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A Note on Super Catalan Numbers

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We show that the super Catalan numbers are special values of the Krawtchouk polynomials by deriving an expression for the super Catalan numbers in terms of a signed set.

KEYWORDS: Catalan number, Krawtchouk polynomial, MacWilliams identities, von Szily's identity, lattice path

The super Catalan numbers,

$$S(m,n) := \frac{(2m)!(2n)!}{m!n!(m+n)!},$$

as designated by Gessel in [9, Eq. (28)] form, hierarchically speaking, special cases of super ballot numbers (cf. [9, pp. 180, 189]). Historically, Gessel points out that these numbers had been observed as early as 1874 and studied by E. Catalan [6]; Aguiar and Hsiao in [1] provide a more detailed account of earlier appearences (cf. [5], [8], [7], and Riordan [14, Chapter 3, Exercise 9, p. 120]). Surprisingly, the innocent looking numbers, $S(m,n)$, seem to have been immune against a combinatorial interpretation for all values of (m,n) over the last century despite limited success stories for particular values (see problem 66(a) in Stanley's bijective open problems compendium [16]). The following references highlight the success cases. In [9], Gessel notes that for $S(1,n)/2$ we obtain the Catalan number C_n; whereas for the case when $m=0$, we yield middle binomial coefficients, $\binom{2n}{n}$. In [10], Gessel and Xin provide a combinatorial interpretation in terms of Dyck paths when $m=2$ or 3. An alternative combinatorial interpretation for the case $m=2$ was provided by Schaeffer in [15] using a method that was introduced in the interpretation to formulas of Tutte for planar maps. A more topologically flavored yet still combinatorial interpretation for the $m=2$ case is also available by Pippenger and Schleich in [13]; they count cubic trees on n interior vertices (or the number of hexagonal trees with n nodes). In 2005, Callan in [4] provided an elegant combinatorial interpretation of the recurrence $S(m,n)/2 = \sum_{k\geq 0} 2^{n-m-2k}\binom{n-m}{2k}S(m,k)/2$ for the case when $m=2$ showing that it enumerates the aligned cubic trees by number of vertices that are neither a leaf nor adjacent to a leaf.

In this note we establish the following expression for super Catalan numbers:

$$S(m,n) = (-1)^m \sum_{P\in\mathcal{P}_{m+n}} (-1)^{h_{2m}(P)}, \tag{1}$$

where the sum is over the set $\mathcal{P}_{m+n}$ of all lattice paths from $(0,0)$ to $(m+n, m+n)$ consisting of unit steps to the right and up, and $h_{2m}(P)$ denotes the height of $P=(P_0, P_1, \ldots, P_{2(m+n)}) \in \mathcal{P}_{m+n}$ after the $2m^{\text{th}}$ step, i.e., the y-coordinate of P_{2m}. Although this is an interpretation of $S(m,n)$ in terms of a signed set only, the right-hand side of (1) is a special value of the Krawtchouk polynomial defined as follows:

$$K_j^d(x) = \sum_{h=0}^{j} (-1)^h \binom{x}{h}\binom{d-x}{j-h}.$$

Then (1) is equivalent to

$$K_{m+n}^{2(m+n)}(2m) = (-1)^m S(m,n). \tag{2}$$

To see the equivalence, observe that each $P\in\mathcal{P}_{m+n}$ has exactly $m+n$ up-steps and that the number of $P\in\mathcal{P}_{m+n}$ with $h_{2m}(P)=h$ is therefore equal to $\binom{2m}{h}\binom{2n}{m+n-h}$.

Krawtchouk polynomials $K_j^d(x)$ appear as the coefficients of the so-called MacWilliams identities (cf. [12, Chap. 5, §2]), and also as the eigenvalues of the distance-j graph of the d-cube (cf. [3, Chap. 3, §2]). The identity shows that $\{(-1)^m S(m,n) \mid m,n\geq 0,\ m+n=N\}$ coincides with the set of non-zero eigenvalues of the distance-N graph of

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the $2N$-cube, which is known as the orthogonality graph and has been studied in connection with pseudo-telepathy in quantum information theory (cf. [11]). Finally, (2) follows immediately from the identity of von Szily (cf. [9, Eq. (29)]):

$$\begin{aligned} S(m,n) &= \sum_{k\in\mathbb{Z}}(-1)^k\binom{2m}{m+k}\binom{2n}{n-k} \\ &= (-1)^m\sum_{h=0}^{m+n}(-1)^h\binom{2m}{h}\binom{2n}{m+n-h} \\ &= (-1)^m K_{m+n}^{2(m+n)}(2m). \end{aligned}$$

We note that (2), as well as the identity of von Szily, is just a restatement of (a special case of) Kummer's evaluation of well-poised ${}_2F_1(-1)$ series.

To obtain a proper interpretation as the size of a set of certain paths, we need to find an injection from the set

$$\{P \in \mathcal{P}_{m+n} \mid h_{2m}(P) \not\equiv m\,(\operatorname{mod} 2)\}$$

to

$$\{P \in \mathcal{P}_{m+n} \mid h_{2m}(P) \equiv m\,(\operatorname{mod} 2)\},$$

and a description of the complement of the image. This is known for the case $m = 1$ (see [2, Section 5.3]), but it seems to be a difficult problem in general.

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