2a^2}Z^2 , \qquad (20.1)$$

where we have put

$$U = \frac{2z'\bar{z}}{a^2+z\bar{z}} .$$

Hence, the condition of immovability of a point is that the following differential equation is satisfied:

$$\frac{dZ}{ds} + \frac{1}{2} + i\kappa Z + \frac{1}{2a^2}Z^2 = 0 . \qquad (20.2)$$

21. The expansion formula for the adjoint curve

Any adjoint curve of a curve is given by

$$Z = Z(s) . \qquad (21.1)$$

Its expansion formula is given also by

$$W = Z + \frac{\delta s}{1!}\frac{DZ}{ds} + \frac{(\delta s)^2}{2!}\frac{D^2Z}{ds^2} + \cdots\cdots\cdots , \qquad (21.2)$$

where $\frac{D^n Z}{ds^n}$ can be computed in turn by virture of the following recurrent formulas:

$$\left.\begin{aligned} A_{n+1} &= \frac{dA_n}{ds} + \frac{1}{2}\delta_{0n} + i\kappa A_n + \frac{1}{2a^2}B_n , \\ A_n &= \frac{D^n Z}{ds^n} , \qquad (A_0 = Z) , \\ B_n &= \sum_{\nu=0}^{n} \binom{n}{\nu} A_\nu A_{n-\nu} \qquad (n = 0, 1, 2, \cdots\cdots) , \end{aligned}\right\} \qquad (21.3)$$

$$\frac{DZ}{ds} = \frac{dZ}{ds} + \frac{1}{2} + i\kappa Z + \frac{1}{2a^2}Z^2 . \qquad (21.4)$$

In particular, in order to get expansion formula of the original curve it is sufficient to put $Z=0$. Therefore writing Z and s instead of W and δs we have the following expansion formula of the given curve $z(s)$:

$$Z = \frac{1}{2}s + \frac{1}{2!}i\kappa\frac{s^2}{2} + \frac{\frac{1}{2a^2} - \kappa^2 + i\kappa'}{2\cdot 3!}s^3 + \cdots\cdots\cdots . \qquad (21.5)$$

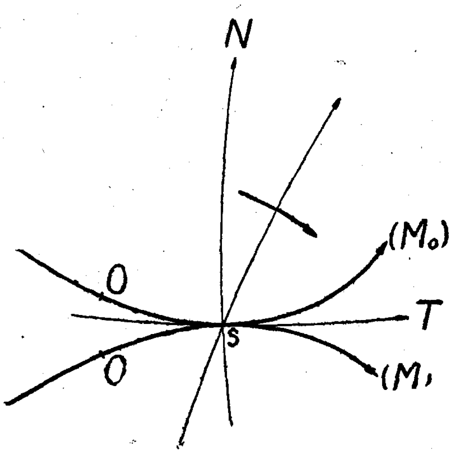

Fig. 13.

22. The problem on Roulettes

Consider the locus of a fixed point with respect to a curve (M_0) when (M_0) rolls over a curve (M). We can represent corresponding points of (M) and (M_0) by the same parameter (arc length) s. If the relative coordinate of a fixed point with respect to (M_0) is Z, the relative coordinate of the same point with respect to (M) is also the same Z. As the point is fixed for (M_0) by hypothesis, we get by virture of the condition of immovability the following relation:

$$\frac{dZ}{ds} + \frac{1}{2} + i\kappa_0 Z + \frac{1}{2a^2} Z^2 = 0 . \tag{22.1}$$

where κ_0 means the geodesic curvature of the curve (M_0). On the other hand we have

$$\frac{DZ}{ds} = \frac{dZ}{ds} + \frac{1}{2} + i\kappa Z + \frac{1}{2a^2} Z^2 . \tag{22.2}$$

From (22.1) and (22.2), we get

$$\frac{DZ}{ds} = i(\kappa - \kappa_0)Z . \tag{22.3}$$

From the last equation we see that the normal great circle of the roulette passes through the contact point of (M) and (M_0).

Various problems concerning on roulettes are solvable by virture of the above equation. However we shall omit the details here.

23. A necessary and sufficient condition for gearing

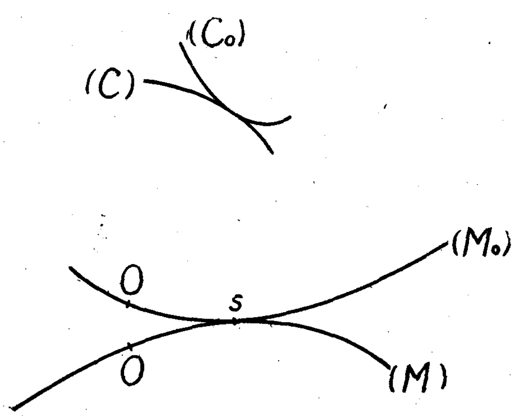

Fig. 14.

Suppose that two curves (M) and (M_0) are rolling in contact each other. What is a necessary and sufficient condition in order that two curves (C) and (C_0) which lie at certain fixed positions with respect to (M) and (M_0) respectively keep always contact with each other?

We shall solve this problem. The contact point of (C) and (C_0) has the same relative coordinate for both curves (M) and (M_0), so we shall denote it by Z. We may consider that the curves (C) and (C_0) are adjoint curves of (M) and (M_0) respectively. These curves may be given by an equation of the following form:

$$Z = Z(s) . \tag{23.1}$$

For the set of curves (M) and (C), we get

$$\frac{DZ}{ds} = \frac{dZ}{ds} + \frac{1}{2} + i\kappa Z + \frac{1}{2a^2} Z^2 , \tag{23.2}$$

and similarly, for the set of curves (M_0) and (C_0) we have

$$\frac{D_0Z}{ds} = \frac{dZ}{ds} + \frac{1}{2} + i\kappa_0 Z + \frac{1}{2a^2} Z^2 , \tag{23.3}$$

where κ and κ_0 are geodesic curvatures of (M) and (M_0) respectively. The required condition is that $\frac{DZ}{ds} \Big/ \frac{D_0Z}{ds}$ is real for any value of s, namely

$$\frac{DZ}{ds} \Big/ \left(\frac{DZ}{ds} - \frac{D_0Z}{ds}\right) = \mu ,$$

where μ is a real function of s. From (23.2) and (23.3) we have

$$\frac{DZ}{ds} - \frac{D_0Z}{ds} = i(\kappa - \kappa_0)Z , \tag{23.4}$$

Hence the required condition is that the following differential equation holds good:

$$\frac{DZ}{ds} = i\mu(\kappa - \kappa_0)Z . \tag{23.5}$$

From (23.5) it is easy to see that the common normal great circle of (C) and (C_0) passes through the contact point of (M) and (M_0).

From (23.4) and (23.5), we obtain

$$\frac{D_0Z}{ds} = i(\mu - 1)(\kappa - \kappa_0)Z , \tag{23.6}$$

i. e.

$$\frac{D_0Z}{ds} = i\mu_0(\kappa_0 - \kappa)Z , \tag{23.7}$$

provided that μ_0 is a real function of s defined by

$$\mu + \mu_0 = 1 . \tag{23.8}$$

If $\kappa \neq \kappa_0$, $\frac{DZ}{ds} = \frac{D_0Z}{ds}$ holds good when and only when $Z=0$. Hence, only the pair of cuves (M), (M_0) keeps the rolling contact and any other pair of curves (C), (C_0) can not keep rolling contact. There arises necessarily sliding.

The specific slidings of (C) and (C_0) are given by

$$\sigma = \left(\frac{DZ}{ds} - \frac{D_0Z}{ds}\right) \Big/ \frac{DZ}{ds} = \frac{1}{\mu} \tag{23.9}$$

and

$$\sigma_0 = \left(\frac{D_0Z}{ds} - \frac{DZ}{ds}\right) \Big/ \frac{DZ}{ds} = \frac{1}{\mu_0} \tag{23.10}$$

respectively. It is evident that

$$\frac{1}{\sigma} + \frac{1}{\sigma_0} = 1 . \tag{23.11}$$

The last equations give meanings of μ and μ_0.

The condition for gearing (23.5) can be written also in the following form:

$$\frac{dZ}{ds} + \frac{1}{2} + i\kappa_h Z + \frac{1}{2a^2} Z^2 = 0 , \tag{23.12}$$

where we have put

$$\kappa_h = \kappa - \mu(\kappa - \kappa_0) = \mu_0 \kappa + \mu \kappa_0 \tag{23.13}$$

$$\kappa_h = \frac{\kappa}{\sigma_0} + \frac{\kappa_0}{\sigma} , \tag{23.14}$$

$$\sigma = \frac{\kappa - \kappa_0}{\kappa - \kappa_h} , \qquad \sigma_0 = \frac{\kappa_0 - \kappa}{\kappa_0 - \kappa_h} . \tag{23.15}$$

24. Tooth profiles with given specific sliding

When the specific sliding is given, we know $\mu = \mu(s)$ and hence $\kappa_h = \kappa_h(s)$. Let us draw a curve (H) whose geodesic curvature is $\kappa_h(s)$, and carry (H) so that the point $s = 0$ of (H) coincides with the point $s = 0$ of the curve (M) and roll (H) on (M), then we obtain a tooth profile (C) as the locus of a fixed point P with respect to (H). In the same way, if we carry (H) so that the point $s = 0$ of (H) coincides with the point $s=0$ of (M_0) and roll (H) on (M_0), then we obtain another tooth profile (C_0) as the locus of the same point P.

Especially, the path of contact of (C) and (C_0) passes through the pitch point when and only when there exists a value s such that $Z = 0$, then the fixed point P lies on the adjoint curve (H). Analytically, any solution $Z = Z(s)$ of the fundamental equation (23.12) gives the path of contact and at the same time it gives tooth profiles (C), (C_0) as adjoint curves of (M), (M_0) respectively.

25. Tooth profiles with constant specific sliding

Now let us consider the case where κ, κ_0 and μ are constants. Then κ_h is also a constant and the adjoint curve (H) is a circle. Hence tooth profiles are epicyclics on the sphere.

Especially, when the path of contact passes through the pitch point, they must belong to the family of cycloids.

26. Determination of tooth profiles with a given path of contact

By hypothesis $Z = Z(t)$ is given. By virture of the fundamental equation (23.12) we have

$$\frac{dZ}{dt}\frac{dt}{ds} + \frac{1}{2} + i\kappa_h Z + \frac{1}{2a^2} Z^2 = 0 , \tag{26.1}$$

Hence we get

$$\Re\left(\frac{1}{Z}\frac{dZ}{dt}\frac{dt}{ds} + \frac{1}{2Z} + \frac{Z^2}{2a^2}\right) = 0 . \tag{26.2}$$

Solving the last differential equation with respect to t, we obtain $t = t(s)$. Then

$$Z = Z(t) = Z(t(s)) ,$$

is the parametric equation of the required tooth profiles (C) and (C_0) regarded as adjoint curves of (M) and (M_0) respectively.

27. Determination of the path of contact and the mating tooth profile (C_0) when a tooth profile (C) is given

By assumption three curves

$$(M) \qquad z = z(s)\,, \tag{27.1}$$

$$(C) \qquad \zeta = \zeta(t)\,, \tag{27.2}$$

and

$$(M_0) \qquad z_0 = z_0(s) \tag{27.3}$$

are given. Then we get

$$(1 - UZ)^2 \frac{d\zeta}{ds} = 2\,\frac{dz}{ds}\,\frac{DZ}{ds}\,, \tag{27.4}$$

where we have put

$$= \frac{(\zeta - z)(z\bar{z} + a^2)}{2z'(\bar{z}\zeta + a^2)} \tag{27.5}$$

$$U = \frac{2\,z'\bar{z}}{z\bar{z} + a^2} \tag{27.6}$$

The fundamental equation of gearing is

$$\frac{DZ}{ds} = i\mu(\kappa - \kappa_0)Z\,. \tag{27.7}$$

Hence we get

$$\frac{(1-UZ)^2 \dfrac{d\zeta}{dt}}{2z'Z} = i\mu(\kappa - \kappa_0)\,\frac{ds}{dt}\,. \tag{27.8}$$

Accordingly, we have

$$\Re\left(\frac{(1-UZ)^2 \dfrac{d\zeta}{dt}}{2z'Z}\right) = 0 \tag{27.9}$$

On the other hand, we have

$$1 - UZ = \frac{z\bar{z} + a^2}{\bar{z}\zeta + a^2}\,,$$

$$\frac{(1 - UZ)^2}{2z'Z} = \frac{z\bar{z} + a^2}{(\zeta - z)(\bar{z}\zeta + a^2)}$$

Consequently (27.9) can be written as follows:

$$\Re\left[\frac{\dfrac{d\zeta}{dt}}{(\zeta - z)(\bar{z}\zeta + a^2)}\right] = 0\,. \tag{27.10}$$

This is an equation of the form $f(s, t) = 0$. Solving it we get $t = t(s)$.

Hence, by virtue of (27.5) we know $Z = Z(s)$, which gives us the required tooth profile (C_0) as an adjoint curve of the curve (M_0).

28. Another form of the fundamental equation

Let us express the relative coordinate Z by the following equation:

$$Z = a \tan \frac{R}{2a} e^{i\Theta} \tag{28.1}$$

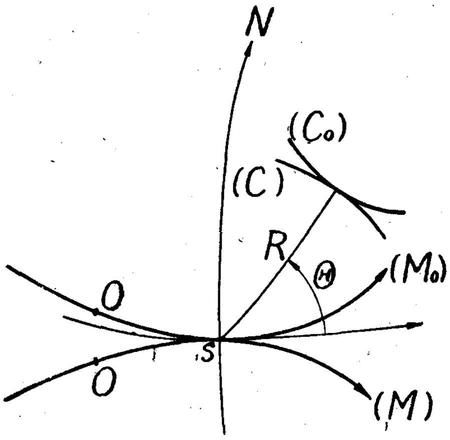

Fig. 15.

where R and Θ are the spherical polar coordinates of the contact point with respect to the frame which consists of the tangent great circle and the normal great circle.

The relations between the spherical polar coordinates R, Θ and rectangular coordinates X_1, X_2, X_3 of the space are given as follows: (cf. Fig. 11)

$$\begin{aligned} X_1 &= a \sin \frac{R}{a} \cos \Theta , \\ X_2 &= a \sin \frac{R}{a} \sin \Theta , \\ X_3 &= a \cos \frac{R}{a} . \end{aligned} \tag{28.2}$$

Substituting (28.1) into the fundamental equation (23.12) and separating the real part and imaginary part, we obtain

$$\frac{dR}{ds} + \cos \Theta = 0 , \tag{28.3}$$

$$a \tan \frac{R}{a} \left(\frac{d\Theta}{ds} + \kappa_h \right) = \sin \Theta . \tag{28.4}$$

These are another form of the fundamental equation of gearing.

29. Deduction of the formula for specific sliding

If we eliminate ds from (28.3) and (28.4), we have

$$\kappa_h \sin \frac{R}{a} = \frac{d}{dR} \left(\sin \Theta \sin \frac{R}{a} \right) \tag{29.1}$$

From the last equation and (23.15), we see that specific slidings of curves (C) and (C_0) are given by

$$\sigma = \frac{\kappa - \kappa_0}{\kappa - \cos \Theta \frac{d\Theta}{dR} - \frac{1}{a} \sin \Theta \cot \frac{R}{a}} , \tag{29.2}$$

$$\sigma_0 = \frac{\kappa_0 - \kappa}{\kappa_0 - \cos \Theta \frac{d\Theta}{dR} - \frac{1}{a} \sin \Theta \cot \frac{R}{a}} , \tag{29.3}$$

respectively.

30. Determination of all pairs of tooth profiles

Let $R = R(s)$ be an arbitrary real function of s. We define κ_h as follows:

$$\kappa_h = 0 \qquad \text{for} \qquad \left(\frac{dR}{ds}\right)^2 = 1\,, \tag{30.1}$$

$$\kappa_h = \frac{\sqrt{1-\left(\frac{dR}{ds}\right)^2}}{a}\cot\frac{R}{a} - \frac{\frac{d^2R}{ds^2}}{\sqrt{1-\left(\frac{dR}{ds}\right)^2}}\,. \tag{30.2}$$

$$\text{for} \qquad \left(\frac{dR}{ds}\right)^2 \neq 1\,.$$

($\sqrt{}$ has two values.)

Now we set

$$Z^* = a\tan\frac{R}{2a}\left(-\frac{dR}{ds} + i\sqrt{1-\left(\frac{dR}{ds}\right)^2}\right). \tag{30.3}$$

Then we have

$$Z = Z^* + \frac{\exp\left[-\int\left(\frac{1}{a^2}Z^* + i\kappa_h\right)ds\right]}{A + \frac{1}{2a^2}\int\exp\left[-\int\left(\frac{1}{a^2}Z^* + i\kappa_h\right)ds\right]ds}\,. \tag{30.4}$$

By the last equation $Z = Z(s)$, we get adjoint curves (C) and (C_0) of (M) and (M_0), respectively.
When $R(s)$ is arbitrary real functions, (C) and (C_0) thus obtained give all pairs of tooth profiles.

31. The relative coordinate for the case of circles

When we choose the coordinate system as is indicated in Fig. 16, the equation of an oriented circle whose geodesic curvature is κ, is given by

$$z = a^2(c-\kappa)\exp(ics)\,, \tag{31.1}$$

where we have put

$$c = \sqrt{\kappa^2 + \frac{1}{a^2}}\,. \tag{31.2}$$

Hence the relative coordinate Z of a point ζ is given by the following equations:

$$Z = \frac{-i}{c-\kappa}\left[\frac{\zeta - a^2(c-\kappa)\exp(ics)}{\zeta + a^2(c+\kappa)\exp(ics)}\right] \tag{31.3}$$

$$\zeta = \frac{a^2(c-\kappa) + iZ}{1 - i(c-\kappa)Z}\exp(ics)\,. \tag{31.4}$$

In particuar, for the case of great circle, we may put $\kappa = 0$.

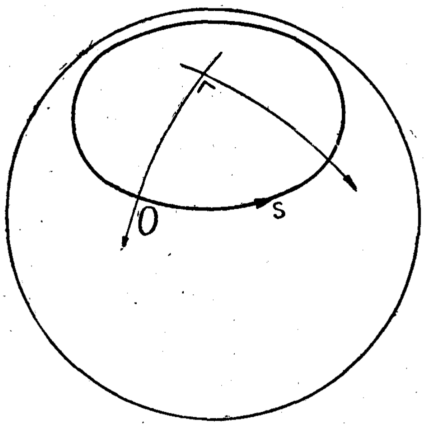

Fig. 16.

32. A remark on the kinematics on a spherical surface

In the developments of the kinematics on a spherical surface, it is often convenient to use the sterographic projection and complex numbers. However, we shall omit the details here. (to be continued)