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## Inverse scattering method for square matrix nonlinear Schrödinger equation under nonvanishing boundary conditions

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Matrix generalization of the inverse scattering method is developed to solve the multicomponent nonlinear Schrödinger equation with nonvanishing boundary conditions. It is shown that the initial value problem can be solved exactly. The multi-soliton solution is obtained from the Gel'fand-Levitan-Marchenko [Amer. Math. Soc. Transl. **1**, 253 (1955)] equation. © 2007 American Institute of Physics.  
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### I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation in the  $(1+1)$  space-time dimension is one of the completely integrable systems, i.e., the soliton equations.<sup>1,2</sup> This model equation has been extensively studied to describe nonlinear dynamics in a wide range of physics from fiber optics<sup>3-5</sup> to Bose-Einstein condensation of cold atoms.<sup>6-8</sup> The initial value problem can be solved exactly via the inverse scattering method (ISM).<sup>9,10</sup> In particular, the reflection-free condition reduces the inverse problem to a set of algebraic equations making it possible to obtain the  $N$ -soliton solution in an explicit way.

One of the major developments in the study of the NLS equation is multicomponent extensions preserving the integrability. Manakov<sup>11</sup> studied a system of the coupled NLS (cNLS) equations on the basis of the ISM and obtained the soliton solutions. While the interaction of vector solitons for the multicomponent focusing NLS equation is elastic in the vector sense, it was shown that during the two-soliton collision exchanges among components of each soliton may occur for particular choices of the parameter values.<sup>11,12</sup> In Ref. 13, the ISM for a matrix generalization of the NLS equation (in general, in a rectangular matrix form) was developed for solving the initial value problem. By assuming the reflection-free condition and *vanishing boundary conditions* (see below) the  $N$ -soliton solution was obtained explicitly. It should be noted that, by appropriate identifications of the matrix-field elements, the matrix NLS equation reduces to the cNLS equations of the Manakov-type<sup>12,13</sup> and remarkably to the spinor-type that is discovered recently<sup>14,15</sup> in connection with Bose-Einstein condensates with the spin degrees of freedom. Results given in Ref. 13 for a general matrix field, such as the  $N$ -soliton solution, conservation laws, and Hamiltonian structure, are directly applicable to the reduced systems. Thus, a further analysis of the matrix NLS equation is desired to deal with multicomponent nonlinear dynamics under different circumstances.

In this paper, we study a square matrix NLS equation under the *nonvanishing boundary conditions* by means of the ISM. The multicomponent system with such boundary conditions is

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regarded as an extension of a basic single-component NLS equation with the self-defocusing nonlinearity studied by Zakharov and Shabat<sup>10</sup> and also that with the self-focusing nonlinearity investigated by Kawata and Inoue.<sup>16</sup> As compared to the case with the vanishing boundary conditions,<sup>13</sup> the conservation laws and Hamiltonian structure are the same, while the Lax pair, the direct and inverse problems, and the  $N$ -soliton solution should be reformulated reflecting the boundary values of the matrix field.

This paper is organized as follows. In Sec. II, nonvanishing boundary conditions for the matrix NLS equation are introduced. The Lax pair is provided to formulate the auxiliary linear system. Then the conservation laws are constructed systematically. In Sec. III, the direct and inverse problems are solved along the ISM procedure and the multisoliton solution is presented. Section IV is devoted to the concluding remarks.

## II. FORMULATION

The matrix NLS equation is expressed as

$$iQ_t + Q_{xx} - 2\varepsilon QQ^\dagger Q = O \quad (\varepsilon = \pm 1), \quad (2.1)$$

where  $Q(x, t)$  and  $O$  are an  $l \times l$  matrix valued function and the zero matrix, respectively,  $Q^\dagger$  is the Hermitian conjugate of  $Q$ , and the subscripts  $t$  and  $x$  denote the partial derivatives.

The case  $\varepsilon = -1$  ( $\varepsilon = +1$ ) of Eq. (2.1) is often referred to as the self-focusing (defocusing) one. In Ref. 13, it was shown that through the ISM the system has an infinite number of conservation laws. The initial value problem was solved and the  $N$ -soliton solution was obtained under the constraint  $\varepsilon = -1$  and the vanishing boundary condition

$$Q(x, t) \rightarrow O \quad \text{as } x \rightarrow \pm \infty. \quad (2.2)$$

Under these conditions each soliton forms the so-called bright soliton with  $l^2$  components. Setting the form

$$Q = \begin{pmatrix} Q_{11} & \cdots & Q_{1l} \\ \vdots & & \vdots \\ Q_{m1} & \cdots & Q_{ml} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad (2.3)$$

with  $m < l$ , one can achieve a rectangular matrix reduction that is compatible with the vanishing boundary condition. In particular, the  $m=1$  case corresponds to the  $l$ -component Manakov model.<sup>12</sup> Other types of reduction can be obtained in a nontrivial way by putting some components equal without breaking consistency of equations.<sup>14,15</sup>

On the other hand, for another integrable constraint  $\varepsilon = +1$ , leading to

$$iQ_t + Q_{xx} - 2QQ^\dagger Q = O, \quad (2.4)$$

the boundary condition should be altered appropriately, which has not been investigated so far. Equation (2.4) is a matrix generalization of the NLS equation for a scalar field  $q(x, t)$  with a self-defocusing nonlinearity,

$$iq_t + q_{xx} - 2|q|^2 q = 0, \quad (2.5)$$

which possesses dark soliton solutions. For the self-defocusing NLS equation (2.5), the boundary condition at  $x \rightarrow \pm \infty$  is assumed to be the nonvanishing one, e.g., a constant,  $|q(x)| \rightarrow \lambda_0$ , rather than the vanishing one,  $|q(x)| \rightarrow 0$ . The ISM procedure was applied to the system (2.5) in Ref. 10. The analysis of the NLS equation under the nonvanishing boundary conditions was extended to the self-focusing case in Ref. 16.

From now on, we concentrate on the analysis of a full-rank  $l \times l$  square matrix NLS equation. We do not include reductions (2.3) in this work, i.e., the vector (Manakov) NLS equation falls outside our considerations. Systematic study of such reductions based on the symmetry argument is an open issue. In this section, we introduce a square matrix type of nonvanishing boundary conditions for Eq. (2.1) and formulate the Lax equation for the ISM.

### A. Nonvanishing boundary condition

We assume that the  $l \times l$  matrix valued function  $Q(x, t)$  satisfies the following nonvanishing conditions:

$$Q(x, t) \rightarrow Q_{\pm} \quad \text{as } x \rightarrow \pm \infty, \quad (2.6)$$

$$Q_{\pm}^{\dagger} Q_{\pm} = Q_{\pm} Q_{\pm}^{\dagger} = \lambda_0^2 I, \quad (2.7)$$

where  $\lambda_0$  is a positive real constant and  $I$  denotes the  $l \times l$  unit matrix. Noting the freedom of unitary transformations,

$$Q' = \mathcal{U} Q \mathcal{V}, \quad (2.8)$$

with  $\mathcal{U}, \mathcal{V}$  unitary matrices, we see that if  $Q$  is a solution of Eq. (2.1), then  $Q'$  is also a solution. By this freedom, without loss of generality, we can fix one side of the boundary condition as

$$Q_+ = \lambda_0 e^{i(kx - \omega t)} I. \quad (2.9)$$

To avoid a complexity, separate the carrier wave part,

$$Q(x, t) = \hat{Q}(x, t) e^{i(kx - \omega t)}, \quad (2.10)$$

where the dispersion relation is determined as

$$\omega = k^2 + 2\varepsilon \lambda_0^2. \quad (2.11)$$

Then the original nonlinear evolution equation (2.1) is equivalent to

$$iQ_t + Q_{xx} + 2ikQ_x + 2\varepsilon(\lambda_0^2 I - QQ^{\dagger})Q = 0. \quad (2.12)$$

Here and hereafter except the final expression (3.100), we drop the hat of  $\hat{Q}$  for a notational simplicity. Accordingly, Eq. (2.9) is rewritten as

$$Q_+ = \lambda_0 I. \quad (2.13)$$

In what follows, we focus on Eq. (2.12) with boundary conditions (2.6), (2.7), and (2.13).

### B. Lax pair

We introduce the Lax matrices in the following forms:

$$U = i\lambda \begin{bmatrix} -I & O \\ O & I \end{bmatrix} + \begin{bmatrix} O & Q \\ \varepsilon Q^{\dagger} & O \end{bmatrix}, \quad (2.14)$$

$$V = i\lambda^2 \begin{bmatrix} -2I & O \\ O & 2I \end{bmatrix} + \lambda \begin{bmatrix} 2ikI & 2Q \\ 2\varepsilon Q^{\dagger} & -2ikI \end{bmatrix} + i \begin{bmatrix} \varepsilon(\lambda_0^2 I - QQ^{\dagger}) & Q_x + 2ikQ \\ \varepsilon(-Q_x^{\dagger} + 2ikQ^{\dagger}) & -\varepsilon(\lambda_0^2 I - Q^{\dagger}Q) \end{bmatrix}, \quad (2.15)$$

where  $\lambda$  is the spectral parameter that is independent of time,  $\lambda_t = 0$ . The potential matrix  $Q$  satisfies nonvanishing boundary conditions (2.6) and (2.7) with Eq. (2.13). In the ISM, we associate a set of linear problems:

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad (2.16)$$

where  $\Psi$  is a  $2l \times l$  matrix function. We also use the following representations:

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad (2.17)$$

where all the entries are  $l \times l$  matrices. Then, the Lax equation is obtained from the compatible condition of Eqs. (2.16),

$$U_t - V_x + UV - VU = 0, \quad (2.18)$$

which is equivalent to the matrix NLS equation (2.12).

### C. Conservation laws

In this subsection, we recapture a systematic method to construct local conservation laws for the matrix NLS equation.<sup>13</sup> The method was originally developed for the scalar field case.<sup>17</sup>

Introduce the quantity

$$\Gamma = \Psi_2 \Psi_1^{-1}. \quad (2.19)$$

From Eqs. (2.14)–(2.18), one can prove that

$$\frac{\partial}{\partial t} \text{tr}(Q\Gamma) = \frac{\partial}{\partial x} \text{tr}(V_{12}\Gamma + V_{11}), \quad (2.20)$$

$$2i\lambda Q\Gamma = -\varepsilon QQ^\dagger + Q\Gamma_x + (Q\Gamma)^2. \quad (2.21)$$

Note that Eq. (2.20) has a form of conservation law. Expand  $Q\Gamma$  in  $\lambda$  as

$$Q\Gamma = \sum_{j=1}^{\infty} \frac{\varepsilon}{(2i\lambda)^j} F_j. \quad (2.22)$$

Then the trace of each coefficient  $F_j$  is a conserved density and Eq. (2.20) represents an infinite number of continuity equations. From Eq. (2.21), we recursively obtain

$$F_1 = -QQ^\dagger, \quad (2.23)$$

$$\text{tr} F_2 = \text{tr}(-QQ_x^\dagger), \quad (2.24)$$

$$\text{tr} F_3 = \text{tr}(-QQ_{xx}^\dagger + \varepsilon QQ^\dagger QQ^\dagger). \quad (2.25)$$

By direct calculation, all elements of  $F_1$  are shown to be conserved densities.

## III. INVERSE SCATTERING METHOD

In this section, we carry out the ISM procedure for Eq. (2.12) based on the Lax pairs (2.14) and (2.15).

### A. Direct problem

We consider the eigenvalue problem,

$$\Psi_x = U\Psi, \quad U = \begin{bmatrix} -i\lambda I & Q \\ \varepsilon Q^\dagger & i\lambda I \end{bmatrix}, \quad (3.1)$$

and define the Jost functions and scattering data for them. Here  $Q$  plays a role of potentials in the eigenvalue problem. In Secs. III A and III B, the analysis will be made with fixed time  $t$ . Under the nonvanishing boundary conditions (2.6),  $U$  has the asymptotic forms:

$$U(\lambda) \rightarrow U^\pm(\lambda) = \begin{bmatrix} -i\lambda I & Q_\pm \\ \varepsilon Q_\pm^\dagger & i\lambda I \end{bmatrix} \quad \text{as } x \rightarrow \pm\infty. \quad (3.2)$$

The characteristic roots of  $U^\pm(\lambda)$  are twofold,

$$i\zeta, \quad -i\zeta, \quad (3.3)$$

where  $\zeta \equiv (\lambda^2 - \varepsilon\lambda_0^2)^{1/2}$ . We introduce  $2l \times 2l$  matrix functions  $T$  and  $T^\pm$  by

$$T(\lambda, \zeta; x) = \begin{bmatrix} -i\tilde{Q}(x) & (\lambda - \zeta)I \\ (\lambda - \zeta)I & i\varepsilon\tilde{Q}^\dagger(x) \end{bmatrix}, \quad (3.4)$$

$$T^\pm(\lambda, \zeta) = \lim_{x \rightarrow \pm\infty} T(\lambda, \zeta; x). \quad (3.5)$$

Here  $\tilde{Q}(x)$  is an  $l \times l$  smooth matrix function that satisfies the same boundary condition as Eq. (2.6),

$$\tilde{Q}(x) \rightarrow Q_\pm \quad \text{as } x \rightarrow \pm\infty, \quad (3.6)$$

and for all  $x$ ,

$$\tilde{Q}(x)\tilde{Q}^\dagger(x) = \tilde{Q}^\dagger(x)\tilde{Q}(x) = \lambda_0^2 I. \quad (3.7)$$

Using  $T^\pm(\lambda, \zeta)$ , we can diagonalize  $U^\pm(\lambda)$  as follows:

$$U^\pm(\lambda) = -i\zeta T^\pm(\lambda, \zeta)(\sigma^z \otimes I)[T^\pm(\lambda, \zeta)]^{-1}, \quad (3.8)$$

where  $\sigma^j$  ( $i=x, y, z$ ) is the Pauli matrix and  $\otimes$  denotes the direct product,

$$\sigma^z \otimes I = \begin{bmatrix} I & O \\ O & -I \end{bmatrix}. \quad (3.9)$$

By using Eq. (3.8), we define matrix Jost functions  $\psi_1^-$ ,  $\psi_2^-$ ,  $\psi_1^+$ , and  $\psi_2^+$  as solutions of Eq. (3.1), whose asymptotic forms are, respectively, given by

$$\psi_1^- \sim T^-(\lambda, \zeta) \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} \quad \text{as } x \rightarrow -\infty, \quad (3.10a)$$

$$\psi_2^- \sim T^-(\lambda, \zeta) \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x} \quad \text{as } x \rightarrow -\infty, \quad (3.10b)$$

$$\psi_1^+ \sim T^+(\lambda, \zeta) \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} \quad \text{as } x \rightarrow +\infty, \quad (3.10c)$$

$$\psi_2^\pm \sim T^\pm(\lambda, \zeta) \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x} \quad \text{as } x \rightarrow +\infty. \quad (3.10d)$$

We note that  $\{\psi_1^+, \psi_2^+\}$  as well as  $\{\psi_1^-, \psi_2^-\}$  constitute fundamental systems of solution. This is easily proven by using the usual Wronskian defined by the determinant. In fact, one can show

$$\frac{d}{dx} \det[\Phi_1, \Phi_2] = 0, \quad (3.11)$$

for any two  $2l \times l$  matrix solutions  $\Phi_1, \Phi_2$  of Eq. (3.1). Checking the value at  $x \rightarrow \pm\infty$ , we have

$$\det[\psi_1^\pm, \psi_2^\pm] = (2\zeta(\lambda - \zeta))^l. \quad (3.12)$$

This indicates the linear independence of  $\psi_1^\pm$  and  $\psi_2^\pm$  except the branch points of  $\zeta$ , i.e.,  $\lambda = \pm \sqrt{\varepsilon\lambda_0}$  at which the solutions degenerate.

If we use a notation  $\boldsymbol{\psi}^\pm \equiv [\psi_1^\pm, \psi_2^\pm]$ , relations (3.10) can be rewritten compactly in the following form;

$$\boldsymbol{\psi}^\pm(\lambda, \zeta; x) \rightarrow T^\pm(\lambda, \zeta) J(\zeta x) \quad \text{as } x \rightarrow \pm\infty, \quad (3.13)$$

where

$$J(\zeta x) \equiv \begin{bmatrix} e^{-i\zeta x} I & O \\ O & e^{i\zeta x} I \end{bmatrix}. \quad (3.14)$$

Then the scattering matrix  $S(\lambda, \zeta)$  is defined by

$$\boldsymbol{\psi}^-(\lambda, \zeta; x) = \boldsymbol{\psi}^+(\lambda, \zeta; x) S(\lambda, \zeta), \quad (3.15)$$

$$S(\lambda, \zeta) = \begin{bmatrix} A(\lambda, \zeta) & \bar{B}(\lambda, \zeta) \\ B(\lambda, \zeta) & \bar{A}(\lambda, \zeta) \end{bmatrix}, \quad (3.16)$$

where all the entries  $A, \bar{A}, B,$  and  $\bar{B}$  represented by  $l \times l$  matrices constitute scattering data.

It is useful to consider another slightly modified eigenvalue problem.<sup>18</sup> Under a transformation,

$$\Phi \equiv T^{-1} \Psi, \quad (3.17)$$

the eigenvalue problem (3.1) becomes

$$\Phi_x = \tilde{U} \Phi, \quad \tilde{U} \equiv T^{-1}(UT - T_x) = -i\zeta \sigma^z \otimes I + W, \quad (3.18)$$

where  $W = (W_{ab})$ ,  $a, b = 1, 2$  with  $l \times l$  matrices,

$$W_{11} = \frac{i\varepsilon}{2\zeta} (Q^\dagger \tilde{Q} + \tilde{Q}^\dagger Q - 2\lambda_0^2 I) - \frac{\varepsilon}{2\zeta(\lambda - \zeta)} \tilde{Q}^\dagger \tilde{Q}_x, \quad (3.19a)$$

$$W_{12} = -\frac{\varepsilon}{2\lambda_0^2} \tilde{Q}^\dagger \left[ \frac{\lambda}{\zeta} (\tilde{Q} Q^\dagger + Q \tilde{Q}^\dagger - 2\lambda_0^2 I) + (Q \tilde{Q}^\dagger - \tilde{Q} Q^\dagger) \right] + \frac{i\varepsilon}{2\zeta} \tilde{Q}_x^\dagger, \quad (3.19b)$$

$$W_{21} = -\frac{1}{2\lambda_0^2} \tilde{Q} \left[ \frac{\lambda}{\zeta} (\tilde{Q}^\dagger Q + Q^\dagger \tilde{Q} - 2\lambda_0^2 I) + (Q^\dagger \tilde{Q} - \tilde{Q}^\dagger Q) \right] - \frac{i}{2\zeta} \tilde{Q}_x, \quad (3.19c)$$

$$W_{22} = -\frac{i\varepsilon}{2\zeta}(\tilde{Q}Q^\dagger + Q\tilde{Q}^\dagger - 2\lambda_0^2 I) - \frac{\varepsilon}{2\zeta(\lambda - \zeta)}\tilde{Q}\tilde{Q}_x^\dagger. \quad (3.19d)$$

As solutions of Eq. (3.18), we introduce new Jost matrices  $\phi^\pm \equiv [\phi_1^\pm, \phi_2^\pm] = T^{-1}\psi^\pm$  with simpler asymptotic forms,

$$\phi^\pm(\lambda, \zeta; x) \rightarrow J(\zeta x) \quad \text{as } x \rightarrow \pm\infty. \quad (3.20)$$

These matrix Jost functions are connected with each other through the same scattering matrix (3.16) as

$$\phi^-(\lambda, \zeta; x) = \phi^+(\lambda, \zeta; x)S(\lambda, \zeta). \quad (3.21)$$

We now examine the properties of the scattering data. For this purpose, define a matrix function for two  $2l \times l$  matrix functions  $\Phi_i(\lambda, \zeta; x)$  ( $i=1, 2$ ) as

$$M[\Phi_1, \Phi_2] \equiv \Phi_1^\dagger(\lambda^*, \zeta^*; x) \begin{bmatrix} I & O \\ O & -\varepsilon I \end{bmatrix} \Phi_2(\lambda, \zeta; x). \quad (3.22)$$

If  $\Phi_1(\lambda, \zeta; x)$ ,  $\Phi_2(\lambda, \zeta; x)$  are solutions of Eq. (3.1), one can show that

$$\frac{d}{dx}M[\Phi_1, \Phi_2] = O. \quad (3.23)$$

From the asymptotic forms (3.10), we obtain the following relations:

$$M[\psi_1^+, \psi_1^+] = -\varepsilon M[\psi_2^+, \psi_2^+] = 2\varepsilon\zeta(\lambda - \zeta)I, \quad (3.24a)$$

$$M[\psi_1^+, \psi_2^+] = O, \quad (3.24b)$$

$$M[\psi_1^+, \psi_1^-] = 2\varepsilon\zeta(\lambda - \zeta)A(\lambda, \zeta), \quad (3.24c)$$

$$M[\psi_2^+, \psi_2^-] = -2\zeta(\lambda - \zeta)\bar{A}(\lambda, \zeta), \quad (3.24d)$$

$$M[\psi_2^+, \psi_1^-] = -2\zeta(\lambda - \zeta)B(\lambda, \zeta), \quad (3.24e)$$

$$M[\psi_1^+, \psi_2^-] = 2\varepsilon\zeta(\lambda - \zeta)\bar{B}(\lambda, \zeta). \quad (3.24f)$$

These relations are rewritten into

$$\begin{bmatrix} A^\dagger(\lambda^*, \zeta^*) & -\varepsilon B^\dagger(\lambda^*, \zeta^*) \\ -\varepsilon \bar{B}^\dagger(\lambda^*, \zeta^*) & \bar{A}^\dagger(\lambda^*, \zeta^*) \end{bmatrix} \begin{bmatrix} A(\lambda, \zeta) & \bar{B}(\lambda, \zeta) \\ B(\lambda, \zeta) & \bar{A}(\lambda, \zeta) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \quad (3.25)$$

which can be shown, e.g., as

$$\begin{aligned} M[\psi_1^-, \psi_1^-] &= A^\dagger(\lambda^*, \zeta^*)M[\psi_1^+, \psi_1^+]A(\lambda, \zeta) + B^\dagger(\lambda^*, \zeta^*)M[\psi_2^+, \psi_2^+]B(\lambda, \zeta) \\ &= 2\varepsilon\zeta(\lambda - \zeta)(A^\dagger(\lambda^*, \zeta^*)A(\lambda, \zeta) - \varepsilon B^\dagger(\lambda^*, \zeta^*)B(\lambda, \zeta)) = 2\varepsilon\zeta(\lambda - \zeta)I. \end{aligned} \quad (3.26)$$

Relation (3.25) leads to the inversion of Eq. (3.15),

$$\psi^+(\lambda, \zeta; x) = \psi^-(\lambda, \zeta; x)[S(\lambda, \zeta)]^{-1}, \quad (3.27)$$



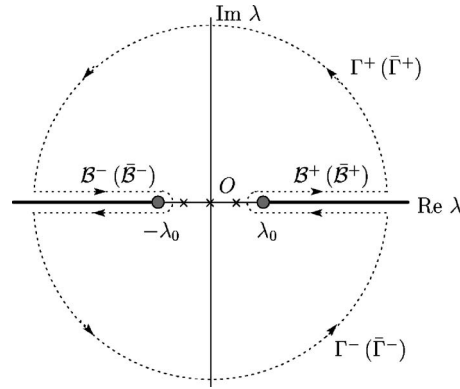


FIG. 1. Cuts, bold lines, and integral contours, dotted lines, on the upper (lower) sheet of the Riemann surface of  $\lambda$  plane for the self-defocusing case,  $\varepsilon = +1$ . The cross denotes the zero for  $\det A$  ( $\det \bar{A}$ ) on the upper (lower) sheet.

$$[S(\lambda, \zeta)]^{-1} = \begin{bmatrix} A^\dagger(\lambda^*, \zeta^*) & -\varepsilon B^\dagger(\lambda^*, \zeta^*) \\ -\varepsilon \bar{B}^\dagger(\lambda^*, \zeta^*) & \bar{A}^\dagger(\lambda^*, \zeta^*) \end{bmatrix}. \tag{3.28}$$

In this system, we have involution relations for the Jost functions

$$\psi^\pm(\lambda, \zeta; x) = \psi^\pm(\lambda, -\zeta; x) \mathcal{P}^\pm(\lambda, \zeta), \tag{3.29}$$

where

$$\mathcal{P}^\pm(\lambda, \zeta) = J(\zeta x) [T^\pm(\lambda, -\zeta)]^{-1} T^\pm(\lambda, \zeta) J(\zeta x) = \frac{1}{\lambda + \zeta} \begin{bmatrix} O & i\varepsilon Q_\pm^\dagger \\ -iQ_\pm & O \end{bmatrix}. \tag{3.30}$$

From the involution, we have another set of relations for the scattering data:

$$S(\lambda, \zeta) = [\mathcal{P}^+(\lambda, \zeta)]^{-1} S(\lambda, -\zeta) \mathcal{P}^-(\lambda, \zeta), \tag{3.31}$$

$$\bar{A}(\lambda, \zeta) = \frac{1}{\lambda_0^2} Q_+ A(\lambda, -\zeta) Q_-^\dagger, \tag{3.32}$$

$$\bar{B}(\lambda, \zeta) = -\frac{\varepsilon}{\lambda_0^2} Q_+^\dagger B(\lambda, -\zeta) Q_-^\dagger. \tag{3.33}$$

By using Eqs. (3.15) and (3.27), we can prove that

$$\det A(\lambda, \zeta) = \det[\psi_1^- \psi_2^+] = \det \bar{A}^\dagger(\lambda^*, \zeta^*) = (\det \bar{A}(\lambda^*, \zeta^*))^*. \tag{3.34}$$

Additionally, taking the determinant of both sides of Eq. (3.32), we have

$$\det \bar{A}(\lambda, \zeta) = \det A(\lambda, -\zeta) \det[\lambda_0^{-2} Q_+ Q_-^\dagger]. \tag{3.35}$$

Combining Eqs. (3.34) and (3.35), we find that  $\det A(\lambda, \zeta) \propto (\det A(\lambda^*, -\zeta^*))^*$ . As a consequence, if  $(\lambda_j, \zeta_j)$  is the zero of  $\det A$ ,  $(\lambda_j^*, -\zeta_j^*)$  is also the zero of  $\det A$ , and the pairs  $(\lambda_j, -\zeta_j)$  and  $(\lambda_j^*, \zeta_j^*)$  are the zeros of  $\det \bar{A}$ . For  $\varepsilon = +1$ , furthermore, the self-adjointness of the eigenvalue problem (3.1) leads to  $\lambda_j = \lambda_j^* \in \mathbb{R}$  and  $\zeta_j = -\zeta_j^* \in i\mathbb{R}$ .

Finally, we make clear the analytical properties of the Jost functions (3.10) in regard to complex  $\lambda$ . To this end, we prepare a two-sheet Riemann surface for  $\lambda$  where  $\zeta \equiv (\lambda^2 - \varepsilon \lambda_0^2)^{1/2}$  is single valued. For  $\varepsilon = +1$ , cuts are made in  $(-\infty, -\lambda_0)$  and  $[\lambda_0, \infty)$  (see Fig. 1). Each sheet is characterized such that  $\text{Im } \zeta > 0$  ( $\text{Im } \zeta < 0$ ) on the upper (lower) sheet. On the other hand, for

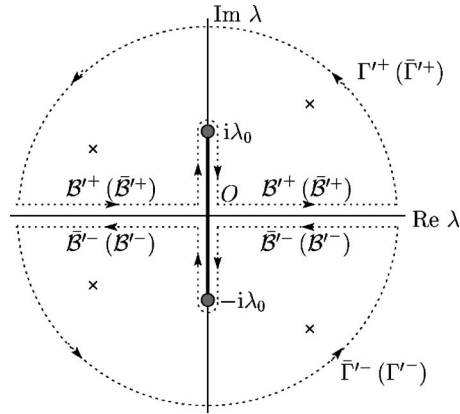


FIG. 2. A cut, bold line, and integral contours, dotted lines, on the upper (lower) sheet of the Riemann surface of  $\lambda$  plane for the self-focusing case,  $\varepsilon=-1$ . The cross denotes the zero for  $\det A$  and  $\det \bar{A}$  on both sheets.

$\varepsilon=-1$ , cuts are made in  $[-i\lambda_0, i\lambda_0]$  (see Fig. 2). It is required that  $\text{Im } \zeta \text{Im } \lambda > 0$  ( $\text{Im } \zeta \text{Im } \lambda < 0$ ) on the upper (lower) sheet. The Jost functions satisfy the following Volterra-type integral equations:

$$\phi^\pm(\lambda, \zeta; x) = J(\zeta x) \left( I + \int_{\pm\infty}^x dy J(\zeta y)^{-1} W \phi^\pm(y) \right). \tag{3.36}$$

Suppose that

$$\int_{-\infty}^{\infty} |(W_{ab})_{ij}| dx < \infty, \tag{3.37}$$

for all  $a, b=1, 2$  and  $i, j=1, \dots, l$ . We may have the Neumann series solution

$$\begin{aligned} \phi^\pm(\lambda, \zeta; x) J(\zeta x)^{-1} &= \sum_{n=0}^{\infty} \int_{\pm\infty}^x dy_1 \int_{\pm\infty}^{y_1} dy_2 \cdots \int_{\pm\infty}^{y_{n-1}} dy_n G(y_1) \cdots G(y_n), \\ &\equiv T^{<(>)} e^{\int_{\pm\infty}^x G(y) dy}, \end{aligned} \tag{3.38}$$

where  $G(y) = J(\zeta(y-x))^{-1} W J(\zeta(y-x))$  and  $T^{<(>)}$  denotes the time-ordered product. Examining the convergence of the Neumann series and its derivatives, it is found that  $\phi_1^-(\lambda, \zeta; x) e^{i\zeta x}$ ,  $\phi_2^+(\lambda, \zeta; x) e^{-i\zeta x}$  are bounded and analytic in the region where  $\text{Im } \zeta > 0$ , and  $\phi_1^+(\lambda, \zeta; x) e^{i\zeta x}$ ,  $\phi_2^-(\lambda, \zeta; x) e^{-i\zeta x}$  are bounded and analytic in the region where  $\text{Im } \zeta < 0$ . Relations (3.24c) and (3.24d) show that  $A(\lambda, \zeta)$  ( $\bar{A}(\lambda, \zeta)$ ) is analytic in the region where  $\text{Im } \zeta > 0$  ( $\text{Im } \zeta < 0$ ). We also have the asymptotic behaviors of the Jost functions and the scattering data as  $\lambda, \zeta \rightarrow \infty$  from the asymptotics of  $W$ :

$$W_{11} = \begin{cases} -\bar{Q}^{-1} \bar{Q}_x + \mathcal{O}(1/|\lambda|) & \text{if } \zeta \simeq \lambda \\ \mathcal{O}(1/|\lambda|) & \text{if } \zeta \simeq -\lambda, \end{cases} \tag{3.39a}$$

$$W_{12} = \mathcal{O}(1), \tag{3.39b}$$

$$W_{21} = \mathcal{O}(1), \tag{3.39c}$$

$$W_{22} = \begin{cases} \tilde{Q}_x \tilde{Q}^{-1} + \mathcal{O}(1/|\lambda|) & \text{if } \zeta \approx \lambda \\ \mathcal{O}(1/|\lambda|) & \text{if } \zeta \approx -\lambda. \end{cases} \quad (3.39d)$$

In calculating Neumann series, the following formulas are useful:

$$T^> e^{-\int_a^b \tilde{Q}^{-1} \tilde{Q}_x dx} = \tilde{Q}^{-1}(b) \tilde{Q}(a), \quad (3.40)$$

$$T^< e^{\int_a^b \tilde{Q}^{-1} \tilde{Q}_x dx} = \tilde{Q}^{-1}(a) \tilde{Q}(b), \quad (3.41)$$

for  $a < b$ . The proof is as follows. Write the right hand side (RHS) of Eq. (3.40) as  $X(a)$ . Differentiating  $X(a)$  gives  $X_a = X \tilde{Q}^{-1} \tilde{Q}_a$ . Solving this equation, we get  $X(a) = A \tilde{Q}(a)$ . Since  $X(b) = I$ , we find  $A = \tilde{Q}^{-1}(b)$ , and we arrive at the desired formula. We get Eq. (3.41) just in the same way. Consequently, when  $\zeta \approx \lambda$ , we have the asymptotics

$$\phi^\pm(\lambda, \zeta; x) J(\zeta x)^{-1} = \begin{bmatrix} \lambda_0^{-2} \tilde{Q}^\dagger(x) Q_\pm & O \\ O & \lambda_0^{-2} \tilde{Q}(x) Q_\pm^\dagger \end{bmatrix} + \mathcal{O}(1/|\lambda|), \quad (3.42a)$$

$$S(\lambda, \zeta) = \begin{bmatrix} \lambda_0^{-2} Q_+^\dagger Q_- & O \\ O & \lambda_0^{-2} Q_+ Q_-^\dagger \end{bmatrix} + \mathcal{O}(1/|\lambda|). \quad (3.42b)$$

On the other hand, when  $\zeta \approx -\lambda$ ,

$$\phi^\pm(\lambda, \zeta; x) J(\zeta x)^{-1} = I + \mathcal{O}(1/|\lambda|), \quad (3.43a)$$

$$S(\lambda, \zeta) = I + \mathcal{O}(1/|\lambda|). \quad (3.43b)$$

Furthermore, one can also show that

$$\psi_1^-(\lambda, \zeta; x) e^{i\zeta x} [A(\lambda, \zeta)]^{-1} - T^+(\lambda, \zeta) \begin{bmatrix} I \\ O \end{bmatrix} = \mathcal{O}(1) \quad \text{Im } \zeta > 0, \quad (3.44)$$

$$\psi_2^-(\lambda, \zeta; x) e^{-i\zeta x} [\bar{A}(\lambda, \zeta)]^{-1} - T^+(\lambda, \zeta) \begin{bmatrix} O \\ I \end{bmatrix} = \mathcal{O}(1) \quad \text{Im } \zeta < 0. \quad (3.45)$$

Note that, however, when  $\lambda = \pm \sqrt{\varepsilon} \lambda_0$  the scattering data may have a singularity of the order  $\mathcal{O}(1/\zeta)$  which can be seen from Eqs. (3.24), while the eigenfunctions are well defined at those branch points in the general situation.<sup>2</sup> It should be remarked that the introduction of  $\tilde{Q}(x)$  is irrelevant in the analysis of the original eigenvalue problem (3.1) and the inverse problem discussed in the subsequent section.

## B. Inverse problem

In this subsection, we derive the Gel'fand-Levitan-Marchenko equations which give the solution of the inverse problem, by using the Jost functions on the complex Riemann surface.

We assume that the Jost functions (3.10) are expressed as

$$\psi^\pm(\lambda, \zeta; x) = T^\pm(\lambda, \zeta) J(\zeta x) + \int_x^{\pm\infty} \mathcal{K}(x, s) T^\pm(\lambda, \zeta) J(\zeta s) ds, \quad (3.46)$$

where the kernel matrix is

$$\mathcal{K}(x,s) = \begin{bmatrix} \mathcal{K}_{11}(x,s) & \mathcal{K}_{12}(x,s) \\ \mathcal{K}_{21}(x,s) & \mathcal{K}_{22}(x,s) \end{bmatrix}, \quad (3.47)$$

with  $\mathcal{K}_{ij}(x,s)$  being  $l \times l$  matrix functions. Substituting expression (3.46) into the eigenvalue problem (3.1), after some calculations, we obtain a linear differential equation for the kernel matrix  $\mathcal{K}(x,s)$ ,

$$\begin{aligned} \partial_x \mathcal{K}(x,s) + (\sigma^z \otimes I) \partial_s \mathcal{K}(x,s) (\sigma^z \otimes I) + (\sigma^z \otimes I) \mathcal{K}(x,s) (\sigma^z \otimes I) [U^\pm(\lambda, \zeta) + i\lambda(\sigma^z \otimes I)] - [U(\lambda, \zeta; x) \\ + i\lambda(\sigma^z \otimes I)] \mathcal{K}(x,s) = O, \end{aligned} \quad (3.48)$$

with boundary conditions

$$2\mathcal{K}_{12}(x,x) = Q_+ - Q(x), \quad (3.49a)$$

$$2\mathcal{K}_{21}(x,x) = \varepsilon(Q_+^\dagger - Q^\dagger(x)), \quad (3.49b)$$

$$\mathcal{K}_{ij}(x,s) \rightarrow O \quad \text{as } s \rightarrow \pm\infty. \quad (3.49c)$$

This type of a linear system is known as the Marchenko equations and can be uniquely solved, which guarantees the existence of expression (3.46).

First, we concentrate on  $\varepsilon=+1$  case. Before going on, we mention several facts. For the characteristic roots (3.3), we have introduced two branch cuts on the real axis,  $(-\infty, -\lambda_0]$  and  $[\lambda_0, \infty)$ , as shown in Fig. 1. We define contour paths  $\mathcal{C}$  enclosing a region in the upper sheet ( $\text{Im } \zeta > 0$ ) of the Riemann surface of  $\lambda$  and  $\bar{\mathcal{C}}$  enclosing a region in the lower sheet ( $\text{Im } \zeta < 0$ ) as (see Fig. 1)

$$\mathcal{C} = \Gamma + \mathcal{B}, \quad \bar{\mathcal{C}} = \bar{\Gamma} + \bar{\mathcal{B}}, \quad (3.50a)$$

$$\Gamma = \Gamma^+ + \Gamma^-, \quad \bar{\Gamma} = \bar{\Gamma}^+ + \bar{\Gamma}^-, \quad (3.50b)$$

$$\mathcal{B} = \mathcal{B}^+ + \mathcal{B}^-, \quad \bar{\mathcal{B}} = \bar{\mathcal{B}}^+ + \bar{\mathcal{B}}^-, \quad (3.50c)$$

where the superscripts  $+$  and  $-$  denote the part of the path  $\Gamma$  ( $\bar{\Gamma}$ ) which exists in the upper and lower half planes of each sheet, respectively, and the part of the path  $\mathcal{B}$  ( $\bar{\mathcal{B}}$ ) which exists in the right and left half planes of each sheet, respectively. The radius of  $\Gamma$  ( $\bar{\Gamma}$ ) is assumed to be large enough for  $\mathcal{C}$  ( $\bar{\mathcal{C}}$ ) to enclose all the zeros of  $\det A$  ( $\det \bar{A}$ ). As noticed in Sec. III A, if  $(\lambda_j, \zeta_j)$  is a zero of  $\det A$ , it holds that  $\lambda_j \in \mathbb{R}$  and  $\zeta_j \in i\mathbb{R}$ . Correspondingly,  $(\lambda_j, -\zeta_j)$  is a zero of  $\det \bar{A}$ . Suppose here that  $\det A$  and  $\det \bar{A}$  have  $N$  zeros, respectively. Along the branch cut in the upper sheet, one can show that

$$\int_{\mathcal{B}} d\lambda \frac{\lambda}{\zeta} e^{i\zeta z} = 4\pi \delta(z), \quad (3.51)$$

$$\int_{\mathcal{B}} d\lambda e^{i\zeta z} = \int_{\bar{\mathcal{B}}} d\lambda \frac{e^{i\zeta z}}{\zeta} = 0, \quad (3.52)$$

where  $z$  is real and  $\delta(z)$  denotes the delta function. Using these formulas, we obtain

$$\frac{1}{4\pi} \int_{\mathcal{B}} d\lambda \frac{e^{i\zeta z}}{\zeta} T^{\pm}(\lambda, \zeta) = \delta(z)(\sigma^x \otimes I). \quad (3.53)$$

In the lower sheet, integral formulas (3.51)–(3.53) hold with replacing  $\mathcal{B} \rightarrow \bar{\mathcal{B}}$ ,  $\zeta \rightarrow -\zeta$ .

Going back to Eq. (3.46), we explicitly write as

$$\psi_1^+(\lambda, \zeta; x) = T^+(\lambda, \zeta) e^{-i\zeta x} \begin{bmatrix} I \\ O \end{bmatrix} + \int_x^{\infty} ds \mathcal{K}(x, s) T^+(\lambda, \zeta) e^{-i\zeta s} \begin{bmatrix} I \\ O \end{bmatrix}, \quad (3.54a)$$

$$\psi_2^+(\lambda, \zeta; x) = T^+(\lambda, \zeta) e^{i\zeta x} \begin{bmatrix} O \\ I \end{bmatrix} + \int_x^{\infty} ds \mathcal{K}(x, s) T^+(\lambda, \zeta) e^{i\zeta s} \begin{bmatrix} O \\ I \end{bmatrix}. \quad (3.54b)$$

We rewrite Eq. (3.15) into

$$\psi_1^-(\lambda, \zeta; x) [A(\lambda, \zeta)]^{-1} = \psi_1^+(\lambda, \zeta; x) + \psi_2^+(\lambda, \zeta; x) B(\lambda, \zeta) [A(\lambda, \zeta)]^{-1}, \quad (3.55a)$$

$$\psi_2^-(\lambda, \zeta; x) [\bar{A}(\lambda, \zeta)]^{-1} = \psi_2^+(\lambda, \zeta; x) + \psi_1^+(\lambda, \zeta; x) \bar{B}(\lambda, \zeta) [\bar{A}(\lambda, \zeta)]^{-1}. \quad (3.55b)$$

Substituting Eq. (3.54a) into Eq. (3.55a), we have a relation

$$\begin{aligned} \psi_1^-(\lambda, \zeta; x) [A(\lambda, \zeta)]^{-1} - T^+(\lambda, \zeta) e^{-i\zeta x} \begin{bmatrix} I \\ O \end{bmatrix} &= \int_x^{\infty} ds \mathcal{K}(x, s) T^+(\lambda, \zeta) e^{-i\zeta s} \begin{bmatrix} I \\ O \end{bmatrix} + \left\{ T^+(\lambda, \zeta) e^{i\zeta x} \begin{bmatrix} O \\ I \end{bmatrix} \right. \\ &\quad \left. + \int_x^{\infty} ds \mathcal{K}(x, s) T^+(\lambda, \zeta) e^{i\zeta s} \begin{bmatrix} O \\ I \end{bmatrix} \right\} B(\lambda, \zeta) [A(\lambda, \zeta)]^{-1}. \end{aligned} \quad (3.56)$$

When we multiply

$$\frac{e^{i\zeta y}}{4\pi\zeta} \quad (y > x) \quad (3.57)$$

on the both sides of Eq. (3.56), the left hand side of Eq. (3.56) becomes analytic on the upper sheet of the Riemann surface ( $\text{Im } \zeta > 0$ ), with the exception of the points  $\lambda_j$ , at which it has simple poles. Here we have assumed that  $1/\det A(\lambda, \zeta)$  has  $N$  isolated simple poles  $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$  in the upper sheet. From Eq. (3.44), we can show that as  $|\lambda| \rightarrow \infty$ , the left hand side of the resultant equation behaves like  $\exp[-\text{Im } \zeta(y-x)] \mathcal{O}(1/|\lambda|)$ . We integrate relation (3.56) multiplied by Eq. (3.57) along the contour  $\mathcal{B}$ . With the integration of the left hand side, the contour can be closed through infinity, i.e.,  $C = \mathcal{B} + \Gamma$ , so that the integral of the left hand side is equal to the sum of the residues at  $\lambda = \lambda_j$ ,

$$\begin{aligned} \frac{1}{4\pi} \int_C d\lambda \frac{e^{i\zeta(y-x)}}{\zeta} \left\{ \psi_1^-(\lambda, \zeta; x) e^{i\zeta x} [A(\lambda, \zeta)]^{-1} - T^+(\lambda, \zeta) \begin{bmatrix} I \\ O \end{bmatrix} \right\} &= \frac{i}{2} \sum_{j=1}^N \frac{e^{i\zeta_j y}}{\zeta_j} \psi_1^-(\lambda_j, \zeta_j; x) \frac{\tilde{A}(\lambda_j, \zeta_j)}{(\det A)'(\lambda_j, \zeta_j)} \\ &= i \sum_{j=1}^N e^{i\zeta_j y} \psi_2^+(\lambda_j, \zeta_j; x) \Pi_j, \end{aligned} \quad (3.58)$$

where  $\tilde{A}$  is the cofactor matrix of  $A$  and  $\Pi_j$  is the residue matrix at  $\lambda = \lambda_j$  defined by

$$\Pi_j = \text{Res}_{\lambda=\lambda_j, \zeta=\zeta_j} \left[ \frac{1}{2\zeta} B(\lambda, \zeta) [A(\lambda, \zeta)]^{-1} \right]. \quad (3.59)$$

In the second equality of Eq. (3.58), we have used a relation deduced from Eq. (3.55a) at  $(\lambda, \zeta) = (\lambda_j, \zeta_j)$ , which means that  $\psi_1^-(\lambda_j, \zeta_j; x)$  is proportional to  $\psi_2^+(\lambda_j, \zeta_j; x)$ . The integral in the right hand side of Eq. (3.58) is transformed into

$$\mathcal{K}(x, y) \begin{bmatrix} O \\ I \end{bmatrix} + \mathcal{F}_c(x+y) \begin{bmatrix} O \\ I \end{bmatrix} + \int_x^\infty ds \mathcal{K}(x, s) \mathcal{F}_c(s+y) \begin{bmatrix} O \\ I \end{bmatrix}, \quad (3.60)$$

where

$$\mathcal{F}_c(z) = \frac{1}{4\pi} \int_B d\lambda \frac{e^{i\xi z}}{\zeta} T^+(\lambda, \xi) \rho(\lambda, \xi), \quad (3.61)$$

$$\rho(\lambda, \xi) = B(\lambda, \xi) [A(\lambda, \xi)]^{-1}. \quad (3.62)$$

From the definition of  $T^+$  in Eq. (3.5), we obtain the following form:

$$\mathcal{F}_c(z) \begin{bmatrix} O \\ I \end{bmatrix} = \begin{bmatrix} i\mathcal{F}'_{1c}(z) + \mathcal{F}_{2c}(z) \\ i\varepsilon Q_+^\dagger \mathcal{F}_{1c}(z) \end{bmatrix}, \quad (3.63)$$

where  $\mathcal{F}'_{1c}(z) \equiv d\mathcal{F}_{1c}(z)/dz$ , and

$$\mathcal{F}_{1c}(z) = \frac{1}{4\pi} \int_{-\infty}^\infty d\xi \frac{e^{i\xi z}}{\lambda} [\rho(\lambda, \xi) - \rho(-\lambda, \xi)], \quad (3.64a)$$

$$\mathcal{F}_{2c}(z) = \frac{1}{4\pi} \int_{-\infty}^\infty d\xi e^{i\xi z} [\rho(\lambda, \xi) + \rho(-\lambda, \xi)]. \quad (3.64b)$$

Using the integral representation (3.54b) in Eq. (3.58), we rewrite Eq. (3.58) into the form (3.63) as follows;

$$\mathcal{F}_d(z) \begin{bmatrix} O \\ I \end{bmatrix} = \begin{bmatrix} i\mathcal{F}'_{1d}(z) + \mathcal{F}_{2d}(z) \\ i\varepsilon Q_+^\dagger \mathcal{F}_{1d}(z) \end{bmatrix}, \quad (3.65)$$

where

$$\mathcal{F}_{1d}(z) = \sum_{j=1}^N i\Pi_j e^{i\xi_j z}, \quad (3.66a)$$

$$\mathcal{F}_{2d}(z) = \sum_{j=1}^N i\lambda_j \Pi_j e^{i\xi_j z}. \quad (3.66b)$$

Combining Eqs. (3.60) and (3.65), we finally obtain the integral equation:

$$\mathcal{K}(x, y) \begin{bmatrix} O \\ I \end{bmatrix} + \mathcal{F}(x+y) \begin{bmatrix} O \\ I \end{bmatrix} + \int_x^\infty ds \mathcal{K}(x, s) \mathcal{F}(s+y) \begin{bmatrix} O \\ I \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix} \quad (y > x). \quad (3.67)$$

Here

$$\mathcal{F}(z) = \mathcal{F}_c(z) - \mathcal{F}_d(z). \tag{3.68}$$

Following the same procedure using Eqs. (3.54b) and (3.55b), we obtain

$$\mathcal{K}(x,y) \begin{bmatrix} I \\ O \end{bmatrix} + \bar{\mathcal{F}}(x+y) \begin{bmatrix} I \\ O \end{bmatrix} + \int_x^\infty ds \mathcal{K}(x,s) \bar{\mathcal{F}}(s+y) \begin{bmatrix} I \\ O \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix} \quad (y > x), \tag{3.69}$$

where

$$\bar{\mathcal{F}}(z) = \frac{1}{4\pi} \int_{\bar{\Gamma}} d\lambda \frac{e^{-i\zeta z}}{\zeta} T^+(\lambda, \zeta) \bar{\rho}(\lambda, \zeta), \tag{3.70}$$

$$\bar{\rho}(\lambda, \zeta) = \bar{B}(\lambda, \zeta) [\bar{A}(\lambda, \zeta)]^{-1}. \tag{3.71}$$

The residue matrix at the zero of  $\det \bar{A}$ , say,  $(\lambda, \zeta) = (\bar{\lambda}_j, \bar{\zeta}_j)$ , is defined by

$$\bar{\Pi}_j = \text{Res}_{\lambda=\bar{\lambda}_j, \zeta=\bar{\zeta}_j} \left[ -\frac{1}{2\zeta} \bar{B}(\lambda, \zeta) [\bar{A}(\lambda, \zeta)]^{-1} \right]. \tag{3.72}$$

Integral equations (3.67) and (3.69) are called the Gel'fand-Levitan-Marchenko equations. Solving these equations as to the kernel  $\mathcal{K}(x, y)$  for given  $\mathcal{F}(z)$  and  $\bar{\mathcal{F}}(z)$ , we can obtain the potential matrix  $Q$  through relations (3.49).

Let us move on to  $\varepsilon = -1$  case. Integral contours are depicted in Fig. 2. The branch cut in the Riemann surface is made along  $[-i\lambda_0, i\lambda_0]$ . Contours playing the same role as in the previous case are named with the same letter with a prime '. We define a contour path  $C'$  ( $\bar{C}'$ ) enclosing a region  $\text{Im } \zeta > 0$  ( $\text{Im } \zeta < 0$ ) as

$$C' = \Gamma' + B', \quad \bar{C}' = \bar{\Gamma}' + \bar{B}', \tag{3.73a}$$

$$\Gamma' = \Gamma'^+ + \Gamma'^-, \quad \bar{\Gamma}' = \bar{\Gamma}'^+ + \bar{\Gamma}'^-, \tag{3.73b}$$

$$B' = B'^+ + B'^-, \quad \bar{B}' = \bar{B}'^+ + \bar{B}'^-, \tag{3.73c}$$

where the superscripts + and - denote the part of paths in the upper and lower half planes of each sheet. The contours  $B'^{\pm}$  and  $\bar{B}'^{\pm}$  are along both the real axis and the branch cut. The radius of  $\Gamma'$  ( $\bar{\Gamma}'$ ) is assumed to be large enough for  $C'$  ( $\bar{C}'$ ) to enclose all the zeros of  $\det A$  ( $\det \bar{A}$ ). In contrast to the previous case, the zero of  $\det A$  is complex and makes a pair. Assume that there are  $N = 2M$  zeros for  $\det A$  and label them such that  $\lambda_{2k} = \lambda_{2k-1}^*$ ,  $\zeta_{2k} = -\zeta_{2k-1}^*$  for  $k = 1, \dots, M$ . Correspondingly, we have the zeros of  $\det \bar{A}$  such that  $\{(\lambda_1, -\zeta_1), (\lambda_1^*, \zeta_1^*), \dots, (\lambda_M, -\zeta_M), (\lambda_M^*, \zeta_M^*)\}$ . The derivation of the integral equation goes in parallel to that for the case  $\varepsilon = +1$ . As a consequence, we arrive at just the same equations (3.67) and (3.69).

We remark on the properties of the matrices  $\Pi_j$  and  $\bar{\Pi}_j$ . First, the determinant is zero. This is proven directly as

$$\det \Pi_j \propto \frac{\det B(\det A)^{l-1}}{[(\det A)']^l} (\lambda_j, \zeta_j) = 0, \tag{3.74}$$

where we have used  $\det A(\lambda_j, \zeta_j) = 0$ . Similarly, we have  $\det \bar{\Pi}_j = 0$ . Second, we have the following relations:

$$\Pi_j = \Pi_j^\dagger = -\bar{\Pi}_j \quad (\varepsilon = +1), \tag{3.75a}$$

$$\Pi_{2j-1} = \Pi_{2j}^\dagger = \bar{\Pi}_{2j-1} = \bar{\Pi}_{2j}^\dagger \quad (\varepsilon = -1). \quad (3.75b)$$

These are proven as follows. From Eqs. (3.25), (3.31), and (3.32), we obtain

$$\bar{B}\bar{A}^{-1}(\lambda, \zeta) = \varepsilon[(BA^{-1})(\lambda^*, \zeta^*)]^\dagger, \quad (3.76)$$

$$\bar{B}\bar{A}^{-1}(\lambda, \zeta) = -\varepsilon BA^{-1}(\lambda, -\zeta). \quad (3.77)$$

For example, we demonstrate  $\bar{\Pi}_j = -\Pi_j^\dagger$  for  $\varepsilon = +1$ . Substituting Eq. (3.76) into the definition of  $\bar{\Pi}_j$  (3.72) we have

$$\begin{aligned} \bar{\Pi}_j &= \operatorname{Res}_{\lambda=\lambda_j, \zeta=-\zeta_j} \left[ -\frac{1}{2\zeta} [(BA^{-1})(\lambda^*, \zeta^*)]^\dagger \right] = - \left\{ \operatorname{Res}_{\lambda^*=\lambda_j^*, \zeta^*=-\zeta_j^*} \left[ \frac{1}{2\zeta^*} (BA^{-1})(\lambda^*, \zeta^*) \right] \right\}^\dagger \\ &= - \left\{ \operatorname{Res}_{\lambda=\lambda_j, \zeta=\zeta_j} \left[ \frac{1}{2\zeta} (BA^{-1})(\lambda, \zeta) \right] \right\}^\dagger = -\Pi_j^\dagger, \end{aligned} \quad (3.78)$$

where in the third equality we have replaced the dummy variables as  $\lambda^* \rightarrow \lambda$ ,  $\zeta^* \rightarrow \zeta$  and used  $\lambda_j^* = \lambda_j$ ,  $\zeta_j^* = -\zeta_j$ . The rests are obtained similarly.

### C. Time dependence of the scattering data

Next, we consider the time dependence of the scattering data. Under the nonvanishing boundary conditions (2.6), the asymptotic forms of the Lax matrix  $V$  are given by

$$V \rightarrow V^\pm = 2(\lambda - k) \begin{bmatrix} -2i\lambda I & Q_\pm \\ \varepsilon Q_\pm^\dagger & 2i\lambda I \end{bmatrix} \quad \text{as } x \rightarrow \pm\infty. \quad (3.79)$$

Operating Eq. (3.79) on the asymptotic forms of the Jost functions (3.13) gives

$$V^\pm T^\pm(\lambda, \zeta) \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x} = -2i(\lambda - k) [\zeta + \lambda(\sigma^z \otimes I)] T^\pm(\lambda, \zeta) \begin{bmatrix} I \\ O \end{bmatrix} e^{-i\zeta x}, \quad (3.80a)$$

$$V^\pm T^\pm(\lambda, \zeta) \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x} = 2i(\lambda - k) [\zeta - \lambda(\sigma^z \otimes I)] T^\pm(\lambda, \zeta) \begin{bmatrix} O \\ I \end{bmatrix} e^{i\zeta x}. \quad (3.80b)$$

Then we define the time-dependent Jost functions  $\psi_j^{\pm(t)}$ ,  $j=1, 2$ , as

$$\psi_1^{\pm(t)} \equiv e^{-2i(\lambda-k)[\zeta+\lambda(\sigma^z \otimes I)]t} \psi_1^\pm \sim e^{-2i(\lambda-k)[\zeta+\lambda(\sigma^z \otimes I)]t-i\zeta x} T^\pm \begin{bmatrix} I \\ O \end{bmatrix} \quad \text{as } x \rightarrow \pm\infty, \quad (3.81a)$$

$$\psi_2^{\pm(t)} \equiv e^{2i(\lambda-k)[\zeta-\lambda(\sigma^z \otimes I)]t} \psi_2^\pm \sim e^{2i(\lambda-k)[\zeta-\lambda(\sigma^z \otimes I)]t+i\zeta x} T^\pm \begin{bmatrix} O \\ I \end{bmatrix} \quad \text{as } x \rightarrow \pm\infty, \quad (3.81b)$$

which obey, respectively,

$$\frac{\partial \psi_1^{\pm(t)}}{\partial t} = V \psi_1^{\pm(t)}, \quad (3.82a)$$

$$\frac{\partial \psi_2^{\pm(t)}}{\partial t} = V \psi_2^{\pm(t)}. \quad (3.82b)$$

These relations give



$$\frac{\partial \psi_1^-}{\partial t} = \{V + 2i(\lambda - k)[\zeta + \lambda(\sigma^z \otimes I)]\} \psi_1^-, \quad (3.83a)$$

$$\frac{\partial \psi_2^-}{\partial t} = \{V - 2i(\lambda - k)[\zeta - \lambda(\sigma^z \otimes I)]\} \psi_2^-. \quad (3.83b)$$

Substituting the definitions of the scattering data (3.15),

$$\psi_1^-(\lambda, \zeta; x, t) = \psi_1^+(\lambda, \zeta; x, t)A(\lambda, \zeta; t) + \psi_2^+(\lambda, \zeta; x, t)B(\lambda, \zeta; t), \quad (3.84a)$$

$$\psi_2^-(\lambda, \zeta; x, t) = \psi_1^+(\lambda, \zeta; x, t)\bar{B}(\lambda, \zeta; t) + \psi_2^+(\lambda, \zeta; x, t)\bar{A}(\lambda, \zeta; t), \quad (3.84b)$$

into Eqs. (3.83) and then taking the limit  $x \rightarrow \infty$ , we find the time dependence of the scattering data as follows:

$$A(\lambda, \zeta; t) = A(\lambda, \zeta; 0), \quad (3.85a)$$

$$B(\lambda, \zeta; t) = B(\lambda, \zeta; 0)e^{4i\zeta(\lambda-k)t}, \quad (3.85b)$$

$$\Pi_j(t) = \Pi_j(0)e^{4i\zeta_j(\lambda_j-k)t}. \quad (3.85c)$$

Using Eqs. (3.85), we obtain explicit time dependence of  $\mathcal{F}_1(z, t)$  and  $\mathcal{F}_2(z, t)$ ,

$$\mathcal{F}_1(z, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi \frac{e^{i\xi z + 4i\xi(\lambda-k)t}}{\lambda} [\rho(\lambda, \xi; 0) - \rho(-\lambda, \xi; 0)] - \sum_{j=1}^N i\Pi_j(0)e^{i\zeta_j z + 4i\zeta_j(\lambda_j-k)t}, \quad (3.86a)$$

$$\mathcal{F}_2(z, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi e^{i\xi z + 4i\xi(\lambda-k)t} [\rho(\lambda, \xi; 0) + \rho(-\lambda, \xi; 0)] - \sum_{j=1}^N i\lambda_j \Pi_j(0)e^{i\zeta_j z + 4i\zeta_j(\lambda_j-k)t}. \quad (3.86b)$$

We have similar time dependence for  $\bar{\mathcal{F}}_1(z, t)$  and  $\bar{\mathcal{F}}_2(z, t)$  with  $\bar{\rho}$  and  $\bar{\Pi}_j$ .

The procedure of the ISM for the initial value problem of the matrix NLS equation (2.1) is summarized as follows. First, we solve the eigenvalue problem (3.1) for the initial value  $Q(x, 0)$  and obtain the scattering data at  $t=0$  (direct problem). Then, with the time-dependent scattering data (3.85), we solve the Gel'fand-Levitan-Marchenko equations (3.67) and (3.69) to obtain  $\mathcal{K}(x, y, t)$  and  $Q(x, t)$  (inverse problem). This procedure provides the direct proof of the complete integrability of the matrix NLS equation (2.1) under the nonvanishing boundary conditions (2.6). Further researches are required to establish each step in a rigorous way.

## D. Soliton solutions

We construct soliton solutions of the matrix NLS equation under the reflection-free condition

$$B(\lambda, \xi) = \bar{B}(\lambda, \xi) = O \quad (\xi: \text{real}), \quad (3.87)$$

whereby the first terms,  $\mathcal{F}_c(z, t)$  parts, in Eqs. (3.86) identically vanish,

$$\mathcal{F}_1(z, t) = - \sum_{j=1}^N i\Pi_j(t)e^{i\zeta_j z}, \quad \mathcal{F}_2(z, t) = - \sum_{j=1}^N i\lambda_j \Pi_j(t)e^{i\zeta_j z}, \quad (3.88)$$

and similarly for  $\bar{\mathcal{F}}_1$  and  $\bar{\mathcal{F}}_2$ . Assume the form

$$\mathcal{K}_{11}(x, y, t) = i\lambda_0 \sum_{j=1}^N \mathcal{K}_{11}^{(j)} \bar{\Pi}_j(t) e^{i\zeta_j(x+y)}, \quad (3.89)$$

$$\mathcal{K}_{12}(x, y, t) = i\lambda_0 \sum_{j=1}^N \mathcal{K}_{12}^{(j)} \Pi_j(t) e^{i\zeta_j(x+y)}. \quad (3.90)$$

Then, the Gel'fand-Levitan-Marchenko equations (3.67) and (3.69) are reduced to a set of linear algebraic equations,

$$i\lambda_0 \mathcal{K}_{12}^{(j)} + i(\zeta_j - \lambda_j) I + \sum_{k=1}^N \frac{\lambda_0(\zeta_j - \lambda_j)}{i(\zeta_j + \zeta_k)} \mathcal{K}_{11}^{(k)} \bar{\Pi}_k(t) e^{i\zeta_k x} + \sum_{k=1}^N \frac{-i\varepsilon\lambda_0^2}{i(\zeta_j + \zeta_k)} \mathcal{K}_{12}^{(k)} \Pi_k(t) e^{i\zeta_k x} = 0, \quad (3.91)$$

$$i\lambda_0 \mathcal{K}_{11}^{(j)} - \lambda_0 I + \sum_{k=1}^N \frac{i\lambda_0^2}{i(\zeta_j + \zeta_k)} \mathcal{K}_{11}^{(k)} \bar{\Pi}_k(t) e^{i\zeta_k x} + \sum_{k=1}^N \frac{-\lambda_0(\zeta_j + \lambda_j)}{i(\zeta_j + \zeta_k)} \mathcal{K}_{12}^{(k)} \Pi_k(t) e^{i\zeta_k x} = 0, \quad (3.92)$$

for  $j=1, \dots, N$ . Those are simplified into

$$\mathcal{K}_{11}^{(j)} = \frac{i\lambda_0}{\zeta_j - \lambda_j} \mathcal{K}_{12}^{(j)}, \quad (3.93)$$

$$-I = \frac{\lambda_0}{\zeta_j - \lambda_j} \mathcal{K}_{12}^{(j)} + \sum_{k=1}^N \frac{\lambda_0^2 e^{2i\zeta_k x}}{i(\zeta_j + \zeta_k)(\zeta_k - \lambda_k)} \mathcal{K}_{12}^{(k)} \bar{\Pi}_k(t) - \sum_{k=1}^N \frac{\varepsilon\lambda_0^2 e^{2i\zeta_k x}}{i(\zeta_j + \zeta_k)(\zeta_j - \lambda_j)} \mathcal{K}_{12}^{(k)} \Pi_k(t). \quad (3.94)$$

Replace  $\lambda_j \rightarrow -\lambda_j$  for notational reason and note relations (3.75) and that the time dependence of  $\Pi_j$  and  $\bar{\Pi}_j$  is

$$\Pi_j(t) = \Pi_j(0) e^{\chi_j}, \quad \bar{\Pi}_j(t) = \bar{\Pi}_j(0) e^{\chi_j}, \quad (3.95)$$

where we introduce the function

$$\chi_j(x, t) = 2i\zeta_j(x - 2(\lambda_j + k)t). \quad (3.96)$$

Then, from Eq. (3.94) we have a matrix form

$$(\mathcal{K}_{12}^{(1)} \dots \mathcal{K}_{12}^{(N)}) \begin{bmatrix} S_{1,1} & \dots & S_{1,N} \\ \vdots & & \vdots \\ S_{N,1} & \dots & S_{N,N} \end{bmatrix} = - \overbrace{(I \dots I)}^N, \quad (3.97)$$

where for  $i, j=1, \dots, N$ ,

$$S_{ij} = \frac{\lambda_0}{\zeta_j + \lambda_j} \delta_{ij} I - \frac{\varepsilon\lambda_0^2}{i(\zeta_i + \zeta_j)} \left( \frac{1}{\zeta_i + \lambda_i} + \frac{1}{\zeta_j + \lambda_j} \right) \Pi_i e^{\chi_i}. \quad (3.98)$$

Solving the linear equations (3.97) and using Eq. (3.49a), we arrive at the multisoliton solution for the modified version of the matrix NLS equation (2.12),

$$Q(x,t) = \lambda_0 I + 2i\lambda_0 \overbrace{(I \cdots I)}^N S^{-1} \begin{bmatrix} \Pi_1 e^{\lambda_1 x} \\ \vdots \\ \Pi_N e^{\lambda_N x} \end{bmatrix}. \quad (3.99)$$

Before concluding, we summarize about the parameters of the solution. For  $\varepsilon = +1$ , the solution (3.99) is the  $N$ -soliton solution.  $\lambda_j$  ( $j=1, \dots, N$ ) is a real constant such that  $-\lambda_0 < \lambda_j < \lambda_0$ .  $\zeta_j = i(\lambda_0^2 - \lambda_j^2)^{1/2}$  is purely imaginary.  $\Pi_j$  is an  $l \times l$  Hermitian matrix. For  $\varepsilon = -1$ , solution (3.99) is the  $M (=N/2)$ -soliton solution.  $\lambda_j$  and  $\zeta_j = (\lambda_j^2 + \lambda_0^2)^{1/2}$  ( $j=1, \dots, N$ ) are complex constants satisfying  $\lambda_{2k} = \lambda_{2k-1}^*$  and  $\zeta_{2k} = -\zeta_{2k-1}^*$  for  $k=1, \dots, N/2$ .  $\Pi_j$  is an  $l \times l$  matrix satisfying  $\Pi_{2k-1} = \Pi_{2k}^\dagger$ .

Although, in a strict sense, we should impose that  $\det \Pi_j = 0$  for all  $j$ , we can relax this condition. The reason is that in the limiting case where two distinct  $\lambda_i$  and  $\lambda_j$  merge in the  $(N+1)$ -soliton solution, the expression, Eq. (3.99), for the solution remains true with replacing formally  $\Pi_i \rightarrow \Pi_i + \Pi_j$ . Recall that  $\det(\Pi_i + \Pi_j) \neq 0$  in general, even when  $\det \Pi_i = 0$  and  $\det \Pi_j = 0$ .

Finally, multiplying the carrier wave part  $e^{i(kx - \omega t)}$  as Eq. (2.10), the multisoliton solution for the original NLS equation (2.1) under nonvanishing boundary conditions (2.6) and (2.9) is obtained:

$$Q(x,t) = \lambda_0 e^{i(kx - \omega t)} \left\{ I + 2i \overbrace{(I \cdots I)}^N S^{-1} \begin{bmatrix} \Pi_1 e^{\lambda_1 x} \\ \vdots \\ \Pi_N e^{\lambda_N x} \end{bmatrix} \right\}. \quad (3.100)$$

As mentioned in Sec. II A, the multisoliton solutions for general nonvanishing boundary conditions are easily obtained through the unitary transformations (2.8).

#### IV. CONCLUDING REMARKS

In this paper, we have studied both the self-defocusing and the self-focusing matrix nonlinear Schrödinger (NLS) equations under nonvanishing boundary conditions. Introducing the Lax pair, we have made the inverse scattering method (ISM) analysis for the systems and shown that the initial value problem is solvable. From the Gel'fand-Levitan-Marchenko equation with the reflection-free condition, the multisoliton solution is obtained explicitly. The conservation laws are given in the same manner as for the vanishing boundary conditions,<sup>13</sup> leading to an infinite number of the conserved quantities. On the other hand, the Lax pair, the direct and inverse problems, and the multisoliton solution are altered due to the boundary conditions.

The multicomponent systems with nonvanishing boundary conditions include several important models in physics, for example, the spinor model for Bose-Einstein condensates with repulsive and antiferromagnetic interactions. The equation for the dynamics of  $F=1$  spinor condensate falls into the  $l=2$  case of the matrix NLS equation. As expected from the applicability of the matrix NLS equation to the spinor bright solitons,<sup>14,15</sup> it is interesting to analyze the spinor "dark" solitons based on the results obtained in this work. We report this issue in a separate paper.<sup>19</sup>

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