Conformal-Projective Geometry of Statistical Manifolds

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Conformal-projective geometry of statistical manifolds, a natural generalization of conformal geometry of Riemannian manifolds, is studied in this paper. In particular, several fundamental results in the geometry are given: a geometric criterion for two statistical manifolds to be conformally-projectively equivalent; conditions for a statistical manifold to be conformally-projectively flat; properties of umbilical hypersurfaces of a conformally-projectively flat statistical manifold.

KEYWORDS:

1. Introduction

The purpose of this paper is to study 'conformal-projective geometry' of statistical manifolds, as a generalization of conformal geometry of Riemannian manifolds.

Let M be a manifold endowed with a torsion-free affine connection ∇ and a Riemannian metric h. We say that (M, ∇, h) is a *statistical manifold* if the (0, 3)-tensor field $C = \nabla h$, which we call the *cubic form*, is symmetric. A statistical manifold with $C = 0$ is a Riemannian manifold with Levi-Civita connection.

Two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are said to be *conformally-projectively equivalent* if there exist two functions φ and ψ on M satisfying that

$$\tilde{h}(X, Y) = e^{\varphi+\psi}h(X, Y), \tag{1.1}$$

$$\tilde{\nabla}_X Y = \nabla_X Y + d\varphi(Y)X + d\varphi(X)Y - h(X, Y)\,\mathrm{grad}_h\,\psi, \tag{1.2}$$

for arbitrary vector fields X and Y. We say that $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ is a *conformal-projective change.*

Conformal-projective equivalence of statistical manifolds was introduced by Matsuzoe [5] to characterize the statistical manifolds that can be realized by centroaffine immersions of codimension two. In the work, he used the fact that a conformal-projective change is a composite of a projective change and a 'dual change' of a projective change.

On the other hand, conformal-projective equivalence can be considered as a natural generalization of conformal equivalence of Riemannian metrics; in fact, for two Riemannian metrics, say h with Levi-Civita connection ∇ and $\tilde{h}$ with $\tilde{\nabla}$, we can easily see that $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ is a conformal-projective change if and only if $h \to \tilde{h}$ is a conformal change of Riemannian metrics.

On a manifold of dimension $n \geq 3$, a change $h \to \tilde{h}$ of Riemannian metrics is conformal if and only if the change preserves the umbilical points of an arbitrary hypersurface (cf. [2]). The corresponding characterization of conformal-projective equivalence is given as follows:

Theorem 1.1. *Suppose that (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are two simply connected statistical manifolds of dimension $n \geq 3$. Then, (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are conformally-projectively equivalent if and only if they satisfy the following two conditions:*

1. $\widetilde{\mathrm{Ric}}\,(X, Y) - \widetilde{\mathrm{Ric}}\,(Y, X) = \mathrm{Ric}\,(X, Y) - \mathrm{Ric}\,(Y, X)$, *where* Ric *and* $\widetilde{\mathrm{Ric}}$ *are the Ricci tensors for ∇ and for $\tilde{\nabla}$, respectively.*
2. *The change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ preserves the tangentially umbilical points and the normally umbilical points of an arbitrary hypersurface of M.*

A statistical manifold (M, ∇, h) is said to be *conformally-projectively flat* if for each point p of M, there exists a conformal-projective change of $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ such that $\tilde{\nabla}$ is flat in a neighbourhood of p. By the fact we remarked above, a Riemannian manifold (M, h) is conformally-projectively flat if and only if it is conformally flat. In this paper, we shall define a tensor field W which plays the role in conformal-projective geometry as the Weyl conformal curvature tensor does in conformal geometry. The tensor field W is invariant under a conformal-projective change; furthermore, we have:

Theorem 1.2. *A statistical manifold (M, ∇, h) of dimension $n \geq 4$ is conformally-projectively flat if and ony if*

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the conformal-projective curvatur tensor W vanishes everywhere on M.

In Section 2, we briefly review preliminary facts on statistical manifolds, on their hypersurfaces, and on their realization in affine space. A few simple propositions about umbilical hypersurfaces are also shown in the section. Section 3 is devoted to proving Theorem 1.1. The definition of the conformal-projective curvature tensor W and a proof of Theorem 1.2 are given in Section 4, with another characterization of the conformally-projectively flat statistical manifolds (Corollary 4.3).

In Section 5, we shall show that a conformally-projectively flat statistical manifold can be embedded into a suitable flat statistical manifold as a statistical submanifold of codimension two (Corollary 5.2). As for this theorem, we do not know whether the corresponding result may hold in Riemannian geometry. However, it may remind us of the following fact: a simply connected, conformally flat Riemannian manifold can be isometrically embedded in a Minkowski space as a space-like submanifold of codimension two.

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2. Preliminaries

2.1. Statistical Manifolds

Let (M, ∇, h) be an n-dimensional statistical manifold. We define the *dual* (or *conjugate*) connection $\bar\nabla$ of ∇ with respect to h by

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \bar\nabla_X Z). \tag{2.1}$$

The connection $\bar\nabla$ is torsion-free since the cubic form $C = \nabla h$ is symmetric. Furthermore, $(M, \bar\nabla, h)$ is also a statistical manifold, with cubic form $-C$. We say that $(M, \bar\nabla, h)$ is the *dual statistical manifold* of (M, ∇, h). If two statistical manifolds (M, ∇, h) and $(M, \tilde\nabla, \tilde h)$ are conformally-projectively equivalent, their dual connections $\bar\nabla$ and $\bar{\tilde\nabla}$ satisfy that

$$\begin{aligned} h(Y, \bar{\tilde\nabla}_X Z) &= e^{-(\varphi+\psi)}\tilde h(Y, \bar{\tilde\nabla}_X Z) \\ &= e^{-(\varphi+\psi)}\{X\tilde h(Y, Z) - \tilde h(\tilde\nabla_X Y, Z)\} \\ &= (d\varphi(X) + d\psi(X))h(Y, Z) + h(\nabla_X Y, Z) + h(Y, \bar\nabla_X Z) - h(\tilde\nabla_X Y, Z) \\ &= h(Y, \bar\nabla_X Z + d\psi(Z)X + d\psi(X)Z - h(X, Z)\operatorname{grad}_h \varphi). \end{aligned}$$

Then the dual statistical manifolds $(M, \bar\nabla, h)$ and $(M, \bar{\tilde\nabla}, \tilde h)$ are also conformally-projectively equivalent.

We denote by R the curvature tensor of ∇. A statistical manifold is said to be *flat* if $R = 0$. It follows from (2.1) that the curvature tensor $\bar R$ of $\bar\nabla$ satisfies

$$h(R(X, Y)Z, V) + h(Z, \bar R(X, Y)V) = 0. \tag{2.2}$$

Then a statistical manifold is flat if and only if its dual statistical manifold is flat. We also see that a statistical manifold is conformally-projectively flat if and only if so is the dual statistical manifold.

A statistical manifold (M, ∇, h) is said to be *1-conformally flat* if for each point p of M there exists a function ψ on M such that the affine connection $\tilde\nabla$ defined by

$$\tilde\nabla_X Y = \nabla_X Y - h(X, Y)\operatorname{grad}_h \psi$$

is flat in a neighbourhood of p. By definition, a 1-conformally flat statistical manifold is conformally-projectively flat.

2.2. Statistical Hypersurfaces

Let (M, ∇, h) be a statistical manifold of dimension $n \geq 3$, and let N be a submanifold of M. At each point p of N, the tangent space $T_p(M)$ is decomposed as the direct sum of the tangent space $T_p(N)$ and its orthogonal complement with respect to h. According to this decomposition, we set

$$\nabla'_X Y = (\text{the } T(N)\text{-component of } \nabla_X Y),$$

where X and Y are vector fields tangent to N. Then, ∇' is a torsion-free affine connection, and N provided with ∇' and the induced metric h' is a statistical manifold, which we call a *statistical submanifold* (see [8]). When N is a hypersurface, we denote by ν a unit normal vector field along N with respect to h, and express $\nabla_X Y$ and $\nabla_X \nu$ as follows:

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y)\nu, \tag{2.3}$$

$$\nabla_X \nu = -\beta^{\#}(X) + \tau(X)\nu. \tag{2.4}$$

We can easily see that α and $\beta^{\#}$ are tensor fields on N of type (0, 2) and of type (1, 1), respectively. Moreover, α

and $\beta = h'(\beta^{\#}(\cdot), \cdot)$ are symmetric; indeed, we have

$$\alpha(X, Y) - \alpha(Y, X) = h(\nabla_X Y, \nu) - h(\nabla_Y X, \nu) = h([X, Y], \nu) = 0,$$

and, by (2.1),

$$\beta(X, Y) = -h(\nabla_X \nu, Y) = h(\nu, \bar{\nabla}_X Y) = \bar{\alpha}(X, Y), \tag{2.5}$$

where $\bar{\alpha}$ denotes α for the hypersurface N in $(M, \bar{\nabla}, h)$.

We say that a point p of N is a *tangentially* (resp. *normally*) *umbilical point* of N in (M, ∇, h) if there exists a constant c such that $\alpha_p = c \cdot h'_p$ (resp. $\beta_p = c \cdot h'_p$). By (2.5), a point is a normally umbilical point of N in (M, ∇, h) if and only if it is a tangentially umbilical point of N in $(M, \bar{\nabla}, h)$.

Proposition 2.1. *A conformal-projective change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ preserves both the tangentially umbilical points and the normally umbilical points of any hypersurface.*

Proof. Let $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ is a conformal-projective change given by (1.1) and (1.2). Since the unit normal vector field along a hypersurface N with respect to $\tilde{h}$ is $\tilde{\nu} = e^{-(\varphi+\psi)/2}\nu$, we have

$$\tilde{\alpha}(X, Y) = \tilde{h}(\tilde{\nabla}_X Y, \tilde{\nu}) = e^{(\varphi+\psi)/2}\{\alpha(X, Y) - h(X, Y)d\psi(\nu)\},$$

and

$$\tilde{\beta}(X, Y) = -\tilde{h}(\tilde{\nabla}_X \tilde{\nu}, Y) = e^{(\varphi+\psi)/2}\{\beta(X, Y) - d\varphi(\nu)h(X, Y)\}.$$

These two equations imply the proposition. □

A hypersurface of a statistical manifold is said to be *tangentially umbilical* if every point is tangentially umbilical, or equivalently, there exists a function f on N such that $\alpha = fh'$. We can also define a *normally umbilical* hypersurface in a similar way.

Remark 2.2. In [4, Corollary 12], we showed that a flat statistical manifold has sufficiently many tangentially umbilical hypersurfaces in the sense that: for a given point p, for a given codimention-one subspace V of the tangent space at p, and for a given real number k, we have a tangentially umbilical hypersurface N such that N goes thorough p, its tangent space at p coincides with V, and $\alpha_p = k \cdot h_p$. Hence Proposition 2.1 implies that a conformally-projectively flat statistical manifold also has enough tangentially umbilical hypersurfaces. However, we do not know whether the property characterizes the conformally-projectively flat statistical manifolds.

Proposition 2.3. *Suppose that two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ of dimension $n \geq 3$ are conformally-projectively equivalent. Then a statistical hypersurface $(N, \tilde{\nabla}', \tilde{h}')$ of $(M, \tilde{\nabla}, \tilde{h})$ is conformally-projectively equivalent to (N, ∇', h') of (M, ∇, h).*

Proof. We assume that $(\tilde{\nabla}, \tilde{h})$ is related to (∇, h) by (1.1) and (1.2). Then we have

$$\tilde{h}' = e^{(\varphi|_N)+(\psi|_N)}h' \tag{2.6}$$

and

$$\tilde{\nabla}'_X Y + \tilde{\alpha}(X, Y)\tilde{\nu} = \nabla'_X Y + \alpha(X, Y)\nu + d\varphi(Y)X + d\varphi(X)Y - h(X, Y)\,\mathrm{grad}_h\,\psi.$$

Since the tangential component of $\mathrm{grad}_h\,\psi$ is given by

$$\mathrm{grad}_h\,\psi - h(\mathrm{grad}_h\psi, \nu)\nu = \mathrm{grad}_{h'}\,(\psi|_N),$$

we obtain

$$\tilde{\nabla}'_X Y = \nabla'_X Y + d(\varphi|_N)(Y)X + d(\varphi|_N)(X)Y - h(X, Y)\,\mathrm{grad}_{h'}\,(\psi|_N). \tag{2.7}$$

Equations (2.6) and (2.7) imply that $(\nabla', h') \to (\tilde{\nabla}', \tilde{h}')$ is a conformal-projective change. □

Proposition 2.4. *Let (M, ∇, h) be a conformally-projectively flat statistical manifold of dimension $n \geq 3$, and let N be a tangentially umbilical hypersurface of (M, ∇, h). Then the statistical hypersurface (M, ∇', h') is conformally-projectively flat.*

Proof. Let p be an arbitrary point of N. By assumption, there exists a conformal-projective change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ such that $\tilde{\nabla}$ is flat on a neighbourhood U of p in M. By virtue of Proposition 2.1, we see that $N \cap U$ is a tangentialy umbilical hypersurface of $(U, \tilde{\nabla}|_U, \tilde{h}|_U)$. In [4, Proposition 6], we showed that such a hypersurface is 1-conformally flat. Hence $N \cap U$ is a conformally-projectively flat statistical hypersurface of $(U, \tilde{\nabla}|_U, \tilde{h}|_U)$. By Proposition 2.3, $N \cap U$ is also conformally-projectively flat as a statistical hypersurface of $(U, \nabla|_U, h|_U)$. Thus we see that (N, ∇', h') is conformally-projectively flat around each point of N. □

2.3. Realization of Statistical Manifolds

Let A^n be an n-dimensional affine space, and R^n an n-dimensional vector space. We denote by D the standard flat connections of these spaces.

Definition 2.5. Let M be an n-dimensional manifold endowed with a torsion-free affine connection ∇ and a Riemannian metric h. We say that (M, ∇, h) *is realized by an equiaffine immersion* (x, ξ) if $x : M \to \mathrm{A}^{n+1}$ is an immersion and ξ is a vector field of A^{n+1} along x such that

1. $x^*T(\mathrm{A}^{n+1}) = x_*T(M) \oplus \mathrm{R}\xi$;
2. According to this decomposition of $x^*T(\mathrm{A}^{n+1})$,
$$(x^*D)_X(x_*Y) = x_*(\nabla_X Y) + h(X, Y)\xi;$$
3. $(x^*D)_X\xi$ has no ξ-component.

Definition 2.6. Let M be an n-dimensional manifold endowed with a torsion-free affine connection ∇ and a Riemannian metric h. We say that *(M, ∇, h) is realized by an equi-centroaffine immersion (x, ξ) of codimension two* if $x : M \to \mathrm{R}^{n+2}$ is an immersion and ξ is a vector field of R^{n+2} along x such that

1. $x^*T(\mathrm{R}^{n+2}) = \mathrm{R}x \oplus x_*T(M) \oplus \mathrm{R}\xi$;
2. According to this decomposition of $x^*T(\mathrm{R}^{n+2})$,
$$(x^*D)_X(x_*Y) = T(X, Y)x + x_*(\nabla_X Y) + h(X, Y)\xi,$$
where T is a certain (0, 2)-tensor field on M;
3. $(x^*D)_X\xi$ has no ξ-component.

For proofs of the following fact, see [4] and [5].

Fact 2.7. *Suppose that M is a simply connected, connected manifold of dimension $n \geq 2$ endowed with a torsion-free affine connection ∇ and a Riemannian metric h. Then,*

1. *(M, ∇, h) is a 1-conformally flat statistical manifold if and only if it is realized by an equiaffine immersion.*
2. *(M, ∇, h) is a conformally-projectively flat statistical manifold if and only if it is realized by an equi-centroaffine immersion of codimension two.*

The theory of centroaffine immersions of codimension two has been systematically developed by Nomizu and Sasaki [6]. We quote a fact from their paper with a suitable modification for later use.

Fact 2.8. *Suppose that a statistical manifold (M, ∇, h) is realized by an equi-centroaffine immersion (x, ξ). Then we have the following formulas:*

$$T(X, Y) = T(Y, X), \tag{2.8}$$
$$R(X, Y)Z = h(Y, Z)S(X) - h(X, Z)S(Y) - T(Y, Z)X + T(X, Z)Y, \tag{2.9}$$
$$(\nabla T)(Y, Z; X) + h(Y, Z)\rho(X) = (\nabla T)(X, Z; Y) + h(X, Z)\rho(Y), \tag{2.10}$$
$$(\nabla S)(Y; X) + \rho(X)Y = (\nabla S)(X; Y) + \rho(Y)X, \tag{2.11}$$
$$T(X, S(Y)) - T(Y, S(X)) = (\nabla\rho)(Y; X) - (\nabla\rho)(X; Y), \tag{2.12}$$
$$h(X, S(Y)) = h(Y, S(X)), \tag{2.13}$$

where ρ and S are the 1-form and the (1, 1)-tensor field determined by

$$(x^*D)_X\xi = \rho(X)x - x_*S(X). \tag{2.14}$$

Conversely, let (M, ∇, h) be a simply connected, connected statistical manifold of dimension $n \geq 2$ with a (0, 2)-tensor field T, a (1, 1)-tensor field S, and a 1-form ρ that satisfy equations (2.8)–(2.13). Then (M, ∇, h) is realized by an equi-centroaffine immersion (x, ξ) satisfying (2.14).

We denote by R_{n+2} the dual vector space of R^{n+2}, and by $\langle \cdot , \cdot \rangle$ the pairing between R_{n+2} and R^{n+2}. For an equi-centroaffine immersion (x, ξ), we define the *dual immersion* $(\bar{x}, \bar{\xi})$ by

$$\langle \bar{x}(p), x(p) \rangle = 0, \qquad \langle \bar{x}(p), x_*X \rangle = 0, \qquad \text{and} \qquad \langle \bar{x}(p), \xi(p) \rangle = 1$$

for all $p \in M$, $X \in T_p(M)$; and

$$\langle \bar{\xi}(p), x(p) \rangle = 1, \qquad \langle \bar{\xi}(p), x_*X \rangle = 0, \qquad \text{and} \qquad \langle \bar{\xi}(p), \xi(p) \rangle = 0$$

for all $p \in M$, $X \in T_p(M)$. In [5], Matsuzoe showed that if an n-dimensional conformally-projectively flat statistical manifold (M, ∇, h) is realized by (x, ξ), then the dual statistical manifold $(M, \bar{\nabla}, h)$ is realized by $(\bar{x}, \bar{\xi})$. Using the result, he defined a function ρ_M on $M \times M$, called the *geometric divergence* of (M, ∇, h), by

$$\rho_M(p, q) = \langle \bar{x}(q), x(p) \rangle, \quad p, q \in M.$$

Note that the definition of ρ_M is independent of the choice of a realization (x, ξ). We refer the reader to [3] and [5] for the role of the geometric divergence in geometry of statistical manifolds, and to [1] for the role in mathematical statistics.

3. Conformal-Projective Equivalence and Umbilical Hypersurfaces

Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be two statistical manifolds, which need not be conformally-projectively equivalent at present. We set $F = \tilde{\nabla} - \nabla$. Then F is a (1, 2)-tensor field on M satisfying $F(X, Y) = F(Y, X)$.

Proposition 3.1. *Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be two statistical manifolds of dimension $n \geq 3$. Suppose that the change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ preserves the tangentially umbilical points of any hypersurface of M. Then, $\tilde{h}$ is conformally equivalent to h, and there exists a 1-form γ on M such that*

$$h(F(X, Y), Z) = \gamma(Z)h(X, Y) \quad \text{if } X, Y \perp Z \text{ with respect to } h. \tag{3.1}$$

Proof. Let p be a point of M, and H an $(n-1)$-dimensional linear subspace of $T_p(M)$. We take a normal coordinate system $(u^1, \ldots, u^n)$ with center at p with respect to the connection ∇ such that $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^{n-1})_p \in H$ and $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$ is an orthonormal basis of $T_p(M)$. For a constant k, the hypersurface N defined by $u^n = (k/2)\sum_{i=1}^{n-1}(u^i)^2$ satisfies that $p \in N$, $T_p(N) = H$, and $\alpha_p = k \cdot h_p$. Hence we have

$$\begin{aligned} h(F(X_p, Y_p), \nu_p) &= h(\tilde{\alpha}(X_p, Y_p)\tilde{\nu}_p - \alpha(X_p, Y_p)\nu_p, \nu_p) \\ &= \tilde{c}(k)\tilde{h}(X_p, Y_p) - kh(X_p, Y_p), \quad X_p, Y_p \in H, \end{aligned} \tag{3.2}$$

where $\tilde{c}(k)$ is a constant depending only on k, not on the choice of X_p and Y_p. Since the left-hand side is independent of the choice of k, $\tilde{h}$ is necessarily proportional to h on H. Since H and p are arbitrary, and since $n \geq 3$, we can see that

$$\tilde{h} = e^f h, \tag{3.3}$$

where f is a function on M. Moreover, by (3.2) and (3.3), we see that there exists a function γ on $T(M)$ such that (3.1) holds. Since h and F are tensor fields, γ is a 1-form. □

Applying Proposition 3.1 to the dual statistical manifolds $(M, \bar{\nabla}, h)$ and $(M, \bar{\tilde{\nabla}}, \tilde{h})$, we have:

Proposition 3.2. *Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be two statistical manifolds of dimension $n \geq 3$. Suppose that the change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ preserves the normally umbilical points of any hypersurface of M. Then, $\tilde{h}$ is conformally equivalent to h, and there exists a 1-form δ on M such that*

$$h(F(X, Y), Z) = -\delta(Y)h(X, Z) \quad \text{if } X, Z \perp Y \text{ with respect to } h. \tag{3.4}$$

As a corollary of Propositions 3.1 and 3.2, we obtain:

Corollary 3.3. *Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be two statistical manifolds of dimension $n \geq 3$. Suppose that the change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ preserves the tangentially umbilical points and the normally umbilical points of any hypersurface of M. Then, $\tilde{h}$ is conformally equivalent to h, and there exist two 1-forms γ and δ on M such that*

$$h(\tilde{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \delta(Y)h(X, Z) - \delta(X)h(Y, Z) + \gamma(Z)h(X, Y). \tag{3.5}$$

Proof. Choosing 1-forms γ and δ as in (3.1) and (3.4), we put

$$T(X, Y, Z) = h(F(X, Y), Z) - \{-\delta(Y)h(X, Z) - \delta(X)h(Y, Z) + \gamma(Z)h(X, Y)\}.$$

By Propositions 3.1 and 3.2, and by the symmetricity of F, $T(X, Y, Z) = 0$ if one of X, Y and Z is orthogonal to the others. Then the following lemma implies that $T = 0$. □

Lemma 3.4. *Let T be a tensor of type (0, 3) over an n-dimensional vector space equipped with an inner product h. Suppose that $n \geq 3$ and that $T(X, Y, Z) = 0$ if one of X, Y and Z is orthogonal to the remaining two vectors. Then we have $T = 0$.*

Proof. We choose a basis $e_1, \ldots, e_n$ of V such that $h(e_i, e_j) = \delta_{ij}$, and set $T_{ijk} = T(e_i, e_j, e_k)$. By assumption, $T_{ijk} = 0$ if at least two of i, j, k are distinct. Therefore it suffices to show that $T_{iii} = 0$. Since $e_j + e_j$ and $e_i - e_j$ are mutually orthogonal, we have

$$0 = T(e_i + e_j, e_i + e_j, e_i - e_j) + T(e_i - e_j, e_i - e_j, e_i + e_j) = 2T_{iii},$$

thereby completing the proof. □

Proof of Theorem 1.1. Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be two simply connected statistical manifolds such that

$$\widetilde{\mathrm{Ric}}(X, Y) - \widetilde{\mathrm{Ric}}(Y, X) = \mathrm{Ric}(X, Y) - \mathrm{Ric}(Y, X), \tag{3.6}$$

and that the change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ preserves the tangentially umbilical points and the normally umbilical points of any hypersurface of M. By Corolary 3.3, we have a function f and two 1-forms γ, δ such that (3.3) and (3.5) hold. Then, we have

$$\begin{aligned} e^{-f}(\tilde{\nabla}\tilde{h})(X, Y; Z) = (\nabla h)(X, Y; Z) &+ \{\delta(Z)h(X, Y) + \delta(X)h(Y, Z) + \delta(Y)h(X, Z)\} \\ &- \{h(Z, X)\gamma(Y) + h(Z, Y)\gamma(X) + h(X, Y)\gamma(Z)\} \\ &+ \{df(Z) + \delta(Z) + \gamma(Z)\}h(X, Y). \end{aligned}$$

Since $\bar{\nabla}\bar{h}$ and ∇h are symmetric, we obtain

$$df + \gamma + \delta = 0. \tag{3.7}$$

By a direct calculation, (3.5) implies that

$$\begin{aligned}\tilde{R}(X, Y)Z = R(X, Y)Z &- (\nabla\delta)(Z; X)Y - (\nabla\delta)(Y; X)Z + h(Y, Z)\nabla_X\gamma^{\#} \\ &+ (\nabla\delta)(Z; Y)X + (\nabla\delta)(X; Y)Z - h(X, Z)\nabla_Y\gamma^{\#} \\ &+ \delta(Y)\delta(Z)X - h(Y, Z)\delta(\gamma^{\#})X + h(Y, Z)h(X, \gamma^{\#})\gamma^{\#} \\ &- \delta(X)\delta(Z)Y + h(X, Z)\delta(\gamma^{\#})Y - h(X, Z)h(Y, \gamma^{\#})\gamma^{\#}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathrm{Ric}}\,(Y, Z) = \mathrm{Ric}\,(Y, Z) &+ n(\nabla\delta)(Z; Y) - (\nabla\delta)(Y; Z) + h(Y, Z)\,\mathrm{trace}\,\nabla\gamma^{\#} \\ &- h(\nabla_Y\gamma^{\#}, Z) + (n-1)\delta(Y)\delta(Z) - (n-1)h(Y, Z)\delta(\gamma^{\#}) \\ &+ h(Y, Z)h(\gamma^{\#}, \gamma^{\#}) - h(\gamma^{\#}, Z)h(Y, \gamma^{\#}),\end{aligned}$$

where $\gamma^{\#}$ is the vector field determined by $h(\gamma^{\#}, \cdot\,) = \gamma$.

Since $h(\nabla_Y\gamma^{\#}, Z) = (\nabla\gamma)(Z; Y) - (\nabla h)(\gamma^{\#}, Z; Y)$, we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}\,(Y, Z) - \widetilde{\mathrm{Ric}}\,(Z, Y) &= \mathrm{Ric}\,(Y, Z) - \mathrm{Ric}\,(Z, Y) + (n-1)\{(\nabla\delta)(Z; Y) - (\nabla\delta)(Y; Z)\} \\ &\quad - \{(\nabla\gamma)(Z; Y) - (\nabla\gamma)(Y; Z)\} \\ &= \mathrm{Ric}\,(Y, Z) - \mathrm{Ric}\,(Z, Y) + 2(n-1)(d\delta)(Y, Z) - 2d\gamma(Y, Z).\end{aligned} \tag{3.8}$$

Equations (3.6), (3.7) and (3.8) imply that $d\gamma = d\delta = 0$. Since M is simply connected, we have two functions φ and ψ on M such that $\gamma = -d\psi$, $\delta = -d\varphi$, and $f = \varphi + \psi$. Substituting these equations into (3.3) and (3.5), we see that the change $(\nabla, h) \to (\tilde{\nabla}, \tilde{h})$ is a conformal-projective change.

The converse is easily deduced from Proposition 2.1 and equation (3.8). □

4. The Conformal-Projective Curvature Tensor and Conformally-Projectively Flat Statistical Manifolds

All the result given in this section are proved under the weaker assumption that the metric of a statistical manifold is pseudo-Riemannian.

For a given statistical manifold (M, ∇, h), we denote by R the curvature tensor, and by Ric the Ricci tensor. The Ricci operator $\mathrm{Ric}^{\#}$ of (M, ∇, h) is the (1, 1)-tensor field determined by

$$h(\mathrm{Ric}^{\#}\,(X), Y) = \mathrm{Ric}\,(X, Y).$$

We call the function $\sigma = \mathrm{trace}_h\,\mathrm{Ric}$ the *scalar curvature* of (M, ∇, h). The corresponding quantities for the dual statistical manifold $(M, \bar{\nabla}, h)$ are denoted with overline.

We define the *conformal-projective curvature tensor* W of (M, ∇, h) by

$$\begin{aligned}W(X, Y)Z = R(X, Y)Z &- \{h(Y, Z)A(X) - h(X, Z)A(Y) + B(Y, Z)X - B(X, Z)Y)\} \\ &- \frac{\sigma}{n(n-1)}\{h(Y, Z)X - h(X, Z)Y\},\end{aligned}$$

where A, B are the tensor fields of type (1, 1) and of type (0, 2), respectively, given by

$$A(X) = \frac{1}{n(n-2)}\{\mathrm{Ric}^{\#}\,(X) + (n-1)\,\overline{\mathrm{Ric}}^{\#}\,(X) - \sigma X\},$$

$$B(X, Y) = \frac{1}{n(n-2)}\{(n-1)\,\mathrm{Ric}\,(X, Y) + \overline{\mathrm{Ric}}\,(X, Y) - \sigma h(X, Y)\}.$$

Then a direct calculation shows:

Proposition 4.1. *If two statistical manifolds of dimension greater than two are conformally-projectively equivalent, then their conformal-projective curvature tensors coincide.*

The conformal-projective curvature tensor on a Riemannian manifold (with Levi-Civita connection) reduces to the usual Weyl conformal curvature tensor, because Ric and $\overline{\mathrm{Ric}}$ coincide in that case.

Proof of Theorem 1.2. The 'only if'-part follows from Proposition 4.1 and the definition of W.

In order to prove the 'if'-part, we may assume that M is simply connected and connected since conformally-projectively flatness is a local notion. Setting

$$S(X) = A(X) - \frac{\sigma}{2n(n-1)}X, \quad T(X, Y) = -B(X, Y) + \frac{\sigma}{2n(n-1)}h(X, Y),$$

$$\rho(X) = -\frac{1}{n-1}\{\mathrm{trace}\,(\nabla S)(\,\cdot\,; X) - \mathrm{trace}\,(\nabla S)(X; \,\cdot\,)\},$$

we shall show that equations (2.8)–(2.13) hold. If it is done, Fact 2.8 implies that (M, ∇, h) is realized by an equi-centroaffine immersion of codimension two. Hence (M, ∇, h) is conformally-projectivey flat by virtue of Fact 2.7.

First, equation (2.9) immediately follows from the assumption that the conformal-projective curvature tensor $W = 0$.

By the first Bianchi identity for R and by (2.9), we have

$$\begin{aligned} 0 &= \mathfrak{S}_{X,Y,Z} R(X, Y)Z = -\mathfrak{S}_{X,Y,Z}\{T(Y, Z)X - T(X, Z)Y\} \\ &= -\mathfrak{S}_{X,Y,Z}\{(T(X, Y) - T(Y, X))Z\}, \end{aligned}$$

where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y and Z. Then we obtain

$$\begin{aligned} 0 &= \mathrm{trace}\,[Z \mapsto \mathfrak{S}_{X,Y,Z}\{(T(X, Y) - T(Y, X))Z\}] \\ &= (n-2)(T(X, Y) - T(Y, X)). \end{aligned}$$

Since $n \geq 3$, this implies equation (2.8). Using the first Bianchi identity for $\bar{R}$ and (2.2), we can verify equaton (2.13) in the same way.

Next, we shall show equations (2.10) and (2.11). By the second Bianchi identity, we obtain

$$\begin{aligned} 0 &= \mathrm{trace}\,[V \mapsto \mathfrak{S}_{X,Y,V}\{(\nabla_V R)(X, Y)Z\}] \\ &= h((\nabla S)(Y; X), Z) - h((\nabla S)(X; Y), Z) + (n-2)\{(\nabla T)(Y, Z; X) \\ &\quad - (\nabla T)(X, Z; Y)\} - (n-1)\{h(X, Z)\rho(Y) - h(Y, Z)\rho(X)\}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} 0 &= \mathrm{trace}_h\,[(Y, Z) \mapsto \mathrm{trace}\,[V \mapsto \mathfrak{S}_{X,Y,V}\{(\nabla_V R)(X, Y)Z\}]] \\ &= (n-2)\{\mathrm{trace}_h\,(\nabla T)(\,\cdot\,, \,\cdot\,; X) - \mathrm{trace}_h\,(\nabla T)(X, \,\cdot\,; \,\cdot\,) + (n-1)\rho(X)\}. \end{aligned}$$

By using the last equation, we have

$$\begin{aligned} 0 &= \mathrm{trace}_h\,[((V, Z) \mapsto \mathfrak{S}_{X,Y,V}\{(\nabla_V R)(X, Y)Z\}] \\ &= (n-2)\{(\nabla S)(Y; X) - (\nabla S)(X; Y)\} + ((\nabla T)(Y, \,\cdot\,; X))^{\#} \\ &\quad - ((\nabla T)(X, \,\cdot\,; Y))^{\#} - (n-1)\{\rho(Y)X - \rho(X)Y\}. \end{aligned} \tag{4.2}$$

Equations (4.1) and (4.2) imply (2.10) and (2.11) because $(n-2)^2 - 1 \neq 0$.

It remains only to prove equation (2.12). By (2.11), we have

$$(\nabla^2 S)(Y; X, Z) - (\nabla^2 S)(X; Y, Z) = (\nabla\rho)(Y; Z)X - (\nabla\rho)(X; Z)Y.$$

We calculate the cyclic sum of the both sides of this equation: for the left-hand side, we have

$$\begin{aligned} \mathfrak{S}_{X,Y,Z}\{(\nabla^2 S)(Y; X, Z) - (\nabla^2 S)(X; Y, Z)\} &= \mathfrak{S}_{X,Y,Z}\{(\nabla^2 S)(Z; Y, X) - (\nabla^2 S)(Z; X, Y)\} \\ &= \mathfrak{S}_{X,Y,Z}\{R(X, Y)(S(Z)) - S(R(X, Y)Z)\} \end{aligned}$$

using (2.9) for the first term and the first Bianchi identity $\mathfrak{S}_{X,Y,Z} S(R(X, Y)Z) = S(\mathfrak{S}_{X,Y,Z} R(X, Y)Z) = 0$ for the second term

$$= \mathfrak{S}_{X,Y,Z}\{h(Y, S(Z))S(X) - h(X, S(Z))S(Y) - T(Y, S(Z))X + T(X, S(Z))Y\}$$

by formula (2.13)

$$= \mathfrak{S}_{X,Y,Z}\{(T(Y, S(X)) - T(X, S(Y)))Z\};$$

while, for the right-hand side, we have

$$\mathfrak{S}_{X,Y,Z}\{(\nabla\rho)(Y; Z)X - (\nabla\rho)(X; Z)Y\} = \mathfrak{S}_{X,Y,Z}\{((\nabla\rho)(X; Y) - (\nabla\rho)(Y; X))Z\}.$$

Thus we obtain

$$\mathfrak{S}_{X,Y,Z}\{(T(Y, S(X)) - T(X, S(Y)))Z\} = \mathfrak{S}_{X,Y,Z}\{((\nabla\rho)(X; Y) - (\nabla\rho)(Y; X))Z\}. \tag{4.3}$$

Equation (2.12) now follows from (4.3) in a manner similar to the proof of (2.8). □

Remark 4.2. 1. The proof above works without any modification even if we set

$$S = A - \left(\frac{\sigma}{2n(n-1)} + \mu\right)I, \quad T = -B + \left(\frac{\sigma}{2n(n-1)} - \mu\right)h,$$

where μ is an arbitrary function on M. This ambiguity corresponds to indefiniteness of the choice of ξ, which can not be uniquely determined with the data of ∇ and h only (cf. [6, Proposition 5.5]). In the case where $\mu = 0$, the corresponding ξ is pre-normalized.

2. In the case of a 3-dimensional statistical manifold, setting S, T and ρ as in the proof of Theorem 1.2, if we assume $W = 0$ and either (2.10) or (2.11), then we can deduce that the statistical manifold is conformally-projectively flat.

As a corollary of Theorem 1.2, we shall give another necessary and sufficient condition for a statistical manifold to be conformally-projectively flat.

Corollary 4.3. *Let (M, ∇, h) be a statistical manifold of dimension $n \geq 4$. Then (M, ∇, h) is conformally-projectively flat if and only if* Ric, $\overline{\mathrm{Ric}}$ *are symmetric and $h(R(X, Y)Z, V) = 0$ for all vector fields X, Y, Z and V which are mutually orthogonal with respect to h.*

Proof. Let W be the conformal-projective curvature tensor. By setting

$$K(X, Y, Z, V) = h(W(X, Y)Z, V),$$

we have

$$K(X, Y, Z, V) = -K(Y, X, Z, V), \tag{4.4}$$

$$K(X, Y, Z, V) + K(Y, Z, X, V) + K(Z, X, Y, V) = 0, \tag{4.5}$$

$$K(X, Y, Z, V) + K(Y, V, Z, X) + K(V, X, Z, Y) = 0, \tag{4.6}$$

$$\mathrm{trace}_h K(\cdot\,, Y, Z, \cdot\,) = \mathrm{trace}_h K(\cdot\,, Y, \cdot\,, V) = 0. \tag{4.7}$$

Moreover, $h(R(X, Y)Z, V) = K(X, Y, Z, V)$ holds for all vector fields X, Y, Z and V which are mutually orthogonal with respect to h. We can therefore deduce the theorem from Theorem 1.2 and the following algebraic lemma. □

Lemma 4.4. *Let K be a tensor of type* (0, 4) *over an n-dimensional vector space provided with an inner product h. Suppose that $n \geq 4$ and K satisfies equations* (4.4)–(4.7). *Then $K = 0$ if $K(X, Y, Z, V) = 0$ for all vectors X, Y, Z and V which are mutually orthogonal with respect to h.*

Proof. We choose a basis $\{e_1, \ldots, e_n\}$ such that $h(e_i, e_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = 1$ or -1, and set $K_{ijkl} = K(e_i, e_j, e_k, e_l)$. From now on, we always assume that i, j, k and l are mutually different indices, and hence $K_{ijkl} = 0$. By (4.4), it suffices to prove that the following five types of $K_{....}$'s are zero:

$$K_{ijki}, \quad K_{ijil}, \quad K_{ijkk}, \quad K_{ijii}, \quad K_{ijji}. \tag{4.8}$$

By assumption, we obtain

$$0 = K(e_i + e_l, e_j, e_k, \varepsilon_i e_i - \varepsilon_l e_l) = \varepsilon_i K_{ijki} - \varepsilon_l K_{ljkl}.$$

Then $\varepsilon_i K_{ijki} = \alpha_{jk}$, where α_{jk} is a constant independent of i. By equation (4.5), we have

$$\varepsilon_k K_{ijkk} = -\varepsilon_k K_{jkik} - \varepsilon_k K_{kijk} = \alpha_{ji} - \alpha_{ij}. \tag{4.9}$$

In the same way, we know that $\varepsilon_i K_{ijil} = \beta_{jl}$ for some constant β_{jl} independent of i. Hence (4.7) implies that

$$0 = \sum_{m=1}^{n} \varepsilon_m K_{mjkm} = (n-2)\alpha_{jk} + \varepsilon_k K_{kjkk},$$

$$0 = \sum_{m=1}^{n} \varepsilon_m K_{mjmk} = (n-2)\beta_{jk} + \varepsilon_k K_{kjkk}.$$

Consequently, we obtain that

$$\varepsilon_k K_{kjkk} = -(n-2)\alpha_{jk} = -(n-2)\beta_{jk}. \tag{4.10}$$

Since four vectors $e_i + \nu e_k$, e_j, $\varepsilon_i e_i - \nu\varepsilon_k e_k + e_l$, $\varepsilon_i e_i - \nu\varepsilon_k e_k - (\varepsilon_i + \varepsilon_k)\varepsilon_l e_l$, where $\nu = \pm 1$, are mutually orthogonal, we obtain

$$0 = K(e_i + \nu e_k, e_j, \varepsilon_i e_i - \nu\varepsilon_k e_k + e_l, \varepsilon_i e_i - \nu\varepsilon_k e_k - (\varepsilon_i + \varepsilon_k)\varepsilon_l e_l)$$

by using (4.9) and (4.10)

$$= \varepsilon_i\{-(n+1)\alpha_{ji} + \alpha_{ij}\} + \nu\varepsilon_k\{-(n+1)\alpha_{jk} + \alpha_{kj}\}.$$

This implies that $\alpha_{jk} = 0$. Again using (4.9) and (4.10), we find that the $K_{....}$'s in (4.8) are zero except for the last one.

Finally, we shall show that $K_{ijji} = 0$. Since four vectors $e_i + e_l$, $e_j + e_k$, $\varepsilon_j e_j - \varepsilon_k e_k$, $\varepsilon_i e_i - \varepsilon_l e_l$ are mutually orthogonal,

$$0 = K(e_i + e_l, e_j + e_k, \varepsilon_j e_j - \varepsilon_k e_k, \varepsilon_i e_i - \varepsilon_l e_l)$$
$$= \varepsilon_i\varepsilon_j K_{ijji} - \varepsilon_i\varepsilon_k K_{ikki} - \varepsilon_j\varepsilon_l K_{ljjl} + \varepsilon_k\varepsilon_l K_{lkkl}.$$

Hence, we have

$$0 = \sum_{k\neq i,j}\sum_{l\neq i,j,k}\{\varepsilon_i\varepsilon_j K_{ijji} - \varepsilon_i\varepsilon_k K_{ikki} - \varepsilon_j\varepsilon_l K_{ljjl} + \varepsilon_k\varepsilon_l K_{lkkl}\}$$
$$= \varepsilon_i\varepsilon_j\{(n(n-3)+1)K_{ijji} + K_{jiij}\}.$$

This equation holds for any mutually distinct indices i and j. Since $n(n-3)+1 > 1$, K_{ijji} must be equal to zero. □

5. Conformally-Projectively Flat Statistical Submanifolds

In this section, we shall study conformally-projectively flat statistical hypersurfaces of a statistical manifold. The main aim of this section is to prove the following 'embedding theorem' for conformally-projectively flat statistical manifolds:

Theorem 5.1. *Let (M, ∇, h) be a simply connected, connected, conformally-projectively flat statistical manifold of dimension $n \geq 2$. Then, there exists a* 1*-conformally flat statistical manifold into which (M, ∇, h) can be embedded as a normally umbilical statistical hypersurface.*

In [7], Uohashi, Ohara and Fujii showed that a given 1-conformally flat statistical manifold can be embedded into a suitable flat statistical manifold as a statistical hypersurface. Hence we have:

Corollary 5.2. *Let (M, ∇, h) be as in Theorem* 5.1. *Then there exists a flat statistical manifold into which (M, ∇, h) can be embedded as a statistical submanifold of codimension two.*

Proof of Theorem 5.1. Let (M, ∇, h) be a simply connected, connected, conformally-projectively flat statistical manifold of dimension $n \geq 2$. By Fact 2.7, we have an equi-centroaffine immersion (x, ξ) of codimension two which realizes (M, ∇, h). We choose ρ and S satisfying equation (2.14).

We define a map $\hat{x} : M \times \mathrm{R} \to \mathrm{R}^{n+2}$ and a vector field $\hat{\xi}$ of R^{n+2} along $\hat{x}$ by

$$\hat{x}(p, t) = (1+t)x(p) + \frac{1}{2}t^2\xi \qquad \text{and} \qquad \hat{\xi}(p, t) = \xi(p),$$

where $(p, t) \in M \times \mathrm{R}$. Identifying $M \times \{0\}$ with M and $T_{(p,t)}(M \times \mathrm{R})$ with $T_p(M) \oplus T_t(\mathrm{R})$, we have

$$\hat{x}_* X = \frac{1}{2}t^2\rho(X)x + x_*\left\{(1+t)X - \frac{1}{2}t^2 S(X)\right\},$$
$$\hat{x}_*\frac{\partial}{\partial t} = x + t\xi.$$

Thus, we obtain

$$\hat{x}_* X = x_* X \qquad \text{and} \qquad \hat{x}_*\frac{\partial}{\partial t} = x \qquad \text{at } (p, 0). \tag{5.1}$$

Since x is an immersion and since $\mathrm{R}x(p) \oplus x_* T_p(M) \oplus \mathrm{R}\xi(p) = T_p(\mathrm{R}^{n+2})$, equation (5.1) implies that there exists an open neighobourhood $\hat{M}$ of M in $M \times \mathrm{R}$ such that $\hat{x}$ is an immersion of $\hat{M}$ into R^{n+2} and

$$\hat{x}^* T(\mathrm{R}^{n+2}) = \hat{x}_* T(\hat{M}) \oplus \mathrm{R}\hat{\xi}.$$

According to this decomposition, we set

$$(\hat{x}^* D)_{\hat{X}}(\hat{x}_*\hat{Y}) = \hat{x}_*(\hat{\nabla}_{\hat{X}}\hat{Y}) + \hat{h}(\hat{X}, \hat{Y})\hat{\xi},$$

where $\hat{X}$ and $\hat{Y}$ are arbitrary vector fields on $\hat{M}$. In a way similar to obtaining (5.1), we have

$$\hat{\nabla}_X Y = \nabla_X Y + T(X, Y)\frac{\partial}{\partial t}, \quad \hat{\nabla}_X\frac{\partial}{\partial t} = X \qquad \text{along } M,$$

and

$$\hat{h}(X, Y) = h(X, Y), \quad \hat{h}\left(X, \frac{\partial}{\partial t}\right) = 0, \quad \hat{h}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 1 \qquad \text{along } M.$$

Hence we see that $\hat{h}$ is a Riemannian metric on $\hat{M}$, $\partial/\partial t$ is the unit normal vector field along M, and (M, ∇, h) is a statistical hypersurface of $(\hat{M}, \hat{\nabla}, \hat{h})$. However, $(\hat{x}^* D)_{\hat{X}}\hat{\xi}$ may have a $\hat{\xi}$-component $\lambda(\hat{X})\hat{\xi}$. To eliminate the component, noting that λ is a 1-form on $\hat{M}$, we change $\hat{\xi}$ to

$$\hat{\hat{\xi}} = \hat{\xi} + \hat{x}_*\hat{V},$$

where $\hat{V}$ is the vector field determined by the condition: $\lambda + \hat{h}(\cdot\,, \hat{V}) = 0$. Then

$$\begin{aligned}(\hat{x}^*D)_{\hat{X}}\hat{\hat{\xi}} &= (\text{the } \hat{x}_*T(\hat{M})\text{-component of } (\hat{x}^*D)_{\hat{X}}\hat{\xi}) + \lambda(\hat{X})\hat{\xi} + \hat{x}_*(\hat{\nabla}_{\hat{X}}\hat{V}) + \hat{h}(\hat{X}, \hat{V})\hat{\xi} \\ &= (\text{the } \hat{x}_*T(\hat{M})\text{-component of } (\hat{x}^*D)_{\hat{X}}\hat{\xi}) + \hat{x}_*(\hat{\nabla}_{\hat{X}}\hat{V}).\end{aligned}$$

Therefore $(\hat{x}^*D)_{\hat{X}}\hat{\hat{\xi}}$ has no $\hat{\hat{\xi}}$-component, and $(\hat{x}, \hat{\hat{\xi}})$ realizes a 1-conformally flat statistical manifold. Moreover, the modification leaves $\hat{h}$ invariant and changes $\hat{\nabla}$ to

$$\hat{\hat{\nabla}}_{\hat{X}}\hat{Y} = \hat{\nabla}_{\hat{X}}\hat{Y} - \hat{h}(\hat{X}, \hat{Y})\hat{V}.$$

Since

$$\begin{aligned}(\hat{x}^*D)_X\hat{\xi} &= (x^*D)_X\xi \qquad \text{at } (p, 0) \\ &= \rho(X)x - x_*S(X) = \hat{x}_*\left(\rho(X)\frac{\partial}{\partial t} - S(X)\right), \\ (\hat{x}^*D)_{\frac{\partial}{\partial t}}\hat{\xi} &= 0,\end{aligned}$$

we have $\lambda = 0$ at $(p, 0)$. This implies that $\hat{V} = 0$ along M. Thus we obtain

$$\hat{\hat{\nabla}}_{\hat{X}}\hat{Y} = \hat{\nabla}_{\hat{X}}\hat{Y} \qquad \text{along } M.$$

As a consequence, the 1-conformally flat statistical manifold $(\hat{M}, \hat{\hat{\nabla}}, \hat{h})$ induces ∇ and h on M, as $(\hat{M}, \hat{\nabla}, \hat{h})$ does.

Finally, since we have

$$\hat{h}\left(\hat{\hat{\nabla}}_X\frac{\partial}{\partial t}, Y\right) = \hat{h}\left(\hat{\nabla}_X\frac{\partial}{\partial t}, Y\right) = h(X, Y) \qquad \text{along } M,$$

the hypersurface M of $\hat{M}$ is normally umbilical. □

Theorem 5.3. *Suppose that a conformally-projectively flat statistical manifold (M, ∇, h) of dimension $n \geq 3$ is realized by an equi-centroaffine immersion (x, ξ). Then, a connected hypersurface N of (M, ∇, h) is tangentially umbilical if and only if there exists a codimension-one subspace V of R^{n+2} such that $x(N) \subset V$.*

Proof. Let v be a constant covector of R_{n+2}. For vector fields X and Y on N, we have

$$(5.2) \qquad X\langle v, x\rangle = \langle v, x_*X\rangle,$$

$$\begin{aligned}(5.3) \qquad X\langle v, x_*Y\rangle &= \langle v, (x^*D)_X(x_*Y)\rangle \\ &= \langle v, T(X, Y)x + x_*(\nabla'_XY + \alpha(X, Y)\nu) + h(X, Y)\xi\rangle \\ &= T(X, Y)\langle v, x\rangle + \langle v, x_*\nabla'_XY\rangle + \alpha(X, Y)\langle v, x_*\nu\rangle + h'(X, Y)\langle v, \xi\rangle.\end{aligned}$$

If $x(N)$ is included in a subspace determined by $\langle v, \cdot\rangle = 0$, we obtain

$$\begin{aligned}0 &= X\langle v, x\rangle = \langle v, x_*X\rangle, \\ 0 &= X\langle v, x_*Y\rangle = \alpha(X, Y)\langle v, x_*\nu\rangle + h'(X, Y)\langle v, \xi\rangle.\end{aligned}$$

Thus we have

$$\alpha(X, Y) = -\frac{\langle v, \xi\rangle}{\langle v, x_*\nu\rangle}h'(X, Y).$$

This implies that the hypersurface N is tangentially umbilical.

Conversely, we assume that N is tangentially umbilical with $\alpha = fh'$. By (5.3), we see that

$$(5.4) \qquad X\langle v, x_*Y\rangle = T(X, Y)\langle v, x\rangle + \langle v, x_*\nabla'_XY\rangle + h'(X, Y)\langle v, fx_*\nu + \xi\rangle.$$

By using (2.14), a direct calculation shows that

$$\begin{aligned}(5.5) \qquad (x^*D)_X(fx_*\nu + \xi) &= df(X)x_*\nu + f\{T(X, \nu)x + x_*(-\beta^\#(X) + \tau(X)\nu) + h(X, \nu)\xi\} + \rho(X)x \\ &\quad - x_*S(X) \\ &= \{df(X) + f\tau(X) - h(S(X), \nu)\}x_*\nu + \{fT(X, \nu) + \rho(X)\}x \\ &\quad - x_*(f\beta^\#(X) + S^\top(X)),\end{aligned}$$

where $S^\top(X) = S(X) - h(S(X), \nu)\nu$, the $T(N)$-component of $S(X)$.

We shall show that the first term on the right-hand side is identially equal to zero. Let R be the curvature tensor of ∇. By using equations (2.3) and (2.4), we have

$$\begin{aligned}h(R(X, Y)Z, \nu) &= (\nabla'\alpha)(Y, Z; X) + \alpha(Y, Z)\tau(X) - (\nabla'\alpha)(X, Z; Y) - \alpha(X, Z)\tau(Y) \\ &= \{df(X) + f\tau(X)\}h'(Y, Z) - \{df(Y) + f\tau(Y)\}h'(X, Z).\end{aligned}$$

On the other hand, it follows from equation (2.9) that

$$h(R(X, Y)Z, v) = h'(Y, Z)h(S(X), v) - h'(X, Z)h(S(Y), v).$$

Comparing these two equations, we obtain

$$df(X) + f\tau(X) = h(S(X), v) \tag{5.6}$$

since $\dim N = n - 1 \geq 2$. Equations (5.5) and (5.6) imply

$$X\langle v, fx_*v + \xi\rangle = \{fT(X, v) + \rho(X)\}\langle v, x\rangle - \langle v, x_*(f\beta^\#(X) + S^\top(X))\rangle. \tag{5.7}$$

We now fix a point p of N, and choose a non-zero covector v of R_{n+2} such that $v = 0$ on the codimension-one subspace $V = \mathrm{R}x(p) \oplus x_*T_p(N) \oplus \mathrm{R}(fx_*v + \xi)_p$ of R^{n+2}. For a curve γ on N with $\gamma(0) = p$ and for a ∇'-parallel base field $(e_1, \ldots, e_{n-1})$ along γ, equations (5.2), (5.4) and (5.7) imply that

$$(\langle v, x(\gamma(t))\rangle, \langle v, x_*e_1(t)\rangle, \ldots, \langle v, x_*e_{n-1}(t)\rangle, \langle v, (fx_*v + \xi)_{\gamma(t)}\rangle) \tag{5.8}$$

satisfies a certain system of homogeneous linear ordinary differential equations with the initial condition $(0, 0, \ldots, 0, 0)$ at $t = 0$. Hence each component of the solution (5.8) is identially zero. By the assumption that N is connected, we have

$$\langle v, x(q)\rangle = 0 \quad \text{for all } q \in N,$$

and consequently, we see that $x(N) \subset V$. □

Remark 5.4. The immersion $x' = x|_N$ into V with the vector field $\xi' = fx_*v + \xi$ is an equi-centroaffine immersion that realizes the conformally-projectively flat statistical hypersurface (N, ∇', h') in V. In fact, calculating directly, we obtain

$$(x'^*D)_X(x'_*Y) = T(X, Y)x' + x'_*(\nabla'_XY) + h'(X, Y)\xi', \tag{5.9}$$

and

$$(x'^*D)_X\xi' = \{\rho(X) + fT(X, v)\}x' - x'_*(S^\top(X) + f\beta^\#(X)). \tag{5.10}$$

Corollary 5.5. *Let (M, ∇, h) be a simply connected, connected, conformally-projectively flat statistical manifold of dimension $n \geq 3$ with geometric divergence ρ_M, and let N be a connected, tangentially umbilical hypersurface of* (M, ∇, h). Then the geometric divergence ρ_N of the statistical hypersurface (N, ∇', h') *coincides with $\rho_M|_{N\times N}$.*

Proof. By Fact 2.7, (M, ∇, h) can be realized by an equi-centroaffine immersion (x, ξ). From the proof of Theorem 5.3 and Remark 5.4, we see that (N, ∇', h') is realized by the equi-centroaffine immersion (x', ξ') into $V = \{\langle v, \cdot\rangle = 0\}$. We denote by $(\bar{x}, \bar{\xi})$ the dual immersion of (x, ξ), and set

$$\bar{x}'(p) = \pi(\bar{x}(p)) \quad \text{and} \quad \bar{\xi}'(p) = \pi(\bar{\xi}(p)) \quad \text{for} \quad p \in N,$$

where $\pi : \mathrm{R}^{n+2} \to \mathrm{R}^{n+2}/\mathrm{R}v$ is the natural projection. Then it is easily verified that $(\bar{x}', \bar{\xi}')$ is the dual immersion of (x', ξ') by identifying the dual space of V with $\mathrm{R}^{n+2}/\mathrm{R}v$ in the canonical way. Hence we have

$$\rho_N(p, q) = \langle \bar{x}'(q), x'(p)\rangle = \langle \bar{x}(q), x(p)\rangle = \rho_M(p, q) \quad \text{for all } p, q \in N.$$

This completes the proof. □

Finally, we shall show another corollary of Theorem 5.3, which gives a characterization of normally umbilical hypersurfaces of a conformally-projectively flat statistical manifold.

Corollary 5.6. *Suppose that a conformally-projectively flat statistical manifold (M, ∇, h) of dimension $n \geq 3$ is realized by an equi-centroaffine immersion (x, ξ). Then, a connected hypersurface N of (M, ∇, h) is normally umbilical if and only if there exists a non-zero constant vector $v \in \mathrm{R}^{n+2}$ such that $v \in \mathrm{R}x(p) \oplus x_*T_p(M)$ for all $p \in N$.*

Proof. Let $(\bar{x}, \bar{\xi})$ be the dual immersion of (x, ξ). By the definition of $\bar{x}$, $v \in \mathrm{R}x(p) \oplus x_*T_p(M)$ for a point $p \in M$ if and only if $\langle \bar{x}(p), v\rangle = 0$. Hence there exists a non-zero constant vector $v \in \mathrm{R}^{n+2}$ such that $v \in \mathrm{R}x(p) \oplus x_*T_p(M)$ for all $p \in N$ if and only if N is a tangentially umbilical hypersurface of the dual statistical manifold $(M, \bar{\nabla}, h)$, or equivalently, N is a normally umbilical hypersurface of (M, ∇, h). □

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