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On Multiplication Maps of Ample Bundles with Nef Bundles on Toric Surfaces*

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We show that the multiplication map of global sections of an ample line bundle and a nef line bundle on a projective toric surface is surjective. We also prove that on a Gorenstein toric 3-fold whose anti-canonical bundle is nef, the multiplication of global sections of an ample line bundle by those of the anti-canonical bundle is surjective.

KEYWORDS: toric variety

1. Introduction

Let X be a nonsingular projective toric surface, and let A and B be an ample line bundle and a globally generated line bundle, respectively, on X. Then Fakhruddin [1] shows that the multiplication map of their global sections

$$\Gamma(X,A) \otimes \Gamma(X,B) \to \Gamma(X,A \otimes B)$$

is surjective. In this article we shall generalize this to the case when X may have singularities.

Theorem 1. *Let X be a projective toric surface may have singularities. Let A be an ample line bundle and B a globally generated line bundle on X. Then the multiplication map*

$$\Gamma(X,A) \otimes \Gamma(X,B) \to \Gamma(X,A \otimes B)$$

is surjective.

Let $T = (k^*)^n$ be an algebraic torus of dimension n defined over an algebraically closed field k. Let denote $M = \mathrm{Hom}_{\mathrm{gr}}(T, k^*)$ the group of characters of T. Then we have $M \cong \mathbb{Z}^n$ and $T = \mathrm{Spec}\ k[M]$. A normal algebraic variety X is called *toric* if it contains an algebraic torus T as a dense open subset, together with an algebraic action $T \times X \to X$ that extends the natural action of T on itself.

Let $\mathcal{O}_X(D)$ be a line bundle on X defined by a Cartier divisor D. If $\mathcal{O}_X(D)$ is globally generated, then there is a convex polytope $P_D = \mathrm{Conv}\{m_1, \ldots, m_r\}$ in $M \otimes \mathbb{R} \cong \mathbb{R}^n$ with some $m_i \in M$ such that

$$\Gamma(\mathcal{O}_X(D)) \cong \bigoplus_{m \in P_D \cap M} k\ \mathbf{e}(m),$$

where $\mathbf{e}(m)$ denotes the character of T corresponding to $m \in M$ (see, for instance, Lemma 2.3 [9] or Section 3.5 [3]). In order to emphasize that all vertices of P_D are lattice points, we call P_D an *integral* convex polytope.

We also know that a line bundle on a complete toric variety is *nef* if and only if it is generated by its global sections (see, for instance, Theorem 3.1 [7]). If D_1 and D_2 are nef divisors on a toric variety X, then the convex polytope corresponding to $D_1 + D_2$ coincides with the Minkowski sum $P_{D_1} + P_{D_2} = \{x_1 + x_2; x_i \in P_{D_i}\ (i = 1, 2)\}$. Moreover, the surjectivity of the multiplication map

$$\Gamma(\mathcal{O}_X(D_1)) \otimes \Gamma(\mathcal{O}_X(D_2)) \longrightarrow \Gamma(\mathcal{O}_X(D_1 + D_2))$$

is equivalent to the equality

$$P_{D_1} \cap M + P_{D_2} \cap M = (P_{D_1} + P_{D_2}) \cap M.$$

We note that the surjectivity of the multiplication map is independent of characteristic of the ground field k.

If D is an ample divisor, then $\dim(P_D) = \dim X$. Conversely, any convex polytope P in $M \otimes \mathbb{R}$ with $\dim P = \mathrm{rank}\, M$ defines a polarized toric variety (X, D) with $\dim X = \dim P$ so that the set of global sections of $\mathcal{O}_X(D)$ is the vector space with a basis $\{\mathbf{e}(m); m \in P \cap M\}$.

By using this correspondence, we can interpret the Theorem 1 as a theorem in combinatrics of convex polygons.

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Theorem 2. *Let $M = \mathbb{Z}^2$ and let P, Q integral convex polygons in $M_{\mathbb{R}}$. Assume that the set of all edges of Q corresponds to a subset of edges of P so that corresponding edges are parallel to each other and they are located in the same direction from their interiors. Then we have*

$$P \cap M + Q \cap M = (P + Q) \cap M.$$

C. Haase, B. Nill, A. Paffenholz and F. Santos [4] also prove Theorem 2 independently. Their proof is different from us.

We have an application of Theorem 1 to Gorenstein toric varieties of dimension three. We remark that on a Gorenstein toric 3-fold Y an ample line bundle L is very ample if $L + K_Y$ is globally generated. This is given in Proposition 1.

A normal Gorenstein variety is called *Fano* if its anti-canonical line bundle is ample. Hence the anti-canonical bundle on a Gorenstein toric Fano 3-fold is very ample. By applying Theorem 1 we have more about the ant-canonical bundle.

Theorem 3. *Let Y be a Gorenstein toric Fano variety of dimension three. Then, for a globally generated line bundle B on Y, the multiplication map*

$$\Gamma(Y, B) \otimes \Gamma(Y, \mathcal{O}_Y(-K_Y)) \to \Gamma(Y, B \otimes \mathcal{O}_Y(-K_Y))$$

is surjective.

In particular, the anti-canonical line bundle $\mathcal{O}_Y(-K_Y)$ is normally generated, that is, the multiplication map

$$\Gamma(Y, \mathcal{O}_Y(-K_Y)) \otimes \Gamma(Y, \mathcal{O}_Y(-kK_Y)) \to \Gamma(Y, \mathcal{O}_Y(-(k+1)K_Y))$$

is surjective for all $k \geq 1$.

We can weaken the assumption on $-K_Y$, but we have to add the condition that B is ample.

Theorem 4. *Let Y be a projective Gorenstein toric variety of dimension three such that $-K_Y$ is nef. Then, for an ample line bundle A on Y, the multiplication map*

$$\Gamma(Y, A) \otimes \Gamma(Y, \mathcal{O}_Y(-K_Y)) \to \Gamma(Y, A \otimes \mathcal{O}_Y(-K_Y))$$

is surjective.

Proofs are given in Section 4.

Ogata [10] also gave an algebro-geometric proof of Fakhruddin's Theorem in terms of blowing-ups and minimal models of rational surfaces.

2. Projective Toric Varieties

In this section we recall some basic notion about toric varieties and line bundles following Oda's book [9], or Fulton's book [3]. Let k be an algebraically closed field of any characteristic. In this article we consider varieties defined over k.

Let N be a free $\mathbb{Z}$-module of rank n, M its dual and $\langle , \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We denote the same $\langle , \rangle$ as the pairing of $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ defined by scalar extension. Let $T_N := N \otimes_{\mathbb{Z}} k^* \cong (k^*)^n$ be the algebraic torus over the field k, where k^* is the multiplicative group of k. Then $M = \mathrm{Hom}_{\mathrm{gr}}(T_N, k^*)$ is the character group of T_N and $T_N = \mathrm{Spec}\ k[M]$. For $m \in M$ we denote $\mathbf{e}(m)$ as the character of T_N. Let Δ be a finite complete fan in N consisting of strongly convex rational polyhedral cones σ in $N_{\mathbb{R}}$, that is, with a finite number of elements $v_1, \ldots, v_r$ in N we can write as

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$$

and it satisfies that $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N\mathrm{emb}(\Delta) := \cup_{\sigma \in \Delta} U_\sigma$ of dimension n (see Section 1.2 [9], or Section 1.4 [3]). Here $U_\sigma = \mathrm{Spec}\ k[\sigma^\vee \cap M]$ and $\sigma^\vee := \{y \in M_{\mathbb{R}}; \langle y, x \rangle \geq 0 \quad \text{for all } x \in \sigma\}$ is the dual cone of σ. For the origin $\{0\} \in \Delta$, the affine open set $U_{\{0\}} = \mathrm{Spec}\ k[M]$ is the unique dense T_N-orbit. We note that a toric variety is always normal.

Set $\Delta(s) := \{\sigma \in \Delta; \dim \sigma = s\}$. Then $\tau \in \Delta(s)$ corresponds to the T_N-orbit $\mathrm{Spec}\ k[\tau^\perp \cap M]$ and its closure $V(\tau)$, which is also a T_N-invariant subvariety of dimension $n - s$. Hence $\Delta(1)$ corresponds to T_N-invariant irreducible divisors. If for any cone $\sigma \in \Delta$ of dimension n there exist a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ in N such that

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n,$$

then the toric variety X is nonsingular.

Let $\mathrm{Pic}(X)$ be the group of all invertible sheaves modulo isomorphisms. The map $D \mapsto \mathcal{O}_X(D)$ gives a homomorphism from the group of Cartier divisors onto $\mathrm{Pic}(X)$. Let $A_{n-1}(X)$ denote the group of all Weil divisors modulo the subgroup of principal divisors $[\mathrm{div}(f)]$. The map $D \mapsto [D]$ determines an injective homomorphism

$$\mathrm{Pic}(X) \hookrightarrow A_{n-1}(X).$$

Since X is toric, any $m \in M$ determines a principal divisor $\mathrm{div}(\mathbf{e}(m))$, which gives a homomorphism from M to the group $\mathrm{Div}(X)$ of T_N-invariant divisors. Let $\{D_1, \ldots, D_d\}$ be the set of all T_N-invariant irreducible divisors. Then we have a commutative diagram with exact rows (see Section 3.4 [3]):

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \mathrm{Div}(X) & \to & \mathrm{Pic}(X) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & \oplus_{i=1}^{d} \mathbb{Z} \cdot D_i & \to & A_{n-1}(X) & \to & 0 \end{array} \tag{1}$$

If X is nonsingular, then the two rows in (1) coincide.

Now let $\Delta(1) = \{\rho_1, \ldots, \rho_d\}$ such that $V(\rho_i) = D_i$ for $i = 1, \ldots, d$. Let $\mathcal{O}_X(D)$ be a line bundle on X defined by a Cartier divisor D. We may write $D = \sum_i a_i D_i$ in $\mathrm{Div}(X)$ up to isomorphisms of line bundles. Then set

$$P_D := \{u \in M_{\mathbb{R}}; \langle u, v_i \rangle \geq -a_i \quad \text{for all } i\},$$

where $v_i \in N$ are the generators of $\rho_i \cap N$. We note that P_D is a rational convex polytope (possibly empty) and that its vertices may not have coordinates of integers. In any case we have

$$\Gamma(\mathcal{O}_X(D)) \cong \bigoplus_{m \in P_D \cap M} k\, \mathbf{e}(m),$$

where $\mathbf{e}(m)$ denotes the character of T_N corresponding to $m \in M$ (see, for instance, Lemma 2.3 [9] or Section 3.5 [3]). If $\mathcal{O}_X(D)$ is globally generated, then there exist $m_1, \ldots, m_r \in M$ such that $P_D = \mathrm{Conv}\{m_1, \ldots, m_r\}$, which we call an *integral* convex polytope.

If D_1 and D_2 are nef T_N-invariant Cartier divisors on a toric variety X, then the convex polytope corresponding to $D_1 + D_2$ coincides with the Minkowski sum $P_{D_1} + P_{D_2} = \{x_1 + x_2; x_i \in P_{D_i}\ (i = 1, 2)\}$. In particular, if $D_1 = D_2 = D$, then we have $P_{2D} = P_D + P_D = 2P_D$. Moreover, the surjectivity of the multiplication map

$$\Gamma(\mathcal{O}_X(D_1)) \otimes \Gamma(\mathcal{O}_X(D_2)) \longrightarrow \Gamma(\mathcal{O}_X(D_1 + D_2))$$

is equivalent to the equality

$$P_{D_1} \cap M + P_{D_2} \cap M = (P_{D_1} + P_{D_2}) \cap M.$$

If $\mathcal{O}_X(D)$ is ample, then it is globally generated and that $\dim(P_D) = \dim X$.

Conversely, any convex polytope P in $M \otimes \mathbb{R}$ with $\dim P = \mathrm{rank}\, M$ defines a polarized toric variety (X, L) with $\dim X = \dim P$. For a vertex v of P, set $C(v) := \mathbb{R}_{\geq 0}(P - v) = \{r(x - v); r \geq 0 \quad \text{and} \quad x \in P\}$. Set $\sigma(v) = C(v)^\vee \subset N_{\mathbb{R}}$ and $\Delta = \{\text{all faces of } \sigma(v); v \in \mathrm{Vert}(P)\}$. Then $X = T_N\mathrm{emb}(\Delta)$ is a toric variety with an ample line bundle L such that

$$\Gamma(X, L) \cong \bigoplus_{m \in P \cap M} k\, \mathbf{e}(m).$$

For $\sigma = \sigma(v)$ the affine variety U_σ is $U(v) := \mathrm{Spec}\, k[C(v) \cap M]$ and X is covered by these open sets $X = \bigcup_{v \in \mathrm{Vert}(P)} U(v)$. If $C(v) \cap M$ is generated by $(P - v) \cap M$ as a semigroup for all vertices $v \in \mathrm{Vert}(P)$, then the ample line bundle L is very ample.

3. Multiplication Maps

In this section we will give a proof of Theorem 1. Let $X = T_N\mathrm{emb}(\Delta)$ be a projective toric surface. Thus $N = \mathbb{Z}^2$ and $M \cong \mathbb{Z}^2$. Let A and B be an ample line bundle and a nef line bundle on X, respectively. Let P and Q be integral convex polytopes corresponding to A and B. We shall show that

$$P \cap M + Q \cap M = (P + Q) \cap M. \tag{2}$$

We generalize the method of Fakhruddin [1].

When $\dim Q = 0$, that is, $B = \mathcal{O}_X$, the equality (2) trivially holds.

When $\dim Q = 1$, there exist two cones $\rho_1, \rho_2 \in \Delta(1)$ such that $\rho_1 = -\rho_2$. In other words, there exist a surjective morphism $f : X \to \mathbb{P}^1$ and a divisor D on $\mathbb{P}^1$ such that $B \cong f^*\mathcal{O}_{\mathbb{P}^1}(D)$. In this case P has two edges F_1, F_2 parallel to Q. Let $L \cong \mathbb{Z}$ be a sublattice of M such that $L_{\mathbb{R}}$ contains F_1. Since L is a direct summand of M, we can choose $m_0, m_1, \ldots, m_r \in M$ so that the union of the line segments $P \cap (L + m_i)$ covers the lattice points $P \cap M$. The following lemma implies the equality (2).

Lemma 1 (Fakhruddin [1]). *Let I, J be closed intervals of the real line $\mathbb{R}$ such that $I \cap \mathbb{Z} \neq \emptyset$ and that $J = [a, b]$ with $a, b \in \mathbb{Z}$. Then we have*

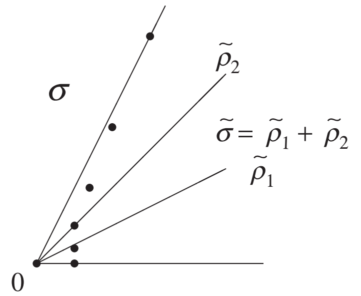
$$(I+J)\cap\mathbb{Z}=I\cap\mathbb{Z}+J\cap\mathbb{Z}.$$

Proof. By a parallel transform of $\mathbb{Z}$, we may assume that I and J are contained in the positive part of $\mathbb{R}$. If $J=J_n:=[n,n+1]$, then it is trivial. We may decompose J into the union $J=\cup_{j=0}^{r}J_{n+j}$ of closed intervals of length one. Then $I+J=\cup_{j=0}^{r}(I+J_{n+j})$. □

Now consider the case when $\dim Q=2$. Define $s:M_{\mathbb{R}}\times M_{\mathbb{R}}\to M_{\mathbb{R}}$ as $s(x,y)=x+y$. For any $m\in(P+Q)\cap M$ we have to find a lattice point in $s^{-1}(m)\cap(P\times Q)$. The second projection maps $s^{-1}(m)\cap(P\times Q)$ to $Q\cap(m-P)$. If $Q\cap(m-P)$ contains a lattice point m_1, then $m_1\in Q\cap M$ and $m-m_1\in P\cap M$. If $\dim Q\cap(m-P)\le 1$, then it contains a lattice point.

Set $\dim Q\cap(m-P)=2$. If a vertex of Q or $m-P$ is contained in $Q\cap(m-P)$, then it is a lattice point. We may assume that $Q\cap(m-P)$ contains no vertices of Q nor $m-P$. Take an edge E of Q such that $E\cap(m-P)\neq\emptyset$. Fakhruddin finds an integral polygon in Q containing E when X is nonsingular. Let $\nu:\tilde{X}=T_N\mathrm{emb}(\tilde{\Delta})\to X$ be the minimal resolution of the singularities. Here $\tilde{\Delta}$ is the fan obtained by the minimal nonsingular refinement of Δ. Then Q corresponds to the nef line bundle ν^*B on $\tilde{X}$. The edge E corresponds to a T_N-invariant divisor $V(\tilde{\rho}_0)$ on $\tilde{X}$ so that $\tilde{\rho}_0$ has the direction perpendicular to E.

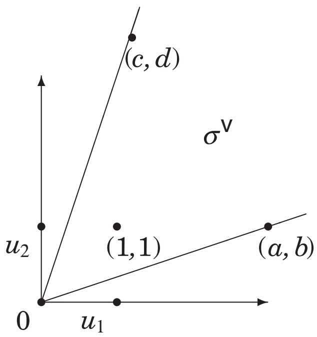
We note that $\rho_0=\tilde{\rho}_0\in\Delta(1)$. By assumption we have $-\rho_0\notin\Delta(1)$. Take the cone $\sigma\in\Delta(2)$ with $\sigma\supset-\rho_0$. Recall the minimal nonsingular subdivision of σ. First take the convex hull of $\sigma\cap N\setminus\{0\}$. Make 1-dimensional cones with apex the origin 0 through the lattice points on the boundary of the convex hull.



Set $\tilde{\sigma}=\tilde{\rho}_1+\tilde{\rho}_2\in\tilde{\Delta}(2)$ the 2-dimensional cone containing $-\rho_0$. Let $v_i\in N$ be the generator of $\tilde{\rho}_i\cap M$ for $i=1,2$. Then $\{v_1,v_2\}$ is a basis of N. Let $\{u_1,u_2\}$ be the basis of M dual to $\{v_1,v_2\}$. Since $\tilde{\sigma}\subset\sigma$, we have $\sigma^\vee\subset\tilde{\sigma}^\vee=\mathbb{R}_{\ge0}u_1+\mathbb{R}_{\ge0}u_2$, hence, we can write

$$\sigma^\vee=\mathbb{R}_{\ge0}(au_1+bu_2)+\mathbb{R}_{\ge0}(cu_1+du_2),$$

with $a>0,b\ge0,d>0$ and $c\ge0$.



We note that $\sigma^\vee$ contains the lattice point u_1+u_2 in its interior. In fact, we can write σ as

$$\sigma=\mathbb{R}_{\ge0}(-bv_1+av_2)+\mathbb{R}_{\ge0}(dv_1-cv_2).$$

Since the line segment connecting v_1 with v_2 is contained in the boundary of the convex hull of $\sigma\cap N\setminus\{0\}$, the line connecting v_1 and v_2 meets with two faces of σ. Thus we have $a>b$ and $c<d$.

Now we devide into two cases: $-\tilde{\rho}_0$ is contained in $\tilde{\Delta}(1)$, or not.

Case I: $-\tilde{\rho}_0\notin\tilde{\Delta}(1)$.

Lemma 2 (Fakhruddin [1]). *In the above notation, when $-\tilde{\rho}_0\notin\tilde{\Delta}(1)$, there exists a integral triangle R contained in Q with the edge E such that other two edges F_1 and F_2 are perpendicular to $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively.*

Proof. Only in this proof, we rename elements of $\tilde{\Delta}(1)$ except $\tilde{\rho}_0$. Set $\tilde{\Delta}(1)=\{\tilde{\rho}_0,\tilde{\rho}_1,\ldots,\tilde{\rho}_d\}$ so that the indices increase anti-clockwise. Let k be the number such that $\tilde{\rho}_k$ and $\tilde{\rho}_{k+1}$ are separeted by $-\tilde{\rho}_0$. Let $v_i\in N$ be the generator of $\tilde{\rho}_i\cap N$. Let denote $D_i=V(\tilde{\rho}_i)$ the T_N-invariant irreducible divisor on the nonsingular toric surface $\tilde{X}$. Set the nef line bundle $\nu^*B=\mathcal{O}_{\tilde{X}}(\sum_i b_iD_i)$. Then we recall

$$Q = \{u \in M_{\mathbb{R}}; \langle u, v_i \rangle \geq -b_i \quad \text{for all } i\}.$$

Let m_1 and m_2 in M be the end points of E. Then we have

$$\langle m_1, v_0 \rangle = \langle m_2, v_0 \rangle = -b_0.$$

Set $\langle m_1, v_1 \rangle > \langle m_2, v_1 \rangle$. Then we have

$$\langle m_1, v_i \rangle > \langle m_2, v_i \rangle \quad \text{for } 1 \leq i \leq k,$$
$$\langle m_1, v_i \rangle < \langle m_2, v_i \rangle \quad \text{for } k+1 \leq i \leq d.$$

Let c_k and c_{k+1} be the integers with $\langle m_2, v_k \rangle = -c_k$ and $\langle m_1, v_{k+1} \rangle = -c_{k+1}$. Set $c_0 = b_0$ and

$$R = \{u \in M_{\mathbb{R}}; \langle u, v_i \rangle \geq -c_i \quad \text{for } i = 0, k, k+1\}.$$

Then R is a triangle with the edge E. Let F_1 and F_2 be the edges of R perpendicular to v_{k+1} and v_k, respectively. We note that the vertex $m' := F_1 \cap F_2$ is a lattice point because $\{v_k, v_{k+1}\}$ is a $\mathbb{Z}$-basis of N.

We have to show $R \subset Q$. Now let

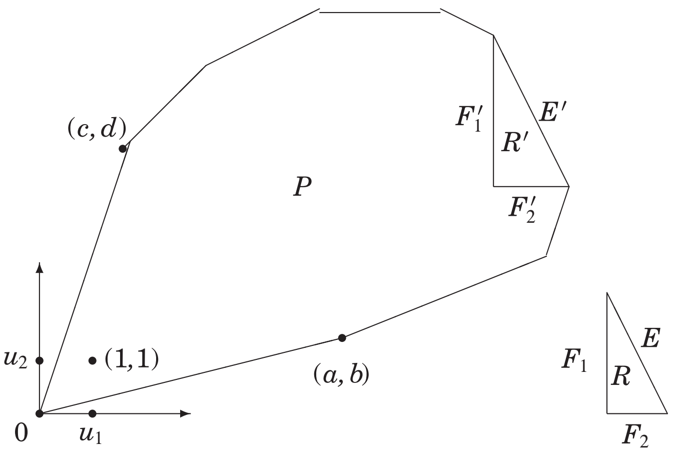
$$R(b_k, b_{k+1}) = \{u \in M_{\mathbb{R}}; \langle u, v_i \rangle \geq -b_i \quad \text{for} \quad i = k, k+1\}$$

the rectangular region. Then $R(b_k, b_{k+1})$ contains Q and the rectangular triangle R. We note that the vertex m'' of $R(b_k, b_{k+1})$ is also a lattice point. Furthermore, we see that m'' is contained in Q because $\nu^* B$ is globally generated (see Theorem 2.7 in [9]). Thus the triangle $\mathrm{Conv}\{m'', E\}$ contains R and is contained in Q. □

In terms of the basis $\{u_1, u_2\}$ dual to $\{v_1, v_2\}$, we can write R in Lemma 2 as $R = \mathrm{Conv}\{0, xu_1, yu_2\}$ with positive integers x, y. Set r the greatest common divisor of x and y. Then we have an integral triangle R_0 with $R = rR_0$.

On the other hand, since A is ample on X, the corresponding polygon P has an edge E' parallel to E. Since $\nu^* A$ is also a globally generated line bundle on $\tilde{X}$, we can apply Lemma 2 to P, and we obtain an integral triangle R' with the edge E'. We may write $R' = sR_0$ with some positive integer s. Let F_1' and F_2' be the edges of R' parallel to F_1 and F_2, respectively.

Since $\sigma^\vee$ is similar to P near some vertex m_0, we can draw the picture of P as the following (we set $m_0 = 0$ by a parallel transform of M). Two edges meeting m_0 have the ends of coordinates (a, b) and (c, d), respectively. From the above argument, we have $a > b \geq 0$ and $0 \leq c < d$.



We may transform the vertex $F_1 \cap F_2$ to the origin 0 by a parallel transform of M. Assume that the face F_i' of R' is contained in the line $\{u \in M_{\mathbb{R}}; \langle u, v_i \rangle = d_i\}$ for $i = 1, 2$. Then we decompose $P + R$ into a union of three parts P_1', P_2' and $R' + R$, where

$$P_i' = (P + R) \cap \{u \in M_{\mathbb{R}}; 0 \leq \langle u, v_i \rangle \leq d_i\} \quad \text{for} \quad i = 1, 2.$$

Let $L_i(d) = \{u \in M_{\mathbb{R}}; \langle u, v_i \rangle = d\}$ for $i = 1, 2$. Then we have

$$P_i' \cap M = \bigcup_{j=0}^{d_i} P_i' \cap L_i(j) \cap M \quad \text{for} \quad i = 1, 2.$$

Since $P_i' \cap L_i(j) = P \cap L_i(j) + F_i$ from the picture, if $P \cap L_i(j) \cap M \neq \emptyset$, then we can apply Lemma 1.

Set $S := \mathrm{Conv}\{0, (a, b), (c, d)\}$ the triangle contained in P. The line $\{(x, x) \in M_{\mathbb{R}}; x \in \mathbb{R}\}$ meets the boundary of S at two points, the origin 0 and the other point (x_0, x_0). Since the point (x_0, x_0) is on the edge of S connecting (a, b) and (c, d), we can write as

$$(x_0, x_0) = \lambda(a, b) + (1 - \lambda)(c, d)$$

with $0 \le \lambda \le 1$. Hence we have

$$x_0 = \lambda a + (1-\lambda)c = c + \lambda(a-c) = a + (1-\lambda)(c-a),$$

in other words, we have $x_0 \ge \min\{a, c\}$. If $a \ge c$, then the lattice points (j, j) $(0 \le j \le c)$ are contained in the triangle $S = \mathrm{Conv}\{0, (a, b), (c, d)\}$, hence, they are contained in P and $P \cap L_1(j)$ contains lattice points for $0 \le j \le c$. Moreover, if $a \ge c$, then P contains the vertical line segment G_1 connecting (c, c) and (c, d). Since P is convex, the convex hull of $F'_1 \cup G_1$ contains at least one lattice point in the intersection with $L_1(j)$ for $c \le j \le d_1$. We can apply Lemma 1. If $c \ge a$, then the lattice points (j, j) $(0 \le j \le a)$ are contained in the triangle $S = \mathrm{Conv}\{0, (a, b), (c, d)\} \subset P$ and P contains the vertical line segment G'_1 connecting (a, a) and (a, b). We can use G'_1 insead of G_1. Thus we have $P'_1 \cap M \subset P \cap M + R \cap M$. It is same for P'_2.

For the part $R' + R = (r+s)R_0$, we can use the fact that

$$\overbrace{R_0 \cap M + \cdots + R_0 \cap M}^{t \text{ times}} = (tR_0) \cap M$$

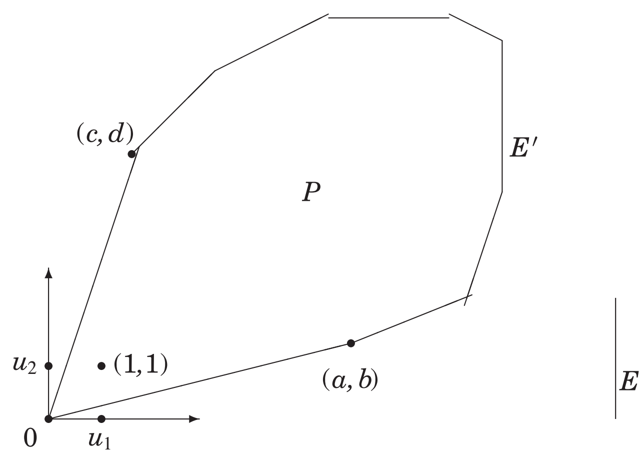
for all $t \ge 1$.

This implies that

$$P \cap M + R \cap M = (P + R) \cap M.$$

Since $P + R \supset P + E \ni m$ (we assumed $E \cap (m - P) \neq \emptyset$), we can find $x \in P \cap M$ and $y \in R \cap M \subset Q \cap M$ with $x + y = m$ in the case I.

Case II: $-\tilde{\rho}_0 \in \tilde{\Delta}(1)$. In this case we may set $-\tilde{\rho}_0 = \tilde{\rho}_1 \subset \tilde{\sigma} = \tilde{\rho}_1 + \tilde{\rho}_2$. We can draw the picture of P as the following.



E' is an edge of P parallel to E and perpendicular to v_1. Assume that E' is contained in the line $\{u \in M_{\mathbb{R}}; \langle u, v_i\rangle = d_1\}$. We may transform the bottom end of E to the origin. Then $P + E$ is contained in the region $\{u \in M_{\mathbb{R}}; 0 \le \langle u, v_i\rangle \le d_1\}$, and we have the decomposition

$$(P + E) \cap M = \bigcup_{j=0}^{d_1} (P \cap L_1(j) + E) \cap M.$$

If we replace E by F_1, then we see that this is the same as P'_1 in the case I and we have

$$P \cap M + E \cap M = (P + E) \cap M.$$

Since $P \cap (m - E) \neq \emptyset$, we can find $x \in P \cap M$ and $y \in E \cap M$ with $m = x + y$ in the case II.

This completes the proof of Theorem 1.

Next we explain how Theorem 1 implies Theorem 2. Let P be an integral convex polygon in $M_{\mathbb{R}}$ and let Q an integral convex polygon whose edges are parallel to some edges of P. We assume that corresponding parallel edges have the same inner normal directions. In other words, there exists an injective map from the set of all edges of Q to the set of all edges of P such that the corresponding edges are parallel and have the same inner normal directions.

Let (X, L) be the polarized toric surface described in Section 2. Set Δ the fan of N defining X, that is, $X = T_N\mathrm{emb}(\Delta)$. The polygon Q also defines a polarized toric surface (X', B), where $X' = T_N\mathrm{emb}(\Delta')$. By assumption we have $\Delta'(1) \subset \Delta(1)$. Since these two fans are complete, we can define a morphism of fans $f : (N, \Delta) \to (N, \Delta')$ by the identity map of N. Then f^*B is a globally generated line bundle on X such that $\Gamma(X, f^*B)$ has a basis $\{\mathbf{e}(m); m \in Q \cap M\}$. Thus we can apply Theorem 1 to L and f^*B.

We remark that this is true only in dimension two. Because $\Delta(1)$ does not determine a fan Δ in higher dimensions.

4. Gorenstein Toric Varieties

Since a toric variety is Cohen-Macaulay, X has the dualizing sheaf, which is a submodule of $k[C(v) \cap M]$ generated by $\{\mathbf{e}(m); m \in (\mathrm{Int}\ C(v)) \cap M\}$ on $U(v)$. Thus $U(v)$ is Gorenstein if and only if $(\mathrm{Int}\ C(v)) \cap M = m_0 + C(v) \cap M$ for some $m_0 \in M$ (see Section 3.2 [9]). Let $D = \sum_i D_i$ be all T_N-invariant divisors on a Gorenstein toric variety X. Then the canonical divisor of X, which is a Cartier divisor by definition, is a Weil divisor $-D = -\sum_i D_i$.

Following Mumford [6], we call an ample line bundle L *normally generated* if the map

$$\Gamma(L)^{\otimes k} \longrightarrow \Gamma(L^{\otimes k})$$

is surjaective for all $k \geq 1$. If an ample line bundle is normally generated, then it is very ample. We know that an ample line bundle on a toric surface is normally generated (see [5] Lemma 1.6.3). We also know that an ample line bundle on a nonsingular toric variety is very ample (cf. Corollary 2.15 [9]).

On the other hand, we know examples of ample but not very ample line bundles on Gorenstein toric 3-folds, whose adjoint bundles have no global sections. For example, $P = \mathrm{Conv}\{0, (1,0,0), (0,1,0), (1,1,2)\}$ defines a polarized Gorenstein toric 3-fold (Y, L). We can easily see that L is not very ample and $\Gamma(L + K_Y) = \{0\}$.

In contrast we have the following proposition.

Proposition 1. *Let Y be a Gorenstein projective toric variety of dimension three. Let L be an ample line bundle on Y such that $L + K_Y$ is generated by global sections. Then L is very ample.*

Before proving the Proposition we remark a property of integral polygons near vertices. Let F be an integral convex polytope of dimension two and v a vertex. Then the semigroup $(\mathbb{R}_{\geq 0}(F - v)) \cap M$ is generated by $(F - v) \cap M$. More precisely, we can find $m_0, m_1, \ldots, m_r \in F \cap M$ such that $\{m_i - v, m_{i-1} - v\}$ is a basis of $(\mathbb{R}F) \cap M \cong \mathbb{Z}^2$ and that the cone $\mathbb{R}_{\geq 0}(F - v)$ is decomposed into a union of cones $\mathbb{R}_{\geq 0}(m_i - v) + \mathbb{R}_{\geq 0}(m_{i-1} - v)$ $(i = 1, \ldots, r)$.

Proof of Proposition 1. Set P the integral convex polytope of dimension three corresponding to the polarized toric variety (Y, L). The condition $\Gamma(L + K_Y) \neq 0$ implies that $(\mathrm{Int}\ P) \cap M \neq \emptyset$.

Take a vertex v of P and a face F of dimension two containing v. Consider the cone $C(v) := \mathbb{R}_{\geq 0}(P - v)$. Since Y is Gorenstein, $(\mathrm{Int}\ C(v)) \cap M = m_0 + C(v) \cap M$ for some $m_0 \in M$. Since $L + K_Y$ is generated by $\Gamma(L + K_Y)$, we see that $m_0 \in \mathrm{Int}\ P$. Since $\mathbb{Z}(F \cap M) \cong \mathbb{Z}^2$ and since we may write $M \cong \mathbb{Z}(F \cap M) \oplus \mathbb{Z}$, we may choose a coordinate system (x, y, z) so that v is the origin, F is contained in the plane $\{z = 0\}$ and that P is contained in the first quadrant $\{x, y, z \geq 0\}$. Since $\mathrm{Int}\ C(v)$ contains a lattice point with its coordinate $z = 1$, the point m_0 has the coordinate $z = 1$, hence, the semigroup $(\mathbb{R}_{\geq 0}(m_0 - v) + \mathbb{R}_{\geq 0}(F - v)) \cap M$ is generated by $m_0 - v$ and $(F - v) \cap M$. By taking all faces containing v, we see that the semigroup $C(v) \cap M$ is generated by $(P - v) \cap M$. We may apply this argument to all vertices, and we know that L is very ample. □

A Gorenstein projective toric variety X is called *Fano* if the anti-canonical divisor $-K_X$ is ample. From Proposition 1 we see that the anti-canonical divisor on a Gorenstein toric Fano 3-fold is very ample. By applying Theorem 1 we have more.

Proposition 2. *Let X be a Gorenstein toric Fano variety of dimension three. For the anti-canonical line bundle $L = \mathcal{O}_X(-K_X)$ and a globally generated line bundle B, the multiplication map*

$$\Gamma(L) \otimes \Gamma(B) \longrightarrow \Gamma(L \otimes B) \tag{3}$$

is surjective.

In particular, the anti-canonical bundle on a Gorenstein toric 3-fold is normally generated.

Proof. We have a short exact sequence

$$0 \to \mathcal{O}_X \to L \to L|_D \to 0.$$

By taking global sections and tensoring with $\Gamma(B)$ we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(\mathcal{O}_X) \otimes \Gamma(B) & \to & \Gamma(L) \otimes \Gamma(B) & \to & \Gamma(L|_D) \otimes \Gamma(B) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma(B) & \to & \Gamma(L \otimes B) & \to & \Gamma(L \otimes B|_D) & \to & 0 \end{array} \tag{4}$$

Since the vertical arrow in the left hand side is isomorphic, we need to show that the vertical arrow in the right hand side is surjective.

Now set P the integral convex polytope in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^3$ corresponding to the polarized toric variety (X, L). And let Q be the integral convex polytope in $M_{\mathbb{R}}$ corresponding B. Then $P + Q$ corresponds to $L \otimes B$. Since $\Gamma(B) = \Gamma(L \otimes B \otimes \mathcal{O}_X(K_X))$ has a basis $\{\mathbf{e}(m); m \in (\mathrm{Int}(P + Q)) \cap M\}$ and since $h^1(B) = 0$, $\Gamma(L \otimes B|_D)$ has a basis $\{\mathbf{e}(m); m \in (\partial(P + Q)) \cap M\}$.

For any $m \in (\partial(P+Q)) \cap M$, we can find a face $F_i \subset P+Q$ of dimension 2 containing m, which corresponds to an irreducible T_N-invariant divisor D_i. And $\Gamma(L \otimes B|_{D_i})$ has a basis $\{\mathbf{e}(m); m \in F_i \cap M\}$. Since $L|_{D_i}$ is ample and $B|_{D_i}$ is generated by global sections on a toric variety of dimension 2, from Theorem 1 we can find $f_1 \in \Gamma(L|_{D_i})$ and $f_2 \in \Gamma(B|_{D_i})$ such that $\mathbf{e}(m) = f_1 \otimes f_2$.

Since the restriction maps $\Gamma(L) \to \Gamma(L|_{D_i})$ and $\Gamma(B) \to \Gamma(B|_{D_i})$ are surjective, we have the surjective map $\Gamma(L|_D) \to \Gamma(L|_{D_i})$ and elements $\tilde{f}_1 \in \Gamma(L|_D), \tilde{f}_2 \in \Gamma(B)$ with $\tilde{f}_1|_{D_i} = f_1, \tilde{f}_2|_{D_i} = f_2$. Then we prove that the vertical map in the right hand side of (4) is surjective, and we have that the map (3) is surjective. □

We remark that in the proof of Proposition 2, we can exchange the assumptions on L and B. Then we have the following.

Proposition 3. *Let X be a Gorenstein projective toric variety of dimension three with nef anti-canonical divisor. Then for any ample line bundle A on X, the multiplication map*

$$\Gamma(\mathcal{O}_X(-K_X)) \otimes \Gamma(A) \longrightarrow \Gamma(\mathcal{O}_X(-K_X) \otimes A)$$

is surjective.

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