Interdisciplinary Information Sciences, Vol. 4, No. 2, pp. 125–127 (1998)

Open Problems in Affine Differential Geometry and Related Topics

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Received July 7, 1997; final version accepted March 23, 1998

The following problems were raised in the workshop "Affine Differential Geometry and Related Topics" at Graduate School of Information Sciences, Tohoku University at December 16–18, 1996.

1. Shun-ichi AMARI (RIKEN, amari@zoo.riken.go.jp)

(a) Consider a quadruplet $\{M, g, \nabla, \nabla^*\}$, where (M, g) is an n-dimensional Riemannian manifold, ∇ and ∇^* are affine connections and mutually dual with respect to g. A quadruplet $\{M, g, \nabla, \nabla^*\}$ is said to be dually flat space if ∇ and ∇^* are flat affine connections (cf. [4]). What is the condition $\{M, g, \nabla, \nabla^*\}$ can be realized by an n-dimensional submanifold in m-dimensional $(m > n)$ dually flat manifold $\{\tilde{M}, \tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*\}$? If it is not true, what quantity should we impose in order to realize the manifold in a finite dimensional manifold?[1] (See [3].)

(b) The set of all smooth probability densities $\{p(x) > 0\}$ on S^1 is diffeomorphic to the manifold of smooth (equiaffine) transformations on S^1 (Friedrich, cf. [6]). How about S^n case? How about $\mathbf{R}^n$ case?

(c) For a given Riemannian manifold (M, g), can we always make the quadruplet $\{M, g, \nabla, \nabla^*\}$ a dually flat space by introducing a symmetric (0, 3)-tensor T? (See [2], [4] and [10].) If the procedure is not unique, what kind of character does the class of those spaces have?

(d) Information geometry is related to the so called large deviation principles whose rate functions are relative entropies (cf. [5]). Investigate the relation between the large deviation principle and the dual connection structure.

(e) The space spanned by probability distributions $\{p(x, \xi)\}$ becomes a statistical manifold (cf. [2] and [13]). Conversely, if a statistical manifold is given, what is the condition that there exists the corresponding probability distributions which coincides the given statistical manifold? How about the dually flat manifolds case?

2. Yoe ITOKAWA (Fukuoka Institute of Technology, itokawa@dontaku.fit.ac.jp)

(a) (D. Gromoll) Determine the fundamental group of a compact affine flat manifold (cf. [12]).

3. Takashi KUROSE (Fukuoka University, sm036447@ssat.fukuoka-u.ac.jp)

(a) Let M, h and ∇ be a manifold, a Riemannian metric and its Levi-Civita connection, respectively. It is known that if the statistical manifold (M, ∇, h) is conformally-projectively flat in Matsuzoe's sense [11], the Riemannian manifold (M, h) is conformally flat. Then investigate the relation between geometry of a conformally flat Riemannian manifold and Matsuzoe's geometric divergence.

(b) We can easily prove that there exist the isothermal coordinates on a minimal surface in Euclidean 3-space (cf. [15]). How about the isothermal coordinates with respect to the affine fundamental form of an affine minimal surface?

(c) Let (M, ∇, h) be a statistical manifold whose curvature tensor is R. If we define a (0, 4)-tensor K by $K(X, Y, Z, W) := h(R(X, Y)Z, W)$, then it satisfies

$$K(X, Y, Z, W) = -K(Y, X, Z, W),$$
$$K(X, Y, Z, W) + K(Y, Z, X, W) + K(Z, X, Y, W) = 0,$$
$$K(X, Y, Z, W) + K(Y, W, Z, X) + K(W, X, Z, Y) = 0.$$

[1] N. Abe [1] showed that if $\{M, g, \nabla, \nabla^*\}$ is a statistical manifold, then it can be realized by a submanifold in the affine space.

Investigate the structure of the set of all (0, 4)-tensors satisfying the above conditions.

(d) For a Riemannian manifold (M, h), the tangent bundle $T(M)$ has a canonical symplectic form ω, which is induced from $T^*(M)$ by h. On the other hand, a torsion-free affine connection ∇ on M determines an almost complex structure J on $T(M)$. Thus determined ω and J are compatible if and only if h and ∇ are compatible. Then, characterize symplectic manifolds with compatible almost complex structure which are locally obtained in this way.

(e) We take an arbitrary point p of an n-dimensional statistical manifold (M, ∇, h) and define a distribution D of rank $(n-1)$ on a ∇-convex neighbourhood U of p as follows: for a point q of $U\backslash\{p\}$, D_q is the h-orthogonal complement of the velocity vector at q of ∇-geodesic joining p and q in U. Is it true that the dual statistical manifold $(M, \bar{\nabla}, h)$ is 1-conformally flat if every distribution D defined in this way is integrable? Note that the converse is true since the level hypersurfaces of $\rho(\cdot, p)$, where ρ is the geometric divergence of a 1-conformally flat statistical manifold, are the integral manifold of D (cf. [9]).

4. Takeshi SASAKI (Kobe University, sasaki@math.kobe-u.ac.jp)

(a) Give estimate of the minimum number of inflection points of a closed curve on a Riemann surface with a projectively flat connection.

(b) Under the same situation, estimate the number of affine vertices. Assuming that the inflection points are isolated, investigate the relation between the number of affine vertices and that of inflection points (cf. [16]).

(c) Prove the existence of a time-global solution to the following equation of motion of curves on the affine plane:

$$\frac{\partial x}{\partial u} = (1 + k_p)x'',$$

where u is a time-parameter, x'' is the affine normal, k is the affine curvature and p is the affine support function of x (cf. [18]).

5. Satoru SHIMIZU (Tohoku University, shimizu@math.tohoku.ac.jp)

(a) Let M be a Hessian manifold with Hessian metric g. Suppose that (M, g) is complete. Is the tangent bundle over M, equipped with a natural complex structure, a Stein manifold? A case of particular interest is when Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines, and Γ is a discrete subgroup of the affine automorphism group of Ω acting freely and properly discontinuously on Ω. Is the tangent bundle over the quotient space Ω/Γ a Stein manifold?

(b) Study the stability of Hessian metrics. More precisely, let M be a compact Hessian manifold and consider a deformation of M in the sense of the deformation of affine structures. Does every small deformation of M admit a Hessian metric? When M is hyperbolic, that is, the universal covering of M is affinely equivalent to a convex domain in $\mathbf{R}^n$ containing no complete straight lines, an affirmative answer to this question has been given by Koszul [7]. On the other hand, as a related result in complex geometry, we have the stability of Kähler metrics, which motivates our question.

(c) Construct examples of compact Hessian manifolds other than direct products of hyperbolic compact Hessian manifolds and flat compact Hessian manifolds. As one of them, we have a compact Hessian manifold admitting a non-trivial fiber bundle structure whose base space is a hyperbolic compact Hessian manifold and whose fiber is a flat compact Hessian manifold, which is constructed in a similar method to that of the construction of Kuga's fiber varieties in the theory of automorphic functions (cf. [8]). Can one construct another example with no fiber bundle structure by using the deformation theory of affine structures?

6. Kwoichi TANDAI

(a) Usually, probability density functions are written as $f(x; \theta)$, where x is a point in the sample space and θ is a parameter. Amari and others introduced and studied a natural Hessian metric on the space of parameters (cf. [4]). The question is to find and study a good geometry on the product space of the sample space and the parameter space. Of course, it might be dependent on the choice of a statistically meaningful θ-coordinate system.

Acknowledgement

We express our gratitude to all the contributors to the workshop, especially Professors S. Amari, Y. Itokawa, T. Kurose, T. Sasaki, S. Shimizu and K. Tandai who raised many interesting open problems in this field.

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