# A Formula to Compute Implied Volatility, with Error Estimate

Song LIANG[1,*] and Yoshihiro TAHARA[2]

[1]*Institute of Mathematics, University of Tsukuba, Japan*
[2]*Credit Risk Management Division, Mitsubishi UFJ Trust and Banking Corporation, Japan*

We derive a simple formula to compute implied volatility approximately, and give an estimate of its relative error, in the framework developed by Black-Scholes. In particular, our error estimate ensures that the relative error of our formula is converging to 0 under certain condition.

KEYWORDS: Implied volatility, Black-Scholes model

## 1. Introduction

Since introduced by Black-Scholes [1] in 1973, Black-Scholes model has been one of the most well-used models in mathematical finance, especially to European options. For the sake of simplicity, let us consider the case with no dividend. Then in the framework of Black-Scholes model, the value of a European call option on a stock is given by

$$C = SN(d_1) - Xe^{-rT}N(d_2), \tag{1.1}$$

with

$$d_1 = \frac{\log(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\log(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

where $N(\cdot)$ stands for the standard normal distribution function:

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

Here, the commodity price, strike price, interest rate and the maturity are denoted by $S$, $X$, $r$ and $T$, respectively, and $\sigma$ represents the volatility, the instantaneous standard deviation of the commodity log-price.

Note that in this model, all model parameters except the volatility are directly observable from market. In practical use, it is always important to estimate the volatility appeared in this model quickly and precisely.

As volatility is the only unknown parameter, by solving the nonlinear equation (1.1) with respect to $\sigma$, with the price of the option given by the practical one decided by the market, we can get implied volatility, which in turn can be used to decide the theoretical price of the other options with the same commodity. Since originally suggested by Latane-Rendleman [4] in 1976, implied volatility has been extensively used in financial research.

Unfortunately, a closed-form solution for an implied volatility of (1.1) is unknown. An numerical solution can be obtained by iterative algorithms, *e.g.*, by using Newton approximation, with the help of computer. However, as pointed out by many authors, iterative algorithms have many shortages such as the error-proneness of the calculation and the cumbersomeness of the spreadsheet implementation, *etc.* Therefore, it is important to find a formula of the approximated implied volatility.

This problem has been discussed by many authors, *e.g.*, Brenner-Subrahmanyan [2] considered the case when the discounted strike price is exactly equal to the present stock price, and Corrado-Miller [3] extended the result. By experimenting using real data, [3] claimed that their approximation is accurate enough in the domain $0.9 < \eta < 1.1$ if the maturity is longer than 3 months, and in the domain $0.95 < \eta < 1.05$ if the maturity is longer than 1 month, where $\eta$ is the ratio of the commodity price to the discounted strike price. However, as far as the authors know, there is no result about the error estimate of the approximation, which guarantees the correctness and the accurateness theoretically.

In this paper, we give a new approximation of the implied volatility, with an error estimate for it, under certain condition. Note that our idea of the error estimate lies on the precedure of our approximation essentially (see Section 3 for details), and can not be applied directly to the one given in [3].

Our main idea is as follows: Instead of using Taylor expansions of $N(d_1)$ and $N(d_2)$ around 0 from the beginning, we first expand them around $d_3 = (d_1 + d_2)/2$. Since $d_3$, when compared to 0, is much closer to $d_1$ and $d_2$ under our assumption, we get an expansion with less error. In particular, by doing so, we are able to get rid of the factor $\frac{1}{1-\eta}$. Notice that as it is common that $\eta$ is around 1, the factor $\frac{1}{1-\eta}$ will enlarge the error, and is not so desirable. (See Section 3 for details).

In the rest of this paper, we derive our formula of the approximated implied volatility in section 2; and in Section 3, we give an estimate of the relative error for it.

## 2. Approximate Solution

In this section, we give the formula of our approximation of implied volatility. As declared in Section 1, our aim is as follows: Find an approximation of the solution $\sigma$ of (1.1), with every parameter except $\sigma$ given. The error estimate of it will be given in Section 3.

We first prepare some notations for simplication. Let $\xi = \frac{1}{2}\sigma\sqrt{T}$, $\eta = \frac{Xe^{-rT}}{S}$, $D = \frac{C}{(1-\eta)S}$, and $m = -\frac{(1+\eta)\log\eta}{1-\eta}$. Note that in practice, we only need to discuss the problem with $\eta$ around 1 and $\xi$ around 0. Also, in this case, $m$ is around 2. Actually, by a simple calculation, we have $m \in (2, 3)$ if $\eta \in (0.1, 1)$ and $m \in (2, 2.1)$ if $\eta \in (1, 2)$. From now on, we assume that $\eta \in (0.1, 2)$.

With the notations above, we have $d_1 = \frac{-\log\eta}{2\xi} + \xi$ and $d_2 = \frac{-\log\eta}{2\xi} - \xi$. Let $d_3 = \frac{-\log\eta}{2\xi}$. Then $d_1 = d_3 + \xi$ and $d_2 = d_3 - \xi$. Divide both sides of (1.1) by $1 - \eta$, and we get that

$$D = \frac{1}{1-\eta}\left(N(d_3 + \xi) - \eta N(d_3 - \xi)\right). \tag{2.1}$$

Note that by definition, $\xi = \frac{-\log\eta}{2d_3}$.

Now, our problem is very clear: find an approximate solution of (2.1) with respect to $d_3$, with $\eta$ and $D$ given.

From now on, we assume that (the real value of) $d_3$ is close to 0 enough and that $\frac{\xi}{|d_3|} \leq 1$. (This certainly implies that $\eta$ is close to 1, therefore, our global assumption $\eta \in (0.1, 2)$ is satisfied.) More precise expression of the domain will be discussed later.

Let $g_1(d_3)$ denote the right hand side of (2.1), i.e.,

$$g_1(d_3) = \frac{1}{1-\eta}\left(N(d_3 + \xi) - \eta N(d_3 - \xi)\right). \tag{2.2}$$

So the real value of $d_3$ is the solution of the equation $g_1(d_3) = D$ with respect to $d_3$.

In the following, we find an approximation $g_3$ of the function $g_1$, and use the solution $\widetilde{d_3}$ of the equation $g_3(d_3) = D$ as our approximation of $d_3$. (So we abuse the notation a little bit by using $d_3$ as a variable for a moment instead of the real value decided by the volatility).

As mentioned in Section 1, instead of expanding around 0, we expand $N(d_1)$ and $N(d_2)$ around $d_3$ at first.

$$\begin{cases} N(d_1) = N(d_3) + N'(d_3)\xi + R_2, \\ N(d_2) = N(d_3) - N'(d_3)\xi + \tilde{R}_2, \end{cases} \tag{2.3}$$

where $R_2$ and $\tilde{R}_2$ are the second remainders of the corresponding Taylor expansions, and can be expressed as

$$R_2 = \frac{1}{2}N''(k)\xi^2, \quad \tilde{R}_2 = \frac{1}{2}N''(l)\xi^2 \tag{2.4}$$

with some number $k$ between $d_1$ and $d_3$, and some number $l$ between $d_2$ and $d_3$. Substitute (2.3) into (2.2), and we get that

$$g_1(d_3) = N(d_3) + \frac{1+\eta}{1-\eta}N'(d_3)\xi + \frac{R_2 - \eta\tilde{R}_2}{1-\eta}. \tag{2.5}$$

Let $g_2(d_3)$ denote the main part of the right hand side of (2.5), i.e.,

$$g_2(d_3) = N(d_3) + \frac{1+\eta}{1-\eta}N'(d_3)\xi. \tag{2.6}$$

Now, Taylor expansions of $N(d_3)$ and $N'(d_3)$ around 0 lead to

$$\begin{cases} N(d_3) = \frac{1}{2} + \frac{d_3}{\sqrt{2\pi}} + Q_3, \\ N'(d_3) = \frac{1}{\sqrt{2\pi}} - \frac{d_3^2}{2\sqrt{2\pi}} + \tilde{Q}_4, \end{cases} \tag{2.7}$$

where $Q_3$ and $\tilde{Q}_4$ are the third and the fourth remainders of the corresponding Taylor expansions, respectively.

By a simple calculation, we have the following estimate with respect to $Q_3$ and $\tilde{Q}_4$.

$$|Q_3| \leq \frac{|d_3|^3}{6\sqrt{2\pi}}, \quad |\tilde{Q}_4| \leq \frac{d_3^4}{8\sqrt{2\pi}}. \tag{2.8}$$

Actually, Taylor expansion of $Q_3$ around 0 gives us that

$$N(d_3) = N(0) + N'(0)d_3 + \frac{1}{2}N''(0)d_3^2 + \frac{1}{3!}N'''(\theta)d_3^3 \tag{2.9}$$

with some $\theta$ between 0 and $d_3$. By the definition of $N(x)$, we have that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $N''(x) = -\frac{1}{\sqrt{2\pi}}xe^{-\frac{x^2}{2}}$, hence $N(0) = \frac{1}{2}$, $N'(0) = \frac{1}{\sqrt{2\pi}}$ and $N''(0) = 0$. Also, $N'''(x) = -\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}(1 - x^2)$, so by a simple calculation, we have $|N'''(x)| \leq \frac{1}{\sqrt{2\pi}}$ for any $x \in \mathbf{R}$. Substituting these to (2.9), we get the first estimate in (2.8). The second estimate in (2.8) can be gotten in the same way. (2.8) will be used in Section 3.

Substituting (2.7) into (2.6), we get that

$$\begin{aligned} g_2(d_3) &= N(d_3) + (1 + \eta) \cdot \frac{\xi}{1 - \eta}N'(d_3) \\ &= \left(\frac{1}{2} + \frac{d_3}{\sqrt{2\pi}} + Q_3\right) + \frac{m}{2d_3}\left(\frac{1}{\sqrt{2\pi}} - \frac{d_3^2}{2\sqrt{2\pi}} + \tilde{Q}_4\right) \\ &= \frac{1}{2} + \frac{4 - m}{4\sqrt{2\pi}}d_3 + \frac{m}{2\sqrt{2\pi}}\frac{1}{d_3} + \left(Q_3 + \frac{m}{2d_3}\tilde{Q}_4\right). \end{aligned} \tag{2.10}$$

Let $g_3(d_3)$ be the main part of the right hand side of (2.10), *i.e.*,

$$g_3(d_3) = \frac{1}{2} + \frac{4 - m}{4\sqrt{2\pi}}d_3 + \frac{m}{2\sqrt{2\pi}}\frac{1}{d_3}. \tag{2.11}$$

Notice that if we multiple by $d_3$ both sides of the equation $g_3(d_3) = D$, we get a quadratic equation with respect to $d_3$, which can be solved easily and precisely, with the two solutions given by

$$\frac{-\sqrt{2\pi}(1 - 2D) \pm \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}}{4 - m}. \tag{2.12}$$

Recall that the real value of $d_3$ is the solution of the equation $g_1(d_3) = D$. Since $g_3$ is an approximation of $g_1$, it is natural to use the solution of $g_3(d_3) = D$ as an approximation of $d_3$. Now, we shall make a decision: which branch of (2.12) should be used?

Our policy is simple: if the two solutions have different signs, take the one that has the same sign with $d_3$; if they have the same sign, take the one with small absolute value. We do so because we are looking for an approximation of $d_3$, which has small absolute value, as assumed. By Lemma 3.1 below, we have that the following decided $\tilde{d}_3$ satisfies this condition.

$$\tilde{d}_3 = \begin{cases} \dfrac{-\sqrt{2\pi}(1 - 2D) - \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}}{4 - m}, & 0.1 < \eta < 1, \\[3mm] \dfrac{-\sqrt{2\pi}(1 - 2D) + \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}}{4 - m}, & 1 < \eta < 2. \end{cases} \tag{2.13}$$

or equivalently, we also have the following expression by a simple calculation.

$$\tilde{d}_3 = \begin{cases} \dfrac{2m}{-\sqrt{2\pi}(1 - 2D) + \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}}, & 0.1 < \eta < 1, \\[3mm] \dfrac{-2m}{\sqrt{2\pi}(1 - 2D) + \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}}, & 1 < \eta < 2. \end{cases} \tag{2.14}$$

We use this as our desired approximation of $d_3$.

With $\tilde{d}_3$ given, since $\sigma = \frac{-\log \eta}{\sqrt{T}}\frac{1}{d_3}$ by definition, our approximation $\tilde{\sigma}$ of $\sigma$ is naturally given by $\tilde{\sigma} = \frac{-\log \eta}{\sqrt{T}}\frac{1}{\tilde{d}_3}$.

## 3.  Error Estimate

In this section, we give an estimate of the relative error of our approximation given in Section 2.

Notice that $|\frac{\tilde{\sigma} - \sigma}{\sigma}| = |\frac{\tilde{\xi} - \xi}{\xi}| = |\frac{\tilde{d}_3 - d_3}{\tilde{d}_3}|$. We estimate the latest expression in the following.

We first show the following.

**Lemma 3.1.**     1. *Assume that $0 < d_3 < 1$, $0.1 < \eta < 1$ and $D > \frac{1}{2}(1 + \sqrt{\frac{15}{2\pi}})$. Then $0 < \tilde{d}_3 < 1$.*

2. *Assume that $-1 < d_3 < 0$, $1 < \eta < 2$ and $D < \frac{1}{2}(1 - \sqrt{\frac{25}{4\pi}})$. Then $-1 < \tilde{d}_3 < 0$.*

**Remark 3.2.** *The conditions with respect to $\eta$ in Lemma 3.1 are always satisfied if $d_3$ (hence also $\xi$, as $\xi \leq |d_3|$ by assumption) is close enough to 0, since $-\log \eta = 2\xi d_3$ by definition.*

**Remark 3.3.** *The conditions with respect to $D$ in Lemma 3.1, i.e., $D > \frac{1}{2}(1 + \sqrt{\frac{15}{2\pi}})$ and $D < \frac{1}{2}(1 - \sqrt{\frac{25}{4\pi}})$ in the corresponding cases, are always satisfied if $d_3$ (hence $\xi$ by assumption) is close enough to 0.*

*Actually, since $D = g_1(d_3)$ by definition, we have*

$$|D| = |g_3(d_3) - g_3(d_3) + g_1(d_3)| \geq |g_3(d_3)| - |g_3(d_3) - g_1(d_3)|.$$

*When $d_3$ and $\xi$ converge to 0, we have by the definition of $g_3$ that the term $|g_3(d_3)|$ converges to $+\infty$, while by Lemma 3.4 below, the term $|g_3(d_3) - g_1(d_3)|$ converges to 0. Therefore, $|D| \to \infty$. Also, notice that $D > 0$ if $0.1 < \eta < 1$, and $D < 0$ if $1 < \eta < 2$. This completes the proof of our assertion.*

*Proof.* Since the proofs are similar, we only give the proof of the first assertion.

By assumption $D > \frac{1}{2}(1 + \sqrt{\frac{15}{2\pi}})$, we have $(1 - 2D)^2 > \frac{15}{2\pi}$. This combined with the assumption $2 < m < 3$ implies $(1 - 2D)^2 > \frac{15}{2\pi} > \frac{m(8-m)}{2\pi}$, hence $2\pi(1 - 2D)^2 - 2m(4 - m) > m^2$, which in turn implies $0 < m < \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}$. Therefore,

$$0 < \frac{m}{\sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}} < 1. \tag{3.1}$$

On the other hand, it is easy to be seen that

$$-\sqrt{2\pi}(1 - 2D) + \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)} > 2\sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}.$$

Combining this with (3.1), we get

$$0 < \widetilde{d_3} = \frac{2m}{-\sqrt{2\pi}(1 - 2D) + \sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}}$$

$$< \frac{m}{\sqrt{2\pi(1 - 2D)^2 - 2m(4 - m)}} < 1.$$

This gives us our first assertion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By Lemma 3.1, we have that $\widetilde{d_3}$ and $d_3$ always have the same sign. Also, it is easy to be seen by the definition of $g_3$ that $g_3$ is continuously differentiable on $\mathbf{R} \setminus \{0\}$. So by mean value theorem, there exists a constant $c$ between $d_3$ and $\widetilde{d_3}$ such that

$$g_3(d_3) - g_3(\widetilde{d_3}) = g_3'(c)(d_3 - \widetilde{d_3}). \tag{3.2}$$

Notice that by definition, we have $g_3(\widetilde{d_3}) = D = g_1(d_3)$. Combining this with (3.2), we get that

$$|d_3 - \widetilde{d_3}| = \frac{1}{|g_3'(c)|} |g_3(d_3) - g_3(\widetilde{d_3})|$$

$$= \frac{1}{|g_3'(c)|} |g_3(d_3) - g_1(d_3)|. \tag{3.3}$$

This transformation, although simple, is important in our error estimate. It helps us to transform the quantity involving both $d_3$ and $\widetilde{d_3}$, which is difficult to be handled, to the one involving $d_3$ only. Notice that by our constitution, $g_3$ is nothing but (twice) Taylor approximation of $g_1$.

Let $(A) = |g_3(d_3) - g_1(d_3)|$ and $(B) = \frac{1}{|g_3'(c)|}$. We estimate them respectively in the following.

We first have the following result with respect to $(A)$.

**Lemma 3.4.**

$$(A)(= |g_3(d_3) - g_1(d_3)|) \leq \frac{17}{48\sqrt{2\pi}} |d_3|^3 + \frac{3}{2\sqrt{2\pi}} \xi.$$

*Proof.* We first have

$$|g_3(d_3) - g_1(d_3)| \leq |g_1(d_3) - g_2(d_3)| + |g_2(d_3) - g_3(d_3)|. \tag{3.4}$$

For the first term on the right hand side of (3.4), we have by (2.5), (2.6) and (2.4) that

$$|g_1(d_3) - g_2(d_3)| = \left| \frac{R_2 - \eta \tilde{R}_2}{1 - \eta} \right|$$

$$= \left| \frac{1}{1 - \eta} \left\{ \frac{N''(k)}{2} \xi^2 - \eta \frac{N''(l)}{2} \xi^2 \right\} \right|$$

$$= \left| \frac{\xi^2}{2(1 - \eta)} \left\{ -\frac{1}{\sqrt{2\pi}} k e^{-k^2/2} + \frac{\eta}{\sqrt{2\pi}} l e^{-l^2/2} \right\} \right|$$

$$\leq \frac{\xi^2}{2|1 - \eta|} \left\{ \frac{|k| + \eta|l|}{\sqrt{2\pi}} \right\} \leq \frac{\xi^2}{2|1 - \eta|} \frac{(1 + \eta)(|d_3| + \xi)}{\sqrt{2\pi}}$$

$$= \frac{m}{2\sqrt{2\pi}} \cdot \frac{2\xi}{|-\log \eta|} \cdot \frac{(|d_3| + \xi)\xi}{2} = \frac{m}{2\sqrt{2\pi}} \cdot \frac{1}{2|d_3|}(|d_3| + \xi)\xi$$

$$= \frac{m}{4\sqrt{2\pi}} \cdot \left( 1 + \frac{\xi}{|d_3|} \right)\xi \leq \frac{m}{2\sqrt{2\pi}}\xi, \tag{3.5}$$

where in the fourth line, we used the fact that $|k|, |l| \leq |d_3| + \xi$, and when passing to the last line, we used the assumption $\xi \leq |d_3|$.

Also, for the second term on the right hand side of (3.4), we have by (2.10), (2.11) and (2.8) that

$$|g_2(d_3) - g_3(d_3)| = \left| Q_3 + \frac{m}{2d_3} \tilde{Q}_4 \right| \leq |Q_3| + \left| \frac{m}{2d_3} \tilde{Q}_4 \right|$$

$$\leq \left| \frac{d_3^3}{6\sqrt{2\pi}} \right| + \left| \frac{m}{16\sqrt{2\pi}} d_3^3 \right| = |d_3|^3 \left\{ \frac{1}{6\sqrt{2\pi}} + \frac{m}{16\sqrt{2\pi}} \right\}. \tag{3.6}$$

Combining the above and the fact that $2 < m < 3$, we get our assertion. □

Especially, by Lemma 3.4, we have that $(A) \to 0$ as $|d_3| \to 0$ (hence $\xi \to 0$).
We next deal with the term $(B)$.

**Lemma 3.5.** *Assume condition in (1) or (2) of Lemma 3.1. Then we have the following.*

$$(B)\left( = \frac{1}{|g_3'(c)|} \right) \leq 2\sqrt{2\pi}(d_3^2 + \tilde{d}_3^2).$$

*Proof.* Differentiate both sides of (2.11), we get that

$$g_3'(x) = \frac{4 - m}{4\sqrt{2\pi}} - \frac{m}{2\sqrt{2\pi}} \frac{1}{x^2}.$$

So the solution of $g_3'(x) = 0$ is given by $\pm\alpha$ with $\alpha = \sqrt{\frac{2m}{4-m}}$. Since $m \in (2, 3)$, we have $\alpha > \sqrt{2}$, in particular, $\alpha > 1$. So $|g_3'(x)|$ is monotone decreasing for $x \in (0, 1)$, and monotone increasing for $x \in (-1, 0)$. Also, $c$ is between $d_3$ and $\tilde{d}_3$. So by Lemma 3.1, we get

$$|g_3'(c)| \geq \min\{|g_3'(d_3)|, |g_3'(\tilde{d}_3)|\}. \tag{3.7}$$

Also, since $m \in (2, 3)$, we have that for any $x \in (-1, 0) \cup (0, 1)$,

$$|g_3'(x)| = \left| \frac{4 - m}{4\sqrt{2\pi}} - \frac{m}{2\sqrt{2\pi}} \frac{1}{x^2} \right| = \frac{m}{2\sqrt{2\pi}} \frac{1}{x^2} - \frac{4 - m}{4\sqrt{2\pi}} \geq \frac{m}{4\sqrt{2\pi}} \frac{1}{x^2}$$

$$\geq \frac{1}{2\sqrt{2\pi}} \frac{1}{x^2}.$$

Therefore,

$$\frac{1}{|g_3'(x)|} \leq 2\sqrt{2\pi}x^2, \quad \text{for any } x \in (-1, 0) \cup (0, 1). \tag{3.8}$$

By (3.7) and (3.8), we get

$$(B) = \frac{1}{|g_3'(c)|} \leq \frac{1}{|g_3'(d_3)|} + \frac{1}{|g_3'(\tilde{d}_3)|} \leq 2\sqrt{2\pi}(d_3^2 + \tilde{d}_3^2).$$

This completes the proof of our assertion. □

Now, we are ready to give the following estimate of the relative error $|\frac{\tilde{d}_3 - d_3}{\tilde{d}_3}|$.

**Lemma 3.6.** *Assume condition in (1) or (2) of Lemma 3.1. Also, assume that $(A) < \frac{1}{4\sqrt{2\pi}}$. Then we have the following.*

$$\left| \frac{\widetilde{d_3} - d_3}{\widetilde{d_3}} \right| \le 8\sqrt{2\pi}(A).$$

**Remark 3.7.** *By Lemma 3.4, $(A) \to 0$ as $\xi, |d_3| \to 0$. So the condition $(A) < \frac{1}{4\sqrt{2\pi}}$ is always satisfied if $|d_3|$ (hence $\xi$) is small enough.*

*Proof.* Let $Y = |\frac{\widetilde{d_3} - d_3}{\widetilde{d_3}}|$. Then since $|d_3| < 1$ by assumption, we have by (3.3) and Lemmas 3.4, 3.5 that

$$
\begin{aligned}
Y &= \frac{1}{|\widetilde{d_3}|} \frac{1}{|g_3'(c)|} |g_3(d_3) - g_1(d_3)| \\
&\le \frac{(A)}{|\widetilde{d_3}|} \cdot 2\sqrt{2\pi}(d_3^2 + \widetilde{d_3}^2) \\
&= 2\sqrt{2\pi}(A) \left( |\widetilde{d_3}| + |d_3| \cdot \left| \frac{d_3 - \widetilde{d_3}}{\widetilde{d_3}} + 1 \right| \right) \\
&\le 2\sqrt{2\pi}(A)(|d_3| + |\widetilde{d_3}|) + 2\sqrt{2\pi}(A)Y.
\end{aligned}
\tag{3.9}
$$

We have by condition $(A) < \frac{1}{4\sqrt{2\pi}}$, so by solving (3.9) with respect to $Y$, we get that

$$Y \le \frac{2\sqrt{2\pi}(A)(|d_3| + |\widetilde{d_3}|)}{1 - 2\sqrt{2\pi}(A)} \le \frac{4\sqrt{2\pi}(A)}{1 - 2\sqrt{2\pi}(A)} \le 8\sqrt{2\pi}(A),$$

where in the second inequality, we used the fact that $d_3, \widetilde{d_3} \in (0, 1) \cup (-1, 0)$. □

Combining Lemmas 3.4, 3.6 and Remarks 3.3, 3.7, we get the following, our main result with respect to the relative error estimate of our approximation given in Section 2.

**Theorem 3.8.** *There exist constants $\xi_0 > 0$ and $d_0 > 0$ such that*

$$\left| \frac{d_3 - \widetilde{d_3}}{\widetilde{d_3}} \right| \le \frac{17}{6} |d_3|^3 + 12\xi. \tag{3.10}$$

*as long as $\xi \in (0, \xi_0]$ and $d_3 \in [-d_0, 0) \cup (0, d_0]$.*

When $|d_3|$ (hence $\xi$) converges to 0, it is trivial that the right hand side of (3.10) converges to 0, therefore, by Theorem 3.8, we get that the relative error of our approximation $\widetilde{d_3}$, hence that of $\widetilde{\sigma}$, given in Section 2, converges to 0.

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