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Distance Trisector of a Segment and a Point

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Motivated by the work of Asano *et al.* [1], we consider the distance trisector problem and *zone diagram* considering segments in the plane as the input geometric objects. As the most basic case, we first consider the pair of curves (distance trisector curves) trisecting the distance between a point and a line, as shown in Figure 1. This is a natural extension of the bisector curve (that is a parabola) of a point and a line. In this paper, we show that these trisector curves C_1 and C_2 exist and are unique.

KEYWORDS: Distane trisector, Zone diagram, Voronoi diagram, Computational geometry

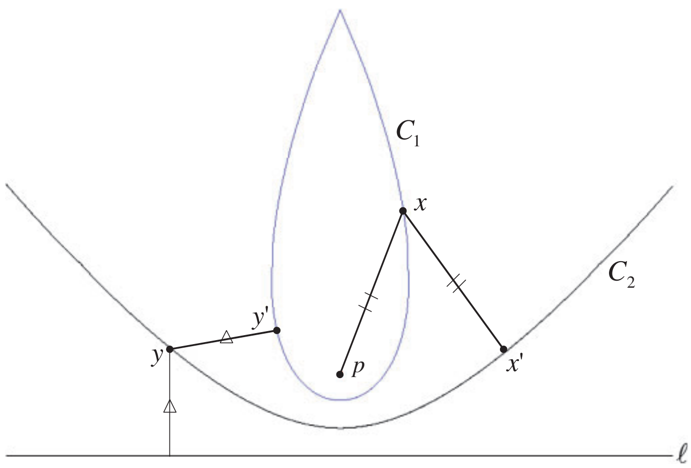

Fig. 1. Distance trisector curves between a point and a line.

1. Introduction

1.1 Distance Trisector

Bisector of two objects play fundamental roles in mathematics and computer science. We are taught in elementary school that the bisector of two points is the perpendicular bisector line, and in junior high school that bisector of a point and a line is a parabola. They are among the most fundamental tools in science and technology. Also, in computational geometry, the Voronoi diagram is a very important geometric structure, that can be considered "generalized bisector of n points" [4].

It is a natural question what happens if the bisection is replaced by trisection (or more). Given two disjoint geometric objects O_1 and O_2, their distance trisector is the pair of two curves C_1 and C_2 separating them such that C_1 is the bisector of O_1 and C_2, while C_2 is the bisector of C_1 and O_2. More generally, the distance k-sector of R_1 and R_2 is a series of $k-1$ curves $C_1, C_2, \ldots, C_{k-1}$ such that C_i is the bisector of C_{i-1} and C_{i+1} for $i = 1, 2, \ldots, k-1$, where we regard $C_0 = O_1$ and $C_k = O_2$. The following story gives some intuition of the distance k-sector: Imagine that $k-1$ robots go through between two objects in parallel. We do not know the timing of the movement of robots, but we can guide the geometric route that each robot moves along. Then, the most safe routes (robot may often out of the route to some extent) of $k-1$ robots form the distance k-sector of two objects.

The concept of trisector curves is proposed by Asano *et al.* [2], and the case where both O_1 and O_2 are points is studied there.

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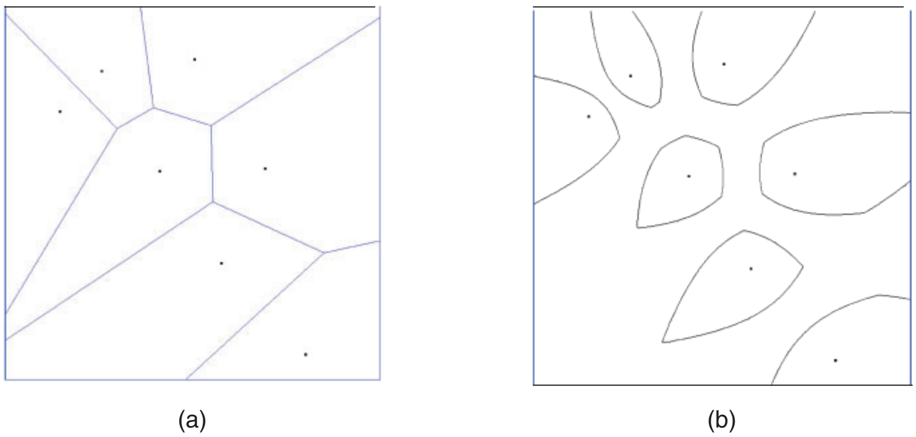


Fig. 2. For points set: (a) the classical Voronoi diagram (b) the Zone diagram.

1.2 Zone diagrams

Extending the idea of a trisector, Asano *et al.* [1] proposed a new kind of planar dissection called *Zone diagram*, which separates n planar points by using n curves such that each curve is the bisector of a point and the union of other curves. The left picture in Figure 2(a) shows the classical Voronoi diagram, and right picture (b) shows the Zone diagram of seven points.

The Voronoi diagram is one of the most popular structures in computational geometry. It is frequently used as a mathematical model to represent a pattern created by a competitive growth process where many bodies grow simultaneously to form a geometric structure together, such as the cell structure of a biological tissue, a crystal-lattice structure, a geographic/geological pattern, an economic/political regional equilibrium, or gravity/electromagnetic field.

There are several generalizations and variations of Voronoi diagrams, and their geometric properties and computational complexities are widely studied; see, e.g., [4, 5]. A common feature of these variations is that they define *partitions* of space into regions (*Voronoi cells*), each of which is the dominating region of an input point or object. However, geometric structures are sometimes observed in the nature in which the union of the cells has a nonempty complement region (called the *neutral zone*). We can regard such a structure as a result of growth process in which the growth terminates before the cell boundaries meet each other, and the termination is due to some non-contact action of other regions. The *Zone diagram* is a way of modeling such a structure.

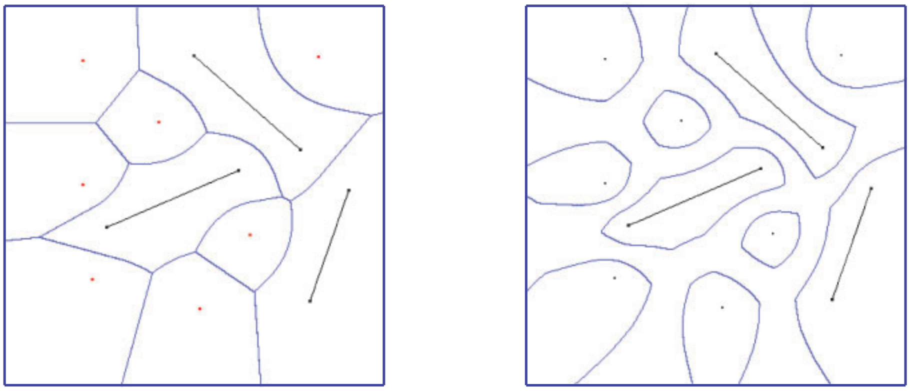
The idea can be explained by a story on equilibrium in the "age of wars". There are n mutually hostile kingdoms. The ith kingdom has a castle at a given location $\mathbf{p}_i$ and a territory R_i around it. The n territories are separated by a no-man's land, the neutral zone. If the territory R_i is attacked from another kingdom, an army departs from the castle $\mathbf{p}_i$ to intercept the attack. The interception succeeds if and only if the defending army arrives at the attacking point on the border of R_i sooner than enemy. However, the attacker can secretly move his troops inside his territory, and the defense army can start from its castle only when the attacker leaves his territory. The Zone diagram is an equilibrium configuration of the territories, such that every kingdom can guard the territory and no kingdom can grow without risk of invasion by other kingdoms. Mathematically, the distance of each point x on the border of the territory R_i to the capital $\mathbf{p}_i$ equals the distance of x to the union of the other R_j, $j \neq i$; this gives the definition of the Zone diagram.

Contribution of this paper

Although a set of points is the most basic geometric object, it is often convenient to consider a more general geometric object, and a set of line segments is frequently discussed in computational geometry. Indeed, Voronoi diagram of line segments is widely studied, and it has been shown that most of algorithmic results on point set Voronoi diagram can be extended. Therefore, it is natural to consider Zone diagram of a set of line segments.

The left picture in Figure 3(a) shows the classical Voronoi diagram, and right picture (b) shows the Zone diagram whit seven points and three line segments.

Distance trisector is a indispensable tool to draw a Zone diagram. In this paper, we first show that the distance trisector of two line segments uniquely exists, and give an algorithm to compute them. Indeed, the main difficulty lies in the case where we consider the trisector between a point and a line, where the bisector is known to be a parabola. This immediately implies that the distance 6-sector of two points exists. The method naturally leads to a practical algorithm for computing the distance trisector, provided that we can compute the bisector of a convex curve and a point, and also bisector of a convex curve and a line.



(a) (b)

Fig. 3. For points set and line segments set: (a) the Voronoi diagram (b) the Zone diagram.

2. Preliminary

Given n points $p_1, \ldots, p_n$ in the plane, each point p_i is assigned a territory R_i.

$$R_i = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, p_i) \leq d(\mathbf{z}, \{R_j : j \neq i\})\},$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance.

For a point $\mathbf{a}$ and a set $X \subseteq \mathbb{R}^2$ we define the *dominance region* of $\mathbf{a}$ with respect to X as

$$\mathrm{dom}(\mathbf{a}, X) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, \mathbf{a}) \leq d(\mathbf{z}, X)\},$$

where $d(\mathbf{z}, X) = \inf_{x \in X} d(\mathbf{z}, x)$. And similarly, for a line l and a set $X \subseteq \mathbb{R}^2$ we define the *dominance region* of l with respect to X as

$$\mathrm{dom}(l, X) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, l) \leq d(\mathbf{z}, X)\}.$$

For a point $\mathbf{a}$ and a set $X \subseteq \mathbb{R}^2$ the *bisector* of $\mathbf{a}$ and X is

$$\mathrm{bisect}(\mathbf{a}, X) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, \mathbf{a}) = d(\mathbf{z}, X)\}.$$

Similarly, for a line l and a set $X \subseteq \mathbb{R}^2$ we define

$$\mathrm{bisect}(\ell, X) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, \ell) = d(\mathbf{z}, X)\}.$$

And more generally, for a set X and a set Y we define

$$\mathrm{dom}(X, Y) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, X) \leq d(\mathbf{z}, Y)\}$$

and

$$\mathrm{bisect}(X, Y) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, X) = d(\mathbf{z}, Y)\}.$$

The following basic properties are given in [2]:

Lemma 1 (Properties of bisector).

(1) $\mathrm{dom}(\mathbf{a}, X)$ *is a closed convex set for every* $\mathbf{a}$ *and every* X.

(2) *(Antimonotonicity) The operator* $\mathrm{dom}(\cdot, \cdot)$ *is antimonotone with respect to the second argument; that is, if* $X \subseteq X'$, *then* $\mathrm{dom}(l, X) \supseteq \mathrm{dom}(l, X')$. *Similarly, for a line* l, *if* $X \subseteq X'$, *then* $\mathrm{dom}(\mathbf{a}, X) \supseteq \mathrm{dom}(\mathbf{a}, X')$.

(3) *If* $\mathbf{z}$ *is a point of* $\mathrm{bisect}(\mathbf{a}, X)$, *then there exists a unique point* $\mathbf{z}' \in X$ *nearest to* $\mathbf{z}$, *and the segment* $\mathbf{z}'\mathbf{z}$ *is an outer normal of* X *at* $\mathbf{z}'$ *(that is, it is perpendicular to some supporting line of* X *at* $\mathbf{z}'$*), and the unique tangent of* $\mathrm{bisect}(\mathbf{a}, X)$ *at* $\mathbf{z}$ *is the perpendicular bisector of the points* $\mathbf{a}$ *and* $\mathbf{z}'$.

3. Existence and uniqueness of distance trisector of segments

We would like to consider distance trisector curves between a pair of line segments s_1 and s_2. The easiest case is that both line segments are complete lines. In that case, the distance trisector curves are naturally the angle trisector lines of the angle between s_1 and s_2 (defined for each of two angles between the lines). Although angle trisecting is a famous problem that cannot be drawn by "ruler and compas", it is easy to compute if we can use trigonometric functions. Indeed, it is anciently known (Archimedes's algorithm) that angular trisector lines can be drawn if we are allowed to use a "ruler with a mark on it".

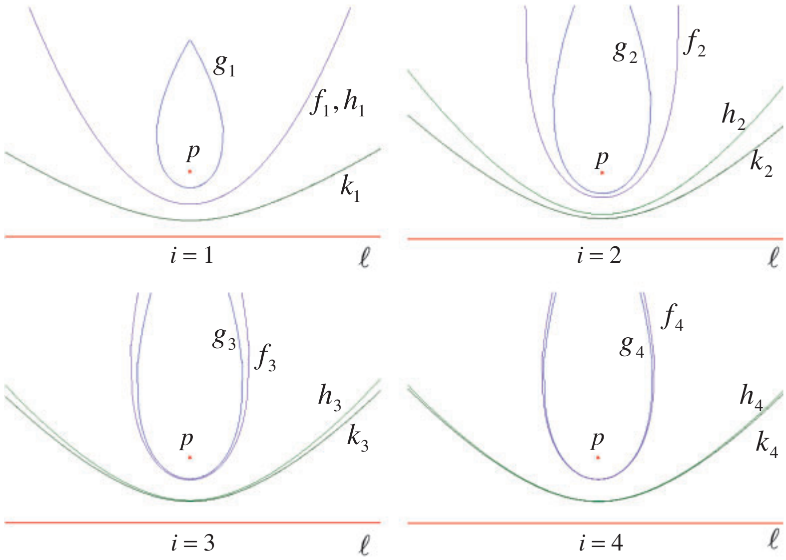


Fig. 4. The curves f_i, g_i, h_i and k_i.

Thus, the difficulty lies in that the segments may have endpoints and the nearest point from one of the trisector curves can be an endpoint. Indeed, it is not difficult to see that if we can prove the existence and uniqueness of distance trisector curves between a point and a line, we can also prove the existence and uniqueness of distance trisector curves between two given segments.

Hence, we focus on the case of a point and a line. Our main result in this section is as follows:

Theorem 1 (Existence and Uniqueness). *Given a point $p = (0, 1)$ and a line $\ell : y = -1$, there exist exactly one pair of curves (C_1, C_2) that trisect them. Moreover, C_1 is the boundary curve of a convex closed region, and C_2 is the image of a convex continuous function as shown in Figure 1.*

Although the theorem is analogous to the one for the pair of points $p = (0, -1)$ and $q = (0, -1)$ given in [2], the major difficulty is that the problem is not symmetric with respect to the x-axis if we consider the point-line problem. In point-point problem, we can show that $C_2 = -C_1$, although we do not have such nice symmetry here. Thus, in some sense, the number of parameters is doubled.

The proof is based on construction of four sequences f_i, g_i, h_i and k_i ($i = 1, 2, \ldots$) of curves each of which converges to one of the trisector curves.

Definition 1. *We define 4 infinite sequences (f_i), (g_i), (h_i) and (k_i) of curves for $i = 1, 2, \ldots$, as follows:*

(1) $f_1 = h_1 = \mathrm{bisect}(p, l)$

(2) $k_i = \mathrm{bisect}(f_i, l)$, $g_i = \mathrm{bisect}(p, h_i)$

(3) $f_{i+1} = \mathrm{bisect}(p, k_i)$, $h_{i+1} = \mathrm{bisect}(g_i, l)$

Let $R(f_i)$ and $R(g_i)$ are regions bounded by f_i and g_i containing p, respectively. Let $R(h_i)$ and $R(k_i)$ are regions bounded by h_i and k_i containing ℓ, respectively. By definition, $R(f_i) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, p) \le d(\mathbf{z}, k_{i-1})\}$ and $R(g_i) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{z}, p) \le d(\mathbf{z}, h_i)\}$. By Lemma 1 the curves f_i and g_i are convex and differentiable.

Figure 4 illustrates the curves of a point p and a line l such that f_i, g_i, h_i and k_i for $i = 1, 2, 3, 4$. Let C_1 and C_2 by any trisectors of p and ℓ. The following lemma is almost trivial.

Lemma 2. *(1) $R(f_i)$ and $R(g_i)$ are closed convex sets.*
(2) h_i and k_i are x-monotone and convex, that is, they can be expressed as images of convex continuous functions.
(3)

$$R(g_1) \subseteq R(g_2) \subseteq R(g_3) \subseteq \ldots \subseteq R(C_1) \subseteq \ldots \subseteq R(f_3) \subseteq R(f_2) \subseteq R(f_1).$$

Also,

$$R(k_1) \subseteq R(k_2) \subseteq R(k_3) \subseteq \ldots \subseteq R(C_2) \subseteq \ldots \subseteq R(h_3) \subseteq R(h_2) \subseteq R(h_1).$$

The following proposition is easy from basic knowledge of analysis that a bounded convex function uniformly converges.

Proposition 1. *For any given $c > 0$, the sequences h_i and k_i uniformly converge to convex continuous functions h and k in the range $-c \le x \le c$, respectively. Also, f_i and g_i uniformly converge to curves f and g, respectively.*

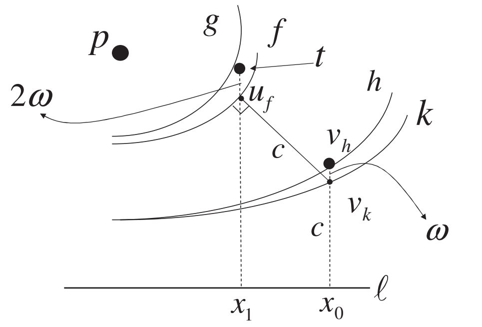


Fig. 5. Proof of Lemma 3.

Then, by definition, $f = \text{bisect}(p, k)$ and $k = \text{bisect}(f, \ell)$. Thus, the pair f, k satisfies the conditions of trisector curves. Similarly, the pair g, h forms trisector curves. Thus, the existence has been proved, and the difficult part is the uniqueness. By Lemma 2 (3), if we show $h \equiv k$ (accordingly, $f \equiv g$), we can conclude that the pair (f, k) are the unique trisector curves, where $h \equiv k$ means that the two curves are identical.

Thus, in the following, we shall show that $h \equiv k$. The key idea of the proof is to show that there exists a constant D such that for any $0 < x_0 < D$ there exists $0 < x' \leq x_0$ such that

$$h(x') - k(x') \geq 2(h(x_0) - k(x_0)).$$

Then, if $h(x_0) \neq k(x_0)$, for any given i there is a point $x'' \leq x_0$ such that $h(x'') - k(x'') \geq 2^i(h(x_0) - k(x_0))$, and we have contradiction to the boundedness of h. Thus, $h(x) = f(x)$ for $0 \leq x \leq D$, and we can easily derive $h \equiv f$ from this by considering the largest value of x that $h(x) = f(x)$ holds.

Thus, it suffices to show the existence of $x' \leq x_0$ satisfying $h(x') - k(x') \geq 2(h(x_0) - k(x_0))$. The proof is based on three lemmas. For any $x_0 \geq 0$, let $u_f = (x_1, f(x_1))$ be the nearest point on f from the point $v_k = (x_0, k(x_0))$. We set a constant D such that for any $0 < x_0 \leq D$, u_f is on the lower boundary curve of $R(f)$. We further assume that $g(x_1) \leq 1$, where $g(x_1)$ is the y-coordinate value of the lower boundary curve of $R(g)$. Such a constant D exists, since $g(0) = 2/3$.

Lemma 3. *If $u_f = (x_1, f(x_1))$ is the nearest point on f from $v_k = (x_0, k(x_0))$, $x_1 < x_0$ and $g(x_1) - f(x_1) \geq 2(h(x_0) - k(x_0))$.*

Proof. See Figure 5. Let $c = k(x_0)$. Since k is the bisector of f and ℓ, $d(u_f, v_k) = c$ by definition of u_f. Consider the point $v_h = (x_0, h(x_0))$ and $\omega = h(x_0) - c$. Let us consider the point $t = (x_1, f(x_1) + 2\omega)$. Then, the distance from v_h to t is less than $c + \omega$, since we have a one-legged path $v_h \to m \to t$ of length $a + \omega$ if m is the vertex of the parallelogram $mu_f v_k v_h$. Since h is the bisector curve of ℓ and g, the distance from v_h to g is $c + \omega$. Therefore, $d(v_h, t) < c + \omega$ implies that t must be below the curve g. Thus, $g(x_1) - f(x_1) \geq 2\omega = 2(h(x_0) - k(x_0))$. □

Lemma 4. *Let $u_g = (x_1, g(x_1))$ such that $g(x_1) \leq 1$, and let $w_h = (x_2, h(x_2))$ be the nearest point on h from u_g. Then, $g(x_1) - f(x_1) < h(x_2) - k(x_2)$.*

Proof. See Figure 6. We set $u_f = (x_1, f(x_1))$, $a = d(p, v_g)$ and $\delta = g(x_1) - f(x_1)$. Since $g(x_1) \leq 1$, $b = d(p, u_f) > d(p, v_g) = a$. Consider the point $s = (x_2, h(x_2) - \delta)$. Since $d(p, u_f) = b > a = d(u_f, s)$ and f is the bisector of p and the curve k, the point s must be above k. Hence, $h(x_2) - k(x_2) > \delta$, and we have the lemma. □

Lemma 5. *Let x_0, x_1 and x_2 be the values such that the assumptions in both of Lemma 3 and Lemma 4 hold. Then, $0 < x_2 \leq x_0$.*

Proof. We assume that $x_2 > x_0$ and derive a contradiction. See Figure 7 (refer to the figure for definitions). We set $u_g = (x_1, g(x_1))$, $u_f = (x_1, f(x_1))$, $v_k = (x_0, f(x_0))$, and $w_h = (x_2, h(x_2))$. By definition, $d(p, u_g) = d(u_g, v_h) = a$. We have $a \leq b$ since $g(x_1) < 1$. Since the distance from the point u_f to k is b, $c = d(u_f, v_k) \geq b$. Similarly, the distance d from w_h to ℓ equals the distance from w_h to g, and hence it is not more than than a. Thus, we have $d \leq a < b \leq c$. However, $c \leq d$ because of convexity of functions, and we have contradiction. □

Now, we have $x_2 \leq x_0$ and $h(x_2) - k(x_2) \geq 2(h(x_0) - k(x_0))$, which we desired to prove. Theorem has been proved.

6-sector of two points

As a direct application of Theorem 1, we give the following:

Theorem 2 (6-sector of two points). *Given two points $p = (0, 1)$ and $q = (0, -1)$, there exist a set of 5 curves to give the 6-sector of p and q. Moreover, this is the unique 6-sector that is symmetric with respect to the x-axis.*

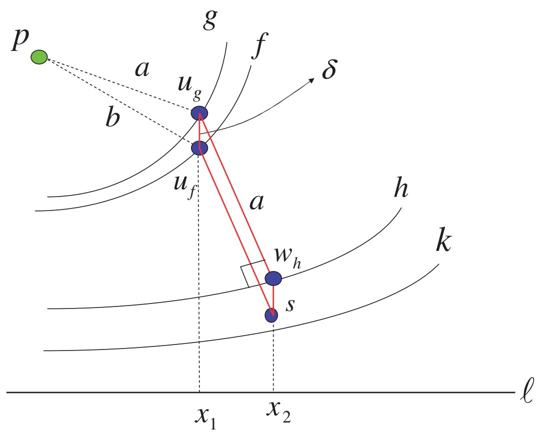


Fig. 6. Proof of Lemma 4.

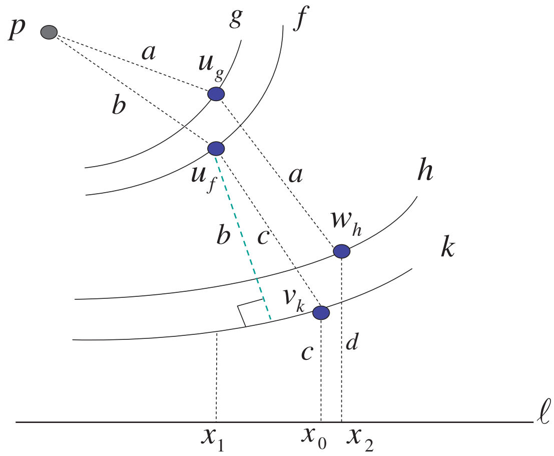


Fig. 7. Proof of Lemma 5.

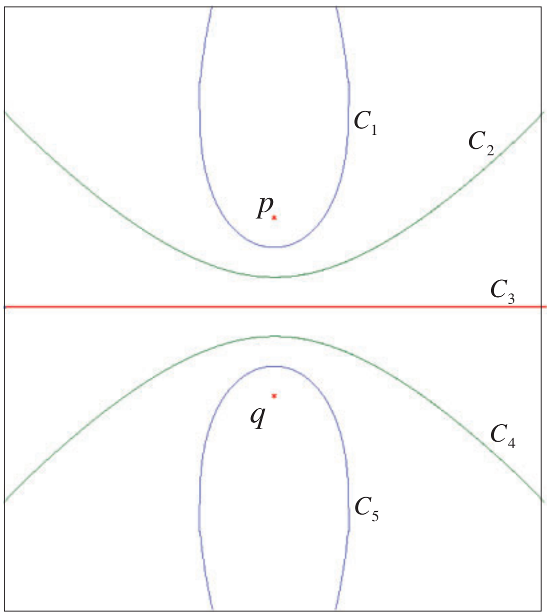


Fig. 8. 6-sector of a point p and a point q.

Note that the existence of k-sector is only known for $k = 2, 3, 4$ before. Although the existence of 4-sector is trivial (x-axis and two parabolas), its uniqueness is not known. Figure 8 illustrates 6-sector of a point p and a point q. This shows symmetricity of 6-sector with respect to the x-axis.

The existence and uniqueness proof naturally leads to the following algorithm for computing the trisector.

Input: point p, line ℓ, error ε
Output: point sets $C(g_i)$, $C(h_i)$ that approximate distance trisector of a point p and a line ℓ for $-n/2 \le x \le n/2, -n/2 \le y \le n/2$

Algorithm *DrawTrisectorCurves*(p, ℓ, ε)

compute bisect(p, l) with $C(f_1)$, $C(h_1)$
$C(g_1) = \text{bisect}(p, C(h_1))$, $C(k_1) = \text{bisect}(C(f_1), \ell)$
$i = 1$
while $(max_{0 \le x \le n/2}(h_i(x) - k_i(x)) > \varepsilon)$
    $i \leftarrow i + 1$
    $C(f_i) = \text{bisect}(p, C(k_{i-1}))$, $C(h_i) = \text{bisect}(C(g_{i-1}), \ell)$
    $C(g_i) = \text{bisect}(p, C(h_i))$, $C(k_i) = \text{bisect}(C(f_i), \ell)$
    return g_i, h_i.

The difficulty that lies in the algorithm is the computation of the bisectors of curves and point or line. Practically, we can do it in the pixel grid by raster-scanning the curves.

4. Concluding remarks

The existence and uniqueness of the Zone diagram of n line segments is nontrivial to show[1]. In higher dimensional space, we can naturally have trisector surfaces between a point and a plane in the space by considering the rotated trajectory of trisector curves.

Notes

[1] Recently, the existence and uniqueness of the Zone diagram of general objects has been proven in [3], extending the idea given in this paper and [2].

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