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An Explicit Formula of the Newman-Coquet Exponential Sum

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In this paper, we first give an explicit formula of the exponential sum of sum of digits with complex coefficients. As an application of this formula, we obtain a simple expression of Newman-Coquet summation formula related to the number of binary digits in a multiple of three.

KEYWORDS: digital sum problem, singular function, multinomial measure, difference equation

1. Introduction

Let p be a positive integer greater than 1 and denote the p-adic expansion of $n \in \boldsymbol{N}$ by $n = \sum_{i \geq 0} \alpha_i(n) p^i$, where $\alpha_i(n) \in \{0, 1, \ldots, p-1\}$. We set

$$s(p, n, l) = \sum_{i \geq 0} \mathbf{1}_{\{j \mid \alpha_j(n) = l\}}(i)$$

for $l = 1, 2, \ldots, p-1$ and

$$\boldsymbol{s}(n)_{(p)} = (s(p, n, 1), s(p, n, 2), \ldots, s(p, n, p-1)).$$

We define the exponential sum:

$$F(\boldsymbol{\xi}, N)_{(p)} = \sum_{n=0}^{N-1} e^{(\boldsymbol{\xi}, \boldsymbol{s}(n)_{(p)})} = \sum_{n=0}^{N-1} e^{\sum_{l=1}^{p-1} \xi_l s(p,n,l)}$$

for $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{p-1}) \in \boldsymbol{C}^{p-1}$ and $N \in \boldsymbol{N}$. In previous paper [7], we gave an explicit formula of $F(\boldsymbol{\xi}, N)_{(p)}$ for $\boldsymbol{\xi} \in \boldsymbol{R}^{p-1}$ by using the distribution function of the multinomial measure.

For the case $p = 2$, set $s(n) = s(2, n, 1)$ and let

$$D(N) = \sum_{n=0}^{N-1} (-1)^{s(3n)}.$$

In [5], Newman noticed that an examination of the multiples of three, $3, 6, 9, 12, 15, 18, 21, 24, 27, \cdots$ written to the two base

$$11, 110, 1001, 1100, 1111, 10010, 10101, 11000, 11011, \cdots$$

shows a definite preponderance of those containing an even number of one digits over those containing an odd number. Newman obtained following inequalities:

$$\frac{3^\alpha}{20} < D(N) N^{-\alpha} < 5.3^\alpha,$$

where $\alpha = \dfrac{\log 3}{\log 4}$. This estimate shows that this strange behaviour persists forever. More precisely, Coquet [1] obtained the following theorem.

Theorem (Coquet [1]).

1. $D(N) = \dfrac{\eta(N)}{3} + N^\alpha F\left(\dfrac{\log N}{\log 4}\right)$

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where $\eta(N) = \begin{cases} 0 & \text{if } N \text{ is even} \\ (-1)^{s(3N-1)} & \text{if } N \text{ is odd} \end{cases}$ and F is a continuous nowhere differentiable function of period 1.

2. $\limsup_{N\to\infty} D(N)N^{-\alpha} = \sup F = \dfrac{55}{3}\left(\dfrac{3}{65}\right)^{\alpha} = 1.601958421\ldots$

$\liminf_{N\to\infty} D(N)N^{-\alpha} = \inf F = \dfrac{2\sqrt{3}}{3} = 1.154700538\ldots$

In this paper, we shall generalize the distribution function $L(\boldsymbol{r},\cdot)$ of the multinomial measure with complex parameters, and give a simple explicit formula of $F(\boldsymbol{\xi},N)_{(p)}$ via a generalized $L(\boldsymbol{r},\cdot)$. Next, as an application of this one, we shall give an explicit expression of $D(N)$.

2. Preliminaries

Assume that we are given a positive integer p greater than 1. Let $I = I_{0,0} = [0,1]$ and

$$I_{n,j} = [j/p^n, (j+1)/p^n), \quad j = 0,1,\ldots,p^n-2,$$
$$I_{n,p^n-1} = [(p^n-1)/p^n, 1]$$

for $n = 1,2,3,\ldots$ Let $\boldsymbol{r} = (r_0, r_1, \ldots, r_{p-2})$ be a vector such that $0 < r_l < 1$ for $l = 0,1,\ldots,p-2$ and $\sum_{l=0}^{p-2} r_l < 1$ and set $r_{p-1} = 1 - \sum_{l=0}^{p-2} r_l$. The probability measure $\mu_{\boldsymbol{r}}$ on I defined by

$$\mu_{\boldsymbol{r}}(I_{n+1,pj+l}) = r_l \mu_{\boldsymbol{r}}(I_{n,j}) \tag{1}$$

for $n = 0,1,2,\ldots, j = 0,1,\ldots,p^n-1, l = 0,1,\ldots,p-1$, is said to be a multinomial measure. If $r_l = 1/p$ for all l, $\mu_{\boldsymbol{r}}$ is the Lebesgue measure on $[0,1]$ and otherwise it is singular with respect to the Lebesgue measure. We denote the distribution function of $\mu_{\boldsymbol{r}}$ by $L(\boldsymbol{r},\cdot)$:

$$L(\boldsymbol{r},x) = \mu_{\boldsymbol{r}}([0,x]), \; x \in I.$$

Following our previous paper [9], we summarize some fundamental properties of $L(\boldsymbol{r},\cdot)$.

Lemma 2.1 ([9, Lemma 2.1.]) *For a given $\boldsymbol{r} = (r_0, r_1, \ldots, r_{p-2})$ and r_{p-1} as above, $L(\mathbf{r},\cdot)$ satisfies the following system of infinitely many difference equations:*

$$\begin{cases} f\left(\dfrac{pj+k+1}{p^{n+1}}\right) - f\left(\dfrac{pj+k}{p^{n+1}}\right) - r_k\left\{f\left(\dfrac{j+1}{p^n}\right) - f\left(\dfrac{j}{p^n}\right)\right\} = 0, \\ f(0) = 0, \; f(1) = 1, \\ n = 0,1,2,\ldots, \; j = 0,1,\ldots,p^n-1, \\ k = 0,1,\ldots,p-1. \end{cases} \tag{2}$$

Lemma 2.2 ([9, Lemma 2.2.]) *Provided that f is continuous, the system* (2) *is equivalent to the following functional equations:*

$$f(x) = \begin{cases} r_0 f(px), & 0 \le x \le \dfrac{1}{p}, \\ r_1 f(px-1) + r_0, & \dfrac{1}{p} \le x \le \dfrac{2}{p}, \\ \cdots & \\ r_k f(px-k) + \displaystyle\sum_{i=0}^{k-1} r_i, & \dfrac{k}{p} \le x \le \dfrac{k+1}{p}, \\ \cdots & \\ r_{p-1} f(px-(p-1)) + \displaystyle\sum_{i=0}^{p-2} r_i, & \dfrac{p-1}{p} \le x \le 1. \end{cases} \tag{3}$$

Moreover, by Hata [2, Corollary 6.6], we have the following lemma.

Lemma 2.3 ([9, Proposition 2.1.]) *$L(\mathbf{r},\cdot)$ is a unique continuous solution of* (3)*, and hence of* (2).

3. Results

Let an integer $p \ge 2$ be given. We now consider the case $\boldsymbol{r} \in \boldsymbol{C}^{p-1}$ and treat the system of infinitely many difference equations (2) with $\boldsymbol{r} = (r_0, r_1, \ldots, r_{p-2})$ such that

$$r_l \in \boldsymbol{C},\ 0 < |r_l| < 1 \text{ for } l = 0, 1, \dots p-2,\ 0 < \left|1 - \sum_{l=0}^{p-2} r_l\right| < 1. \tag{4}$$

Let $r_{p-1} = 1 - \sum_{l=0}^{p-2} r_l$. Then we can immediately prove that Lemma 2.2 holds and the functional equation (3) has a unique continuous solution. Therefore, under the condition (4), we set $L(\boldsymbol{r}, \cdot)$ the continuous solution of (3) anew. We can also get an exact form of the k-th derivative of $L(\boldsymbol{r}, \cdot)$ with respect to the parameters r_l $(l = 0, 1, \cdots, p-2)$ in a similar manner to [9, Section 5.].

Theorem 1 *Suppose that* $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{p-1}) \in \boldsymbol{C}^{p-1}$ *satisfies the condition* $|1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}| > 1$ *and* $N \in \boldsymbol{N}$, *then we have*

$$F(\boldsymbol{\xi}, N)_{(p)} = \frac{1}{r_0^{[t]+1}} L\left(\boldsymbol{r}, \frac{1}{p^{1-\{t\}}}\right), \tag{5}$$

where

$$r_0 = \frac{1}{1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}} \quad \text{and} \quad r_l = \frac{e^{\xi_l}}{1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}}$$

for $l = 1, 2, \dots, p-1$, *and* $\{t\}$ *denotes the fractional part of* $t = \log_p N$.

Theorem 2 *For the quantum* $D(N)$ *in Introduction, we have*

$$D(N) = \frac{1}{3} \sum_{n<3N} (-1)^{s(n)} + \frac{2}{3} 3^{[\tilde{t}]+1} ReL\left(\boldsymbol{r}, \frac{1}{4^{1-\{\tilde{t}\}}}\right), \tag{6}$$

where $\boldsymbol{r} = (r_0, r_1, r_2) = \left(\frac{1}{3}, -\frac{w}{3}, -\frac{w^2}{3},\right)$, $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}$ *(i.e. a cubic root of unity) and* $\tilde{t} = \frac{\log 3N}{\log 4}$. $[\tilde{t}]$ *is the integral part of* $\tilde{t}$.

Remark 3.1 *Since the set* $\{3N/4^{[\tilde{t}]+1} : N = 0, 1, \cdots\}$ *is dense in* $[1/4, 1]$, *we obtain the next estimates:*

$$\limsup_{N\to\infty} D(N)N^{-\alpha} = \max_{1/4\le x\le 1} 2\ 3^{\alpha-1}x^{-\alpha}ReL(\boldsymbol{r}, x)$$
$$\liminf_{N\to\infty} D(N)N^{-\alpha} = \min_{1/4\le x\le 1} 2\ 3^{\alpha-1}x^{-\alpha}ReL(\boldsymbol{r}, x)$$

We illustrate the graph of $g(x) = 2\ 3^{\alpha-1}x^{-\alpha}ReL(\boldsymbol{r}, x)$ in Fig. 1.

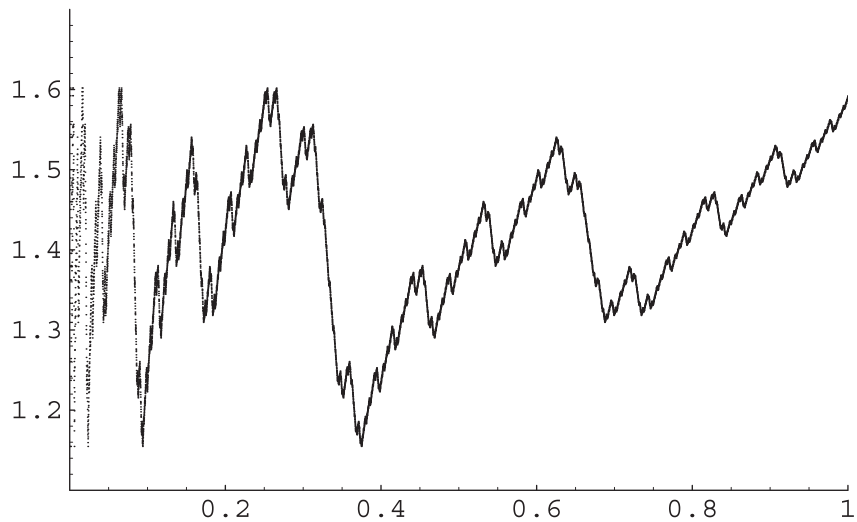

Fig. 1. $g(x)$

4. Proofs.

Proof of the Theorem 1. As $L(\boldsymbol{r}, 0) = 0$, we have

$$
\begin{aligned}
L\left(\boldsymbol{r}, \frac{1}{p^{1-\{t\}}}\right) &= L\left(\boldsymbol{r}, \frac{N}{p^{[t]+1}}\right) \\
&= \sum_{n=0}^{N-1}\left\{L\left(\boldsymbol{r}, \frac{n+1}{p^{[t]+1}}\right) - L\left(\boldsymbol{r}, \frac{n}{p^{[t]+1}}\right)\right\} \\
&= \sum_{n=0}^{N-1}\left\{L\left(\boldsymbol{r}, \frac{\sum_{i\geq 0}\alpha_i(n)p^i + 1}{p^{[t]+1}}\right) - L\left(\boldsymbol{r}, \frac{\sum_{i\geq 0}\alpha_i(n)p^i}{p^{[t]+1}}\right)\right\}.
\end{aligned}
$$

By using (2), we get

$$
\begin{aligned}
&L\left(\boldsymbol{r}, \frac{\sum_{i\geq 0}\alpha_i(n)p^i + 1}{p^{[t]+1}}\right) - L\left(\boldsymbol{r}, \frac{\sum_{i\geq 0}\alpha_i(n)p^i}{p^{[t]+1}}\right) \\
&= L\left(\boldsymbol{r}, \frac{\left(\sum_{i\geq 1}\alpha_i(n)p^{i-1}\right)p + \alpha_0(n) + 1}{p^{[t]+1}}\right) - L\left(\boldsymbol{r}, \frac{\left(\sum_{i\geq 1}\alpha_i(n)p^{i-1}\right)p + \alpha_0(n)}{p^{[t]+1}}\right) \\
&= r_{\alpha_0(n)}\left\{L\left(\boldsymbol{r}, \frac{\left(\sum_{i\geq 2}\alpha_i(n)p^{i-2}\right)p + \alpha_1(n) + 1}{p^{[t]}}\right) - L\left(\boldsymbol{r}, \frac{\left(\sum_{i\geq 2}\alpha_i(n)p^{i-2}\right)p + \alpha_1(n)}{p^{[t]}}\right)\right\} \\
&= r_{\alpha_0(n)}r_{\alpha_1(n)}\left\{L\left(\boldsymbol{r}, \frac{\left(\sum_{i\geq 3}\alpha_i(n)p^{i-3}\right)p + \alpha_2(n) + 1}{p^{[t]-1}}\right) - L\left(\boldsymbol{r}, \frac{\left(\sum_{i\geq 3}\alpha_i(n)p^{i-3}\right)p + \alpha_2(n)}{p^{[t]-1}}\right)\right\}.
\end{aligned}
$$

To iterate this procedure, we find

$$
\begin{aligned}
L\left(\boldsymbol{r}, \frac{n+1}{p^{[t]+1}}\right) - L\left(\boldsymbol{r}, \frac{n}{p^{[t]+1}}\right) &= r_{\alpha_0(n)}r_{\alpha_1(n)}\cdots r_{\alpha_{[t]}(n)} \\
&= r_0^{[t]+1-\sum_{l=1}^{p-1}s(p,n,l)}r_1^{s(p,n,1)}r_2^{s(p,n,2)}\cdots r_{p-1}^{s(p,n,p-1)}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
L\left(\boldsymbol{r}, \frac{1}{p^{1-\{t\}}}\right) &= \sum_{n=0}^{N-1}\left\{r_0^{[t]+1-\sum_{l=1}^{p-1}s(p,n,l)}r_1^{s(p,n,1)}r_2^{s(p,n,2)}\cdots r_{p-1}^{s(p,n,p-1)}\right\} \\
&= r_0^{[t]+1}\sum_{n=0}^{N-1}\left\{\left(\frac{r_1}{r_0}\right)^{s(p,n,1)}\left(\frac{r_2}{r_0}\right)^{s(p,n,2)}\cdots\left(\frac{r_{p-1}}{r_0}\right)^{s(p,n,p-1)}\right\}.
\end{aligned}
$$

Setting $r_0 = \dfrac{1}{1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}}$ and $r_l = \dfrac{e^{\xi_l}}{1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}}$ for $l = 1, 2, \ldots, p-1$, we obtain the Theorem 1. □

Proof of the Theorem 2. As

$$
\sum_{n<3N}(-1)^{s(n)}w^n = \sum_{n<N}(-1)^{s(3n)} + \sum_{n<N}(-1)^{s(3n+1)}w + \sum_{n<N}(-1)^{s(3n+2)}w^2,
$$

we have

$$
\begin{aligned}
Re\sum_{n<3N}(-1)^{s(n)}w^n &= \sum_{n<N}(-1)^{s(3n)} - \frac{1}{2}\sum_{n<N}(-1)^{s(3n+1)} - \frac{1}{2}\sum_{n<N}(-1)^{s(3n+2)} \\
&= \frac{3}{2}D(N) - \frac{1}{2}\sum_{n<N}\left\{(-1)^{s(3n)} + (-1)^{s(3n+1)} + (-1)^{s(3n+2)}\right\}.
\end{aligned}
$$

Therefore, we obtain

$$
D(N) = \frac{1}{3}\sum_{n<3N}(-1)^{s(n)} + \frac{2}{3}Re\sum_{n<3N}(-1)^{s(n)}w^n.
$$

Set the 4-adic expansion of $n \in \boldsymbol{N}$ by $n = \sum_{i\geq 0}\alpha_i(n)4^i$, where $\alpha_i(n) \in \{0, 1, 2, 3\}$. Then we have

$$
w^n = w^{\sum_{i\geq 0}\alpha_i(n)(3+1)^i} = w^{\sum_i \alpha_i(n)} = w^{s(4,n,1)+2s(4,n,2)+3s(4,n,3)}.
$$

On the other hand, as

$$s(n) = s(4,n,1) + s(4,n,2) + 2s(4,n,3),$$

we get

$$\begin{aligned}(-1)^{s(n)} w^n &= (-1)^{s(4,n,1)+s(4,n,2)+2s(4,n,3)} w^{s(4,n,1)+2s(4,n,2)+3s(4,n,3)} \\ &= (-w)^{s(4,n,1)} (-w^2)^{s(4,n,2)}.\end{aligned}$$

Therefore, we have

$$\sum_{n=0}^{3N-1} (-1)^{s(n)} w^n = \sum_{n=0}^{3N-1} e^{\sum_{l=1}^{3} \xi_l s(4,n,l)} = F(\boldsymbol{\xi}, 3N)_{(4)}$$

where $\xi_1 = \log(-w)$, $\xi_2 = \log(-w^2)$ and $\xi_3 = 0$. Since

$$r_0 = \frac{1}{1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}} = \frac{1}{3},\ r_1 = \frac{e^{\xi_1}}{1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}} = -\frac{w}{3},$$
$$r_2 = \frac{e^{\xi_2}}{1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}} = -\frac{w^2}{3},\ r_3 = \frac{e^{\xi_3}}{1+e^{\xi_1}+e^{\xi_2}+e^{\xi_3}} = \frac{1}{3},$$

we obtain our formula by Theorem 1. □

Remark 4.1 *For $z \in \boldsymbol{C}$, let $\overline{z}$ be the conjugate complex. Then by Lemma* 2.2*, we know the following functional equations hold:*

(7)
$$L(\boldsymbol{r},x) = \begin{cases} \frac{1}{3} L(\boldsymbol{r},4x) & \text{if } 0 \le x \le \frac{1}{4}, \\ -\frac{w}{3} L(\boldsymbol{r},4x-1) + \frac{1}{3} & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ -\frac{w^2}{3} L(\boldsymbol{r},4x-2) + \frac{1}{3} - \frac{w}{3} & \text{if } \frac{1}{2} \le x \le \frac{3}{4}, \\ \frac{1}{3} L(\boldsymbol{r},4x-3) + \frac{1}{3} - \frac{w}{3} - \frac{w^2}{3} & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$

for $\boldsymbol{r} = (\frac{1}{3}, -\frac{w}{3}, -\frac{w^2}{3})$, and

(8)
$$L(\boldsymbol{r}',x) = \begin{cases} \frac{1}{3} L(\boldsymbol{r}',4x) & \text{if } 0 \le x \le \frac{1}{4}, \\ -\frac{\overline{w}}{3} L(\boldsymbol{r}',4x-1) + \frac{1}{3} & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ -\frac{\overline{w}^2}{3} L(\boldsymbol{r}',4x-2) + \frac{1}{3} - \frac{\overline{w}}{3} & \text{if } \frac{1}{2} \le x \le \frac{3}{4}, \\ \frac{1}{3} L(\boldsymbol{r}',4x-3) + \frac{1}{3} - \frac{\overline{w}}{3} - \frac{\overline{w}^2}{3} & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$

for $\boldsymbol{r}' = (\frac{1}{3}, -\frac{\overline{w}}{3}, -\frac{\overline{w}^2}{3})$. It is well-known that above each system has a unique continuous solution and the range $G_1([0,1])$ (resp. $G_2([0,1])$) is the Koch curve on the lower half-plane (resp. the upper half-one) of $\boldsymbol{C}$. Set $\beta = \frac{1-w}{3} = \frac{1}{2} - \frac{\sqrt{3}}{6}\sqrt{-1}$. Then, noticing equalities

$$\beta\overline{\beta} = \frac{1}{3},\ \beta(1-\overline{\beta}) = -\frac{w}{3},\ (1-\beta)\overline{\beta} = -\frac{w^2}{3},\ (1-\beta)(1-\overline{\beta}) = \frac{1}{3},$$

we can easily show that $L(\boldsymbol{r},x)$ and $L(\boldsymbol{r}',x)$ satisfy the following systems of functional equations:

$$L(\boldsymbol{r},x) = \begin{cases} \beta\overline{L(\boldsymbol{r},2x)} & \text{if } 0 \le x \le \frac{1}{2} \\ (1-\beta)\overline{L(\boldsymbol{r},2x-1)} + \beta & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$
$$L(\boldsymbol{r}',x) = \begin{cases} \overline{\beta}\overline{L(\boldsymbol{r}',2x)} & \text{if } 0 \le x \le \frac{1}{2} \\ (1-\overline{\beta})\overline{L(\boldsymbol{r}',2x-1)} + \overline{\beta} & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Kawamura [4] *studied general type of above functional equations and investigated those fractal properties. We finally remark that Coquet* [1] *also pointed out that the function F of Theorem* 1 *concerned with the classical fractal scheme.*

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