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Notes on Transference of Continuity from Maximal Fourier Multiplier Operators on R^n to Those on T^n

Makoto KANEKO* and Enji SATO†

**Division of Mathematical Structures, Graduate School of Information Sciences, Tohoku University, Katahira 2-1-1, Sendai 980-8577, Japan*
(e-mail, kaneko@math.is.tohoku.ac.jp)
†Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata 990-8560, Japan
(e-mail, esato@kszaoh3sci.kj.yamagata-u.ac.jp)

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Given a sequence $\{\phi_j\}$ of bounded functions on the dual group Γ of a locally compact abelian group G, we have a family of Fourier multiplier operators each element of which is made from a component ϕ_j of the given sequence. On the other hand, the restrictions $\phi_j|\Lambda$ of ϕ_j to a subgroup Λ of Γ build Fourier multiplier operators on $G/\Lambda^{\perp}$. We are interested in the transference of continuity from the maximal operator constructed by the family of Fourier multiplier operators composed of $\{\phi_j\}$ to the counterpart maximal operator corresponding to $\{\phi_j|\Lambda\}$. For the study, it is a powerful tool that, if $k \in L^1(\Gamma)$, then the maximal operator corresponding to $\{k*\phi_j\}$ inherits the strong or weak typeness (p, q) from the one associated with $\{\phi_j\}$. First we give a method of showing it. Our result contains the case $p=q=1$ and our proof is simpler and more straightforward than the one in [2]. Next we consider the case of $G=\boldsymbol{R}^n$ and $\Lambda=\boldsymbol{Z}^n$, and develop arguments over Lorentz spaces and Hardy spaces.

KEYWORDS: Fourier multiplier, maximal operator, weak type (p, q), Lorentz space, Hardy space

1 Introduction

Let G be a locally compact abelian group with dual group Γ, and m and μ be the Haar measures on G and Γ, respectively. For $\phi \in L^\infty(\Gamma)$, the corresponding Fourier multiplier transform is denoted by T_ϕ: $T_\phi f = (\phi \hat{f})^\vee$ for $f \in L^2(G)$, where $\hat{f}$ means the Fourier transform of f and $\check{g}$ does the inverse Fourier transform of g.

When a normed or quasi-normed space $X(G)$ of functions or distributions on G is given, $T_\phi f$ is defined for $f \in X(G) \cap L^2(G)$. We are interested in the continuity ot T_ϕ as a mapping from $X(G) \cap L^2(G)$ equipped with the topology induced from $X(G)$ to some normed or quasi-normed function space on G.

If ϕ is continuous at every points of a closed subgroup Λ or Γ, then the restriction $\phi|\Lambda$ induces a Fourier multiplier transform $\tilde{T}_{\phi|\Lambda}$ on $L^2(\hat{\Lambda})$, where $\hat{\Lambda}$ is the dual group of Λ and may be identified with the quotient group G/H of G by the annihilator $H = \Lambda^\perp$ of Λ in G.

Our interest is to ask if $\tilde{T}_{\phi|\Lambda}$ inherits the continuity from T_ϕ. In the following we study the problem in more general setting. Let ϕ_j $(j = 1, 2, \dots)$ be bounded on Γ and continuous at each point in Λ. We treat the maximal operator $T^* = T^*_{\{\phi_j\}} = \sup_j |T_{\phi_j}(\,\cdot\,)|$ and the counterpart maximal operator $\tilde{T}^* = \tilde{T}^*_{\{\phi_j|\Lambda\}} = \sup_j |\tilde{T}_{\phi_j|\Lambda}(\,\cdot\,)|$. And we shall show that the boundedness of T^* is passed down to $\tilde{T}^*$ in some cases.

To say precisely, we introduce the following notations. For a linear space $X(G)$ of functions or distributions on G and a function space $Y(G)$ on G, each of which is equipped with a norm or a quasi-norm $\|\cdot\|_{X(G)}$ and $\|\cdot\|_{Y(G)}$, respectively, we define $N_{X(G),Y(G)}(\{\phi_j\})$ by

$$N_{X(G),Y(G)}(\{\phi_j\}) = \sup \|T^*f\|_{Y(G)} / \|f\|_{X(G)},$$

where the supremum is taken over all non-zero $f \in X(G) \cap L^2(G)$. In the same way, we also define $N_{\tilde{X}(G/H),\tilde{Y}(G/H)}(\{\phi_j|\Lambda\})$ by

$$N_{\tilde{X}(G/H),\tilde{Y}(G/H)}(\{\phi_j|\Lambda\}) = \sup_{F \neq 0} \|\tilde{T}^*F\|_{\tilde{Y}(G/H)} / \|F\|_{\tilde{X}(G/H)}$$

for the mapping $\tilde{T}^*$ from a space $\tilde{X}(G/H)$ of functions or distributions on G/H to another similar space $\tilde{Y}(G/H)$. When the case of single multiplier is handled, in other words, when the case of $\phi_j = \phi$ $(j = 1, 2, \dots)$ with some $\phi \in L^\infty(\Gamma)$ is under consideration, we write them $N_{X(G),Y(G)}(\phi)$ and $N_{\tilde{X}(G/H),\tilde{Y}(G/H)}(\phi|\Lambda)$ instead of $N_{X(G),Y(G)}(\{\phi\})$ and $N_{\tilde{X}(G/H),\tilde{Y}(G/H)}(\{\phi|\Lambda\})$, respectively.

De Leeuw [8] gets the famous inequality

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$$N_{L^p(T^n),L^p(T^n)}(\phi\,|\,Z^n) \le N_{L^p(R^n),L^p(R^n)}(\phi)$$

in the case of $G = R^n$ and $\Lambda = Z^n$. This inequality is extended to the context of locally compact abelian groups by Saeki [20].

Coifman-Weiss [6, 7] treat this problem as an object to which so called transference method is applicable. After that their ideas have been progressed by Kenig-Tomas [17], Kaneko [16], Asmar-Berkson-Bourgain [1] and Asmar-Berkson-Gillespie [2, 3]. In these works the role of the Fourier multiplier transform $T_{k*\phi_j}$ corresponding to $k*\phi_j$ with some $k \in L^1(\Gamma)$ is important and it is tried and succeeded to have the following inequality

$$N_{L^p(G),L^p(G)}(\{k*\phi_j\}) \le C\|k\|_{L^1(\Gamma)}N_{L^p(G),L^p(G)}(\{\phi_j\})$$

or

$$N_{L^p(G),wL^p(G)}(\{k*\phi_j\}) \le C\|k\|_{L^1(\Gamma)}N_{L^p(G),wL^p(G)}(\{\phi_j\}), \tag{1}$$

where $wL^p(G)$ means the *weak* $L^p(G)$ space and C is a constant. Hereafter we use the character C to represent constants, but they may be different in their occurences. The inequality (1) induces

$$N_{L^p(G/H),wL^p(G/H)}(\{\phi_j|\Lambda\}) \le CN_{L^p(G),wL^p(G)}(\{\phi_j\}) \tag{2}$$

as well as the strong type cases.

When we try to establish the inequality (1) for $p = 1$, we face to technically difficult problems which never arise in the case of $1 < p < +\infty$. Untill Asmar-Berkson-Bourgain [1] solved the problems and obtained the inequality (1) for $p = 1$, $G = R^n$ and $\Lambda = Z^n$, it had been left as an open problem wether $N_{L^1(R^n),wL^1(R^n)}(\phi) < +\infty$ implies $N_{L^1(T^n),wL^1(T^n)}(\phi\,|\,Z^n) < +\infty$ (Pełczyński [19]). Asmar-Berkson-Bourgain's result was extended to the case of locally compact abelian groups by Asmar-Berkson-Gillespie [2]. Their method to have (1) for any locally compact abelian group G and $p = 1$ is almost as following. First they prove (1) for any compact abelian group adapting the strategy of Asmar-Berkson-Bourgain. Next they develop some argument over the relationship among the Fourier multipliers defined on the dual group Γ, its discretized group Γ_d and the quotient group Γ/Λ. And then they get finally the desired inequality for arbitrary locally compact abelian groups with the aid of the construction theory for locally compact abelian groups.

In §2, we shall give an alternative proof of (1), where we use the weak type version theorem of Marcinkiewicz-Zygmund introduced in [12] as a powerful tool.

The authors are grateful for the late Professor Akihito Uchiyama* for pointing out the applicability of the theorem to our task.

To show the heredity of continuity from T_ϕ to $\tilde{T}_{\phi|\Lambda}$, if we make our way through the transference method refered to above, then there appears a constant in the dominating side like (2). But, in the context of R^n and Z^n, Woźniakowski [22] shows

$$N_{L^1(T^n),wL^1(T^n)}(\phi\,|\,Z^n) \le N_{L^1(R^n),wL^1(R^n)}(\phi)$$

without the existence of any constant in the right-hand side. In §3, we shall improve it such as the form containing the case of maximal operators

$$N_{L^{(p,r)}(T^n),L^{(p,s)}(T^n)}(\{\phi_j|Z^n\}) \le N_{L^{(p,r)}(R^n),L^{(p,s)}(R^n)}(\{\phi_j\}),$$

where $L^{(p,r)}$ and $L^{(p,s)}$ are the Lorentz spaces on the measure spaces expressed in the each parenthesis. Our result includes the case $p = r = 1$ and $s = +\infty$. It asserts that the property of weak type (1, 1) of T^* passes to $\tilde{T}^*$. Fan [9] has already taken related theorems within the frameworks of Lorentz spaces. But his results do not contain the weak type cases.

Fan-Wu [10, 11] study the transmission of the continuity on Hardy spaces and has obtained the strong type results. We shall give a weak type version of theirs in §4.

2 Convolution theorem

In this section, we give an alternative proof of a theorem in Asmar-Berkson-Gillespie [2] which gives a weak type (1, 1)-estimate for the maximal operator defined by a sequence of the multipliers convoluted by an $L^1(\Gamma)$-function. We shall give it in a little more general form. For the sake of notational simplicity, we denote $N_{L^p(G),wL^q(G)}(\{\phi_j\})$ by $N^{(w)}_{p,q}(\{\phi_j\})$.

Theorem 2.1 *Let G be a locally compact abelian group with the dual group Γ, and assume that $1 \le p < +\infty$ and $0 < q < +\infty$. Suppose $k \in L^1(\Gamma)$ and $\phi_j \in L^\infty(\Gamma)$ $(j = 1, 2, \ldots)$. Then*

$$N^{(w)}_{p,q}(\{k*\phi_j\}) \le C_{p,q}\|k\|_{L^1(\Gamma)}N^{(w)}_{p,q}(\{\phi_j\}),$$

* The authors feel more sorry than we can say to inform that Professor Akihito Uchiyama passed away on August 11, 1997.

where $C_{p,q}$ is a positive constant depending only on p and q.

Proof. According to Fatou's lemma, it is sufficient to prove the existence of a constant $C_{p,q}$ depending only on p and q such that the distributional inequality

$$m(\{x \in G \mid (M_k^{(J)}f)(x) > t\}) \le \left(\frac{C_{p,q}\|k\|_{L^1(\Gamma)}N_{p,q}^{(w)}(\{\phi_j\})}{t} \|f\|_{L^p(G)} \right)^q \tag{3}$$

holds for all $t > 0$ and $f \in L^p(G) \cap L^2(G)$ with respect to the auxiliary maximal functions $M_k^{(J)}f$ ($J = 1, 2, \ldots$) defined by

$$(M_k^{(J)}f)(x) = \max_{1 \le j \le J} |(T_{k*\phi_j}f)(x)|.$$

We consider the case $N_{p,q}^{(w)}(\{\phi_j\}) < \infty$, and take a $t > 0$ and fix it hereafter.

We assume first that f is continuous and integrable over G, and that the support of the Fourier transform $\hat{f}$ of f is compact. Under the assumptions, we get the integrability of both $\phi_j\hat{f}$ and $(k*\phi_j)\hat{f}$. Therefore we may assume that $(M_k^{(J)}f)(x) < \infty$ for all $x \in G$. So we define $F_j(\varepsilon)$ by

$$F_j(\varepsilon) = \{x \in G \mid (M_k^{(J)}f)(x) - \varepsilon < |(T_{k*\phi_j}f)(x)|\}$$

for any $\varepsilon > 0$ and $j = 1, 2, \ldots, J$, and set

$$E_1(\varepsilon) = F_1(\varepsilon), \quad E_j(\varepsilon) = F_j(\varepsilon) \setminus \bigcup_{i=1}^{j-1} E_i(\varepsilon) \quad (j = 2, 3, \ldots, J).$$

Then $G = \bigcup_{j=1}^J E_j(\varepsilon)$ and the right-hand side is a disjoint union. By making use of the decomposition of G, we define an operator K_ε by

$$(K_\varepsilon g)(x) = \sum_{j=1}^J (T_{k*\phi_j}g)(x)\chi_{E_j(\varepsilon)}(x) \quad (g \in L^2(G)),$$

where $\chi_{E_j(\varepsilon)}$ is the characteristic function of $E_j(\varepsilon)$. Then

$$(M_k^{(J)}f)(x) - \varepsilon < |(K_\varepsilon f)(x)| \le (M_k^J f)(x) \quad (x \in G).$$

Therefore

$$(M_k^{(J)}f)(x) = \lim_{\varepsilon \to +0} |(K_\varepsilon f)(x)| \quad (x \in G). \tag{4}$$

On the other hand, we can easily observe the equality

$$(T_{k*\phi_j}f)(x) = \int_\Gamma k(\gamma)\gamma(x)T_{\phi_j}(\bar{\gamma}f)(x)d\mu(\gamma)$$

holds, because of that the integrabilities of k and $\hat{f}$ permit us to change the order of integration in the iterated integrals appearing in the process of the calculation. Now we introduce a linear operator L_ε defined by

$$(L_\varepsilon g)(x) = \sum_{j=1}^J (T_{\phi_j}g)(x)\chi_{E_j(\varepsilon)}(x) \quad (g \in L^2(G)).$$

By the inequality $|(L_\varepsilon g)(x)| \le (T^*g)(x)$ and the finiteness of $N_{p,q}^{(w)}(\{\phi_j\})$, we see that L_ε is a linear operator of weak type (p, q) and its operator norm is less than or equal to $N_{p,q}^{(w)}(\{\phi_j\})$. Furthermore the equation

$$(K_\varepsilon f)(x) = \int_\Gamma k(\gamma)\gamma(x)L_\varepsilon(\bar{\gamma}f)(x)d\mu(\gamma)$$

holds. Applying the Cauchy-Schwarz inequality to the right-hand side, then

$$|(K_\varepsilon f)(x)| \le \|k\|_1^{1/2}\left\{\int_\Gamma |L_\varepsilon(\bar{\gamma}f)(x)|^2|k(\gamma)|d\mu(\gamma)\right\}^{1/2}. \tag{5}$$

We try to replace the L^2-norm with respect to the finite measure $d\mu_k(\gamma) = |k(\gamma)|d\mu(\gamma)$ in the right-hand side of (5) with a limit of a discrete l^2-norm stated below (10). By the regularity of μ_k, we have such compact sets $\Gamma_n \subset \Gamma$ as $\mu_k(\Gamma \setminus \Gamma_n) < 1/n$ for any positive integer n. For each n we have the following. Since $\hat{f}$ is a compactly supported continuous function, there exists a symmetric neighborhood V of 0 in Γ such that

$$\int_\Gamma |\hat{f}(\gamma + \gamma_1) - \hat{f}(\gamma)|d\mu(\gamma) < 1/n \quad (\gamma_1 \in V). \tag{6}$$

For such a V, we take a symmetric neighborhood U of 0 satisfying $U + U \subset V$. Because of the compactness of Γ_n, we can choose a finite covering $\{U_s\}$ of Γ_n, each of which has such a form as $U_s = U + \xi_s$ with some $\xi_s \in \Gamma_n$.

Let $\Gamma_{n,1} = \Gamma_n \cap U_1$, $\Gamma_{n,s} = \Gamma_n \cap (U_s \backslash \cup_{i=1}^{s-1} U_i)$ $(s = 2, 3, \ldots)$. Then $\Gamma_n = \cup_s \Gamma_{n,s}$, where the right-hand side is a finite union of a disjoint family. We select a point $\gamma_{n,s} \in \Gamma_{n,s}$ from each $\Gamma_{n,s}$, and set $g_{n,s} = \sqrt{\mu_k(\Gamma_{n,s})}\bar{\gamma}_{n,s}f$. Then we have the inequality

$$(7)\qquad \left|\left\{\int_\Gamma |L_\varepsilon(\bar{\gamma}f)(x)|^2|k(\gamma)|d\mu(\gamma)\right\}^{1/2} - \{\sum_s |(L_\varepsilon g_{n,s})(x)|^2\}^{1/2}\right| \leq \left\{\int_\Gamma |L_\varepsilon(\bar{\gamma}f)(x) - \sum_s L_\varepsilon(\bar{\gamma}_{n,s}f)(x)\chi_{\Gamma_{n,s}}(\gamma)|^2 d\mu_k(\gamma)\right\}^{1/2}.$$

Let j be such an integer that $x \in E_j(\varepsilon)$, then

$$(8)\qquad |L_\varepsilon(\bar{\gamma}f)(x) - \sum_s L_\varepsilon(\bar{\gamma}_{n,s}f)(x)\chi_{\Gamma_{n,s}}(\gamma)| = |T_{\phi_j}(\bar{\gamma}f)(x)| \leq \|\phi_j\|_\infty \|\hat{f}\|_1$$

for $\gamma \in \Gamma\backslash\Gamma_n$, and, for a point γ in some $\Gamma_{n,s'}$,

$$(9)\qquad |L_\varepsilon(\bar{\gamma}f)(x) - \sum_s L_\varepsilon(\bar{\gamma}_{n,s}f)(x)\chi_{\Gamma_{n,s}}(\gamma)| \leq \|\phi_j\|_\infty \int_\Gamma |\hat{f}(\gamma_1 + \gamma) - \hat{f}(\gamma_1 + \gamma_{n,s'})|d\mu(\gamma_1) \leq \|\phi_j\|_\infty/n$$

holds by reason of (6). The relations (7), (8), (9) and the feature of Γ_n induce the estimate

$$\left|\left\{\int_\Gamma |L_\varepsilon(\bar{\gamma}f)(x)|^2|k(\gamma)|d\mu(\gamma)\right\}^{1/2} - \{\sum_s |L_\varepsilon g_{n,s}(x)|^2\}^{1/2}\right| \leq \{(\|\phi_j\|_\infty\|\hat{f}\|_1)^2/n + (\|\phi_j\|_\infty^2\|k\|_1)/n^2\}^{1/2}.$$

This implies the point wise relation

$$(10)\qquad \left\{\int_\Gamma |L_\varepsilon(\bar{\gamma}f)(x)|^2|k(\gamma)|d\mu(\gamma)\right\}^{1/2} = \lim_{n\to+\infty} \{\sum_s |(L_\varepsilon g_{n,s})(x)|^2\}^{1/2}.$$

This is the replacement aimed for in (5). By making use of (4), (5) and (10) together with Fatou's lemma, we have

$$(11)\qquad m(\{x \in G | (M_k^{(J)}f)(x) > t\}) \leq \liminf_{\varepsilon\to+0} \liminf_{n\to+\infty} m(\{x \in G | \{\sum_s |(L_\varepsilon g_{n,s})(x)|^2\}^{1/2} > t/\|k\|_1^{1/2}\}).$$

Applying the weak version theorem of Marcinkiewicz-Zygmund [12, p. 486, Th.2.9] to the distribution on the right-hand side, then

$$(12)\qquad m(\{x \in G | \{\sum_s |(L_\varepsilon g_{n,s})(x)|^2\}^{1/2} > t/\|k\|_1^{1/2}\}) \leq \{C_{p,q}N_{p,q}^{(w)}(\{\phi_j\})\|k\|_1^{1/2}t^{-1}\|(\sum_s |g_{n,s}|^2)^{1/2}\|_p\}^q \leq \{C_{p,q}N_{p,q}^{(w)}(\{\phi_j\})\|k\|_1 t^{-1}\|f\|_p\}^q,$$

where $C_{p,q}$ is just the constant appears in the weak version theorem of Marcinkiewicz-Zygmund, which is determined only by p and q. The above two relations (11) and (12) give the desired estimate (3) for such a function f that satisfies the conditions stated at the begining.

The task left to us is to remove the restrictions imposed on f. Let $f \in L^p(G) \cap L^2(G)$. As shown below, we can choose a sequence $\{f_N\}$ satisfying the following properties:

(i) $f_N \in C(G) \cap L^1(G)$ and supp $\widehat{f_N}$ is compact.
(ii) $\lim_{N\to+\infty} (T_{k*\phi_j}f_N)(x) = (T_{k*\phi_j}f)(x)$ a.e. $(j = 1, \ldots, J)$.
(iii) $\lim_{N\to+\infty} \|f - f_N\|_{L^p(G)} = 0$.

Applying the result already established above to each f_N, and then using Fatou's lemma, then we see that the distributional inequality (3) still holds for f concerned.

To construct such functions f_N $(N = 1, 2, \ldots)$ that satisfy **(i)**, **(ii)** and **(iii)**, we take first compact sets K_ν $(\nu = 1, 2, \ldots)$ satisfying $\lim_{\nu\to+\infty} \|f\chi_{K_\nu^c}\|_{L^p(G)} = \lim_{\nu\to+\infty} \|f\chi_{K_\nu^c}\|_{L^2(G)} = 0$, where K_ν^c means the complement of K_ν. And then set $g_\nu = f\chi_{K_\nu}$. Then $g_\nu \in L^1(G)$. Next we take an approximate identity $\{u_\delta\}$ in $L^1(G)$ such that each supp $\widehat{u_\delta}$ is compact (Hewitt-Ross [13, vol. II, p. 298, (33.12)]), and write $g_{\nu,\delta} = g_\nu * u_\delta$. Then $g_{\nu,\delta} \in C(G) \cap L^1(G)$ and supp $\widehat{g_{\nu,\delta}}$ is compact. Furthermore $g_{\nu,\delta} \to g_\nu$ in $L^2(G)$. So we can take δ_ν such as $\|g_{\nu,\delta_\nu} - g_\nu\|_{L^2(G)} < 1/\nu$ for each ν. Then the sequence of the functions $h_\nu = g_{\nu,\delta_\nu}$ $(\nu = 1, 2, \ldots)$ conveges to f in $L^2(G)$. For each j, $T_{k*\phi_j}h_\nu \to T_{k*\phi_j}f$ in $L^2(G)$. Therefore we can choose such a subsequence $\{\nu_N\}$ from $\{\nu\}$ that $T_{k*\phi_j}h_{\nu_N} \to T_{k*\phi_j}f$ a.e. as $N \to +\infty$ for all $j = 1, 2, \ldots, J$. The functions $f_N = h_{\nu_N}$ $(N = 1, 2, \ldots)$ could be the candidates for the desired functions. It is easy to see that they satisfy **(i)** and **(ii)**. And the relations

$$\|f - h_\nu\|_{L^p(G)} = \|(f - f*u_{\delta_\nu}) + (f - g_\nu)*u_{\delta_\nu}\|_{L^p(G)} \leq \|f - f*u_{\delta_\nu}\|_{L^p(G)} + \|f - g_\nu\|_{L^p(G)}$$

assure that the f_N's also satisfy **(iii)**, since we may assume that $u_\delta \geq 0$ and $\lim_\delta \int_{W^c} u_\delta(x)dm(x) = 0$ for any neighborhood W and 0, and may use the fact that the set of all continuous functions on G with compact supports is dense in $L^p(G)$ ([13, vol. I, p. 140, (12.10)]).

3 On the case of $G=R^n$

In this section, we study the case $G=R^n$. The additive group R^n can be covered by the cubes $Q_m = 2\pi m + Q \cong R^n/(Z^n)^\perp$, $Q = [0, 2\pi)^n$, $m \in Z^n$. This property is an advantage for the following argument. By making use of this feature, Fan had $N_{L^{(p,q)}(T^n),L^{(p,q)}(T^n)}(\{\phi_j|\Lambda\}) \leq CN_{L^{(p,q)}(R^n),L^{(p,q)}(R^n)}(\{\phi_j\})$ in [9] with $\phi_j(\xi) = \phi(\lambda_j\xi)$ $(j = 1, 2, \ldots)$, and Woźniakowski proved

$$N_{L^1(T^n),wL^1(T^n)}(\phi\,|\,Z^n) \leq N_{L^1(R^n),wL^1(R^n)}(\phi)$$

and pointed out that we are able to take 1 as the constant in the right-hand side of (2) in Introduction within the framework of single Fourier multiplier in [22]. He also said in Remark at the end of his paper that the same inequality holds even if we replace L^1 by $L(r, p)$, $p \geq 1$, $0 < r < +\infty$ $(1 \leq r$ for $p = 1)$ and wL^1 by $L(s, p)$, $0 < s < +\infty$ (probably $L(r, p)$ and $L(s, p)$ should be read $L(p, r)$ and $L(p, s)$, respectively). In the statement of the remark, the case of weak type is excluded. The aim of this section is to extend Woźniakowski's result to the case of maximal operators and to have a result containing the weak type cases. But we depend on the ideas of them for the proof.

To state our theorem, we introduce the following notations.

For a measurable function f on a measure space (M, m), we denote by λ_f the distribution function of f defined by $\lambda_f(t) = m(\{x \in M|\,|f(x)| > t\})$ for $0 < t < +\infty$. We write the non-decreasing rearrangement of f by f^* which is defined by $f^*(t) = \inf\{s > 0\,|\,\lambda_f(s) \leq t\}$. Following Hunt [15], we denote by $L(p, q)$ the function space of all functions f on M satisfying

$$\|f\|^*_{p,q} = (q/p)^{1/q}\|t^{1/p}f^*(t)\|_{L^q(dt/t)} = q^{1/q}\|t\lambda_f(t)^{1/p}\|_{L^q(dt/t)} < +\infty,$$

where $L^q(dt/t)$ means the Lebesgue L^q space on $(0, +\infty)$ with respect to the dilation invariant measure dt/t. Hereafter we shall use such an ambiguity that $q^{1/q} = 1$, if $q = +\infty$. To express the underlying measure space M, we also use the notation $L^{(p,q)}(M)$ instead of $L(p, q)$.

The measure space M treated in this section is R^n or $Q \cong T^n$, and the measure m is the Lebesgue measure on each of them. For a subset E of R^n or Q, the Lebesgue measure of E is designated as $|E|$.

For a 2π-periodic function F, we denote the distribution function of F on Q by Λ_F instead of λ_F to clarify the difference between underlying measure spaces Q and R^n.

Our assertion is the following.

Theorem 3.1 *Assume that each $\phi_j \in L^\infty(R^n)$ is continuous at every points of Z^n. Then*

$$N_{L^{(p,r)}(T^n),L^{(p,s)}(T^n)}(\{\phi_j|Z^n\}) \leq N_{L^{(p,r)}(R^n),L^{(p,s)}(R^n)}(\{\phi_j\})$$

for $0 < p < +\infty$, $0 < r < +\infty$ and $0 < s \leq +\infty$.

Proof. As a matter of convenience, we introduce such a notation that

$$(\tilde{T}^*_J F)(x) = \max\nolimits_{1\leq j\leq J}|(\tilde{T}_j F)(x)| \tag{13}$$

for $J = 1, 2, \ldots,$ where we set $\tilde{T}_j = \tilde{T}_{\phi_j|Z^n}$ $(j = 1, 2, \ldots)$. We try to estimate the distribution function

$$\Lambda_{\tilde{T}^*_J F}(t) = |\{\theta \in Q\,|\,(\tilde{T}^*_J F)(\theta) > t\}|$$

of $\tilde{T}^*_J F$ for $t > 0$ at first under the assumption that F is a trigonometric polynomial. And assume that $F(x) = \sum_{m\in Z^n}\hat{F}(m)\,e^{im\cdot x}$. Let χ_k be a non-negative smooth function bounded by 1 from above on R^1 with compact support such that $\chi_k(\tau) = 1$ $(|\tau| \leq 2\pi k)$ and $\chi_k(\tau) = 0$ $(|\tau| \geq 2\pi(k + 1))$ for $k = 1, 2, \ldots,$ and set

$$\psi^\varepsilon_k(x) = \psi_k(\varepsilon x), \quad \psi_k(x) = \prod_{\nu=1}^n \chi_k(x_\nu) \quad (\varepsilon > 0, x = (x_1, \ldots, x_n) \in R^n).$$

With these functions we write

$$r_{j;\varepsilon,k}(x) = \psi^\varepsilon_k(x)(\tilde{T}_j F)(x) - T_j(\psi^\varepsilon_k F)(x),$$

where $T_j = T_{\phi_j}$. Then $r_{j;\varepsilon,k}$ is the inverse Fourier transform of $\hat{r}_{j;\varepsilon,k} \in L^1(R^n)$. Therefore

$$|r_{j;\varepsilon,k}(x)| \leq (2\pi)^{-n/2}\|\hat{r}_{j;\varepsilon,k}\|_1 \quad (x \in R^n).$$

Since $\hat{r}_{j;\varepsilon,k}$ has such a form as

$$\hat{r}_{j;\varepsilon,k}(\xi) = \sum_{m\in Z^n}\hat{F}(m)\{\phi_j(m) - \phi_j(\xi)\}\frac{1}{\varepsilon^n}\hat{\psi}_k\left(\frac{\xi - m}{\varepsilon}\right),$$

we have the L^1-estimate

$$\|\hat{r}_{j;\varepsilon,k}\|_1 \leq \sum_{m\in Z^n}|\hat{F}(m)|\int_{R^n}|\phi_j(m) - \phi_j(m + \varepsilon\xi)||\hat{\psi}_k(\xi)|d\xi.$$

So we put the right-hand side into $\delta_{j;\varepsilon,k}$ and set

$$\Delta_{J;\varepsilon,k} = \max_{1 \le j \le J} \delta_{j;\varepsilon,k}.$$

Then we have

$$\psi_k^\varepsilon(x)(\tilde{T}_J^* F)(x) \le T^*(\psi_k^\varepsilon F)(x) + (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}.$$

Because of the facts that each ϕ_j is bounded and continuous at every points $m \in Z^n$, $\hat{\psi}_k \in L^1(R^n)$, and $\hat{F}(m) = 0$ except for a finite number of m, we see $\lim_{\varepsilon\to 0} \delta_{j;\varepsilon,k} = 0$. Therefore $\lim_{\varepsilon \to 0} \Delta_{J;\varepsilon,k} = 0$ for any fixed J and k. For sufficiently small ε such that $(2\pi)^{-n/2}\Delta_{J;\varepsilon,k} < t$, we have $\lambda_{\psi_k^\varepsilon(\tilde{T}_J^*F)}(t) \le \lambda_{T^*(\psi_k^\varepsilon F)}(t - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k})$. Therefore

$$\begin{aligned}
\|t\lambda_{\psi_k^\varepsilon(\tilde{T}_J^*F)}(t)^{1/p}\|_{L^s((a,+\infty),dt/t)} &\le \|t\lambda_{T^*(\psi_k^\varepsilon F)}(t - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k})^{1/p}\|_{L^s((a,+\infty),dt/t)} \\
&\le \|(t + (2\pi)^{-n/2}\Delta_{J;\varepsilon,k})\lambda_{T^*(\psi_k^\varepsilon F)}(t)^{1/p}\|_{L^s((a-(2\pi)^{-n/2}\Delta_{J;\varepsilon,k},+\infty),dt/t)} \\
&\le \left(1 + \frac{(2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}{a - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}\right)\|t\lambda_{T^*(\psi_k^\varepsilon F)}(t)^{1/p}\|_{L^s(dt/t)}
\end{aligned}$$

for any a and ε such that $a > (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}$, where $L^s((b, +\infty), dt/t)$ means the Lebesgue L^s space on $(b, +\infty)$ with respect to the measure dt/t. The last inequality in the above chain of relations follows from $(t + (2\pi)^{-n/2}\Delta_{J;\varepsilon,k})/t \le 1 + (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}/(a - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k})$ for $t > a - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}$. The $L^s(dt/t)$-norm in the last term is equal to $s^{-1/s}$ times $\|T^*(\psi_k^\varepsilon F)\|_{p,s}^*$. And it is bounded by $N_{L(p,r),L(p,s)}(\{\phi_j\})\|\psi_k^\varepsilon F\|_{p,r}^*$. Therefore

$$\|t\lambda_{\psi_k^\varepsilon(\tilde{T}_J^*F)}(t)^{1/p}\|_{L^s((a,+\infty),dt/t)} \le s^{-1/s}\left(1 + \frac{(2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}{a - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}\right)N_{L(p,r),L(p,s)}(\{\phi_j\})\|\psi_k^\varepsilon F\|_{p,r}^*. \tag{14}$$

By the way, $\psi_k^\varepsilon(x) \le 1$ and supp $\psi_k^\varepsilon \subset [-2\pi(k+1)/\varepsilon, 2\pi(k+1)/\varepsilon]^n$. The number of the muti-indices m satisfying $Q_m \cap [-2\pi(k+1)/\varepsilon, 2\pi(k+1)/\varepsilon]^n \ne \emptyset$ is not greater than $[2\{\varepsilon^{-1}(k+1)+1\}]^n$. Therefore

$$\lambda_{\psi_k^\varepsilon F}(t) \le |\cup\{Q_m(|F| > t) | Q_m \cap [-2\pi(k+1)/\varepsilon, 2\pi(k+1)/\varepsilon]^n \ne \emptyset\}| \le \left\{2\left(\frac{k+1}{\varepsilon} + 1\right)\right\}^n \Lambda_F(t),$$

where we have used the periodicity of F and the notation $Q_m(|F| > t) = \{x \in Q_m \mid |F(x)| > t\}$. By this relation, we have

$$\begin{aligned}
\|\psi_k^\varepsilon F\|_{p,r}^* = r^{1/r}\|t\lambda_{\psi_k^\varepsilon F}(t)^{1/p}\|_{L^r(dt/t)} &\le \left\{2\left(\frac{k+1}{\varepsilon} + 1\right)\right\}^{n/p} r^{1/r}\|t\Lambda_F(t)^{1/p}\|_{L^r(dt/t)} \\
&= \left\{2\left(\frac{k+1}{\varepsilon} + 1\right)\right\}^{n/p} \|F\|_{p,r}^*.
\end{aligned} \tag{15}$$

On the other hand, $\psi_k^\varepsilon(x) = 1$ for $x \in [-2\pi k/\varepsilon, 2\pi k/\varepsilon]^n$. So we consider the multi-indices m satisfying $Q_m \subset [-2\pi k/\varepsilon, 2\pi k/\varepsilon]^n$. The number of such multi-indices m is at least $\{2(\varepsilon^{-1}k - 1)\}^n$. Taking account of the periodicity of $\tilde{T}_J^*F$, we have

$$|\lambda_{\psi_k^\varepsilon(\tilde{T}_J^*F)}(t)| \ge |\cup\{Q_m(\tilde{T}_J^*F > t) | Q_m \subset [-2\pi k/\varepsilon, 2\pi k/\varepsilon]^n\}| \ge \left\{2\left(\frac{k}{\varepsilon} - 1\right)\right\}^n \Lambda_{\tilde{T}_J^*F}(t) \tag{16}$$

with the same notations as above. Therefore the left-hand side of (14) can be estimated from below by

$$\left\{2\left(\frac{k}{\varepsilon} - 1\right)\right\}^{n/p} \|t\Lambda_{\tilde{T}_J^*F}(t)^{1/p}\|_{L^s((a,+\infty),dt/t)}. \tag{17}$$

Combining (14), (17) and (15), we have

$$\begin{aligned}
\left\{2\left(\frac{k}{\varepsilon} - 1\right)\right\}^{n/p} s^{1/s}\|t\Lambda_{\tilde{T}_J^*F}(t)^{1/p}\|_{L^s((a,+\infty),dt/t)} &\le \left\{2\left(\frac{k+1}{\varepsilon} + 1\right)\right\}^{n/p}\left(1 + \frac{(2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}{a - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}\right) \\
&\quad \times N_{L(p,r),L(p,s)}(\{\phi_j\})\|F\|_{p,r}^*.
\end{aligned} \tag{18}$$

Dividing the both sides of above inequality by $\{2(\varepsilon^{-1}k - 1)\}^{n/p}$ and letting $\varepsilon \to +\infty$, and then pushing k to infinity, then we get

$$s^{1/s}\|t\Lambda_{\tilde{T}_J^*F}(t)^{1/p}\|_{L^s((a,+\infty),dt/t)} \le N_{L(p,r),L(p,s)}(\{\phi_j\})\|F\|_{p,r}^*.$$

Finally we let a tend to 0, then we have the inequality

$$s^{1/s}\|t\Lambda_{\tilde{T}_J^*F}(t)^{1/p}\|_{L^s(dt/t)} \le N_{L(p,r),L(p,s)}(\{\phi_j\})\|F\|_{p,r}^* \tag{19}$$

on condition that F is a trigonometric polynomial.

Now we remove the restriction posed on F. Let $F \in L^{(p,r)}(Q) \cap L^\infty(Q)$. The Cesàro means F_N, $F_N(\theta)$

$= \sum_{|m_\nu| \le N} \prod_{\nu=1}^{n} (1 - |m_\nu|/N)\hat{F}(m)\, e^{i\theta\cdot m}$, of F converge to F a.e. Since each $\tilde{T}_j F_N$ is equal to the Cesàro mean of $\tilde{T}_j F$, $\lim_{N\to+\infty} \tilde{T}_J^* F_N = \tilde{T}_J^* F$ a.e. Therefore

$$(20) \qquad s^{1/s}\|t\Lambda_{\tilde{T}_J^*F}(t)^{1/p}\|_{L^s(dt/t)} \le \liminf_{N\to+\infty} s^{1/s}\|t\Lambda_{\tilde{T}_J^*F_N}(t)^{1/p}\|_{L^s(dt/t)}.$$

Let now $E = \{\theta \in Q \mid \lim_{N\to+\infty} F_N(\theta) = F(\theta)\}$. Then $|E| = (2\pi)^n$. With the notation $E_G(t) = \{\theta \in E \mid |G(\theta)| > t\}$, we have the relation $E_F(t) \subset \liminf_{N\to+\infty} E_{F_N}(t) \subset \limsup_{N\to+\infty} E_{F_N} \subset E_F(t-\varepsilon)$ for any $t > \varepsilon > 0$. By these relations, we can easily obtain

$$\Lambda_F(t) \le \liminf_{N\to+\infty} \Lambda_{F_N}(t) \le \limsup_{N\to+\infty} \Lambda_{F_N}(t) \le \Lambda_F(t-\varepsilon)$$

for any $t > \varepsilon > 0$. Therefore, at the continuous points t of Λ_F, and hence at almost all points t, we have $\lim_{N\to+\infty} \Lambda_{F_N}(t) = \Lambda_F(t)$. Furthermore $0 \le \Lambda_{F_N}(t) \le (2\pi)^n \chi_{(0,\|F\|_\infty)}(t)$, where $\chi_{(0,\|F\|_\infty)}$ means the characteristic function of the interval $(0, \|F\|_\infty)$. The Lebesgue dominated convergence theorem assures that

$$(21) \qquad \lim_{N\to+\infty} \|F_N\|_{p,r}^* = \lim_{N\to+\infty} r^{1/r}\|t\Lambda_{F_N}(t)^{1/p}\|_{L^r(dt/t)} = r^{1/r}\|t\Lambda_F(t)^{1/p}\|_{L^r(dt/t)} = \|F\|_{p,r}^*.$$

The relations (20) and (21), and the inequality (19) for F_N guarantee (19) to hold for $F \in L^{(p,r)}(Q) \cap L^\infty(Q)$.

Lastly we consider the case $F \in L^{(p,r)}(Q) \cap L^2(Q)$. For such F, we set $F_N(\theta) = F(\theta)$, if $|F(\theta)| \le N$, and $F_N(\theta) = 0$, if $|F(\theta)| > N$. Then each F_N satisfies the inequality (19). Since $|F_N| \le |F|$, we have $\|F_N\|_{p,r}^* \le \|F\|_{p,r}^*$. On the other hand, since $F_N \to F$ in $L^2(Q)$, we can extract a subsequence $\{N_\nu\}$ from $\{N\}$ such that $\tilde{T}_J^* F_{N_\nu} \to \tilde{T}_J^* F$ a.e. By these reasons, we see that (19) holds for $F \in L^{(p,r)}(Q) \cap L^2(Q)$ too.

Since $\Lambda_{\tilde{T}_J^*F}(t) \uparrow \Lambda_{\tilde{T}^*F}(t)$ as $J \uparrow +\infty$, we get the desired inequality

$$\|\tilde{T}^*F\|_{p,s}^* = s^{1/s}\|t\Lambda_{\tilde{T}^*F}(t)^{1/p}\|_{L^s(dt/t)} \le N_{L(p,r),L(p,s)}(\{\phi_j\})\|F\|_{p,r}^*$$

for $F \in L^{(p,r)}(Q) \cap L^2(Q)$.

Remarks. Here we give some remarks for replacing $L(p, s)$ by $L(q, s)$, $q \ne p$, in Theorem 3.1. By tracing the argument inducing (18), we get

$$\left\{2\left(\frac{k}{\varepsilon} - 1\right)\right\}^{n/q} s^{1/s}\|t\Lambda_{\tilde{T}_J^*F}(t)^{1/q}\|_{L^s((a,+\infty),dt/t)} \le \left\{2\left(\frac{k+1}{\varepsilon} + 1\right)\right\}^{n/p}\left(1 + \frac{(2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}{a - (2\pi)^{-n/2}\Delta_{J;\varepsilon,k}}\right) \times N_{L(p,r),L(q,s)}(\{\phi_j\})\|F\|_{p,r}^*$$

for any trigonometric polynomial F.

We consider the case $p > q$ first. In this case, after the same process as stated at just below (18), we have

$$s^{1/s}\|t\Lambda_{\tilde{T}_J^*F}(t)^{1/q}\|_{L^s((a,+\infty),dt/t)} = 0$$

under the assumption $N_{L(p,r),L(q,s)}(\{\phi_j\}) < +\infty$. By the arbitrarity of a, this implies $\tilde{T}_J^*F = 0$ for any trigonometric polynomial F. And so it follows that every ϕ_j must be $\phi_j = 0$ on Z^n, if $N_{L(p,r),L(q,s)}(\{\phi_j\}) < +\infty$. Taking account of $N_{L(p,r),L(q,s)}(\{\phi_j(\varepsilon\cdot)\}) = \varepsilon^{n(1/q-1/p)}N_{L(p,r),L(q,s)}(\{\phi_j\})$ and assuming the continuity of each ϕ_j on $\boldsymbol{R}^n$, we finally have the conclusion that, if $p > q$, then $N_{L(p,r),L(q,s)}(\{\phi_j\}) < +\infty$ implies $\phi_j = 0$ $(j = 1, 2, \ldots)$. This is similar to the result of Hörmander [14, p. 96, Th.1.1] for (L^p, L^q)-multipliers.

Next we show that, when $p < q$, we can not necessarily get the conclusion that the finiteness of $N_{L(p,r),L(q,s)}(\{\phi_j\})$ implies the one of the counterpart $N_{L(p,r),L(q,s)}(\{\phi_j | Z^n\})$. For the purpose we take first a smooth function ψ on $\boldsymbol{R}^1$ such that $\operatorname{supp}\psi \subset [-1/2, 1/2]$ and $\psi(0) = 1$. And set

$$\phi(\xi) = \sum_{m\in Z} \psi(\varepsilon_m^{-1}(\xi - m)),$$

where $\varepsilon_m = 2^{-|m|}$ $(m \in Z)$. To see that the inverse Fourier transform $\check{\phi}$ of ϕ is in $L(p', r')$ for such p' and r' that $1 < p' < +\infty$ and $1 \le r' \le +\infty$ or that $p' = r' = +\infty$, we adopt the norm $\|\cdot\|_{p',r'}$ instead of $\|\cdot\|_{p',r'}^*$, which is defined by

$$\|f\|_{p',r'} = \|f^{**}\|_{p',r'}^*, \quad f^{**}(t) = \sup_{t\le|E|} \frac{1}{|E|}\int_E |f(x)|dx.$$

It is known that $\|f\|_{p',r'}^* \le \|f\|_{p',r'} \le \{p'/(p'-1)\}\|f\|_{p',r'}^*$ (Hunt [15, pp. 257–258], O'Neil [18, p. 136]). Since $\|\cdot\|_{p',r'}$ is a norm and $\|\varepsilon_m\check{\psi}(\varepsilon_m\cdot)\|_{p',r'} = \varepsilon_m^{1-1/p'}\|\check{\psi}\|_{p',r'}$, we have

$$\|\check{\phi}\|_{p',r'} \le \sum_{m\in Z} \|\varepsilon_m\check{\psi}(\varepsilon_m\cdot)\|_{p',r'} = \left(\sum_{m\in Z} \varepsilon_m^{1-1/p'}\right)\|\check{\psi}\|_{p',r'} < +\infty.$$

Therefore we get $\check{\phi} \in L^{(p',r')}(\boldsymbol{R}^1)$ for such p' and r' that are in the range stated above. When $1 < p < q < +\infty$, $1 \le r < +\infty$ and $1 \le s \le +\infty$, we are able to take such a p_0 that satisfies $1 < p_0 < +\infty$ and $1/q = 1/p + 1/p_0 - 1$. And evidently $1/s \le 1/r + 1$. Therefore we can use Young's inequality and have

(22) $$\|T_\phi f\|_{q,s} = \|\check{\phi}*f\|_{q,s} \le C\|\check{\phi}\|_{p_0,1}\|f\|_{p,r}$$

with some constant C (O'Neil [18, pp. 137–138, Th.2.6]). In other interesting case of $1 < p < q = s = +\infty$ and $1 \le r < +\infty$ or the case $1 = r = p < q = s = +\infty$, we have the relation

$$|(T_\phi f)(x)| = |(\check{\phi}*f)(x)| \le \int_0^{+\infty} \{\check{\phi}(x - \cdot)\}^*(t)f^*(t)dt \le C\|\check{\phi}(x - \cdot)\|^*_{p',r'}\|f\|^*_{p,r} = C\|\check{\phi}\|^*_{p',r'}\|f\|^*_{p,r},$$

where C is a constant depending only on p and r, and the indices p' and r' are the conjugates of p and r, respectively (Hunt [15, p. 257, (1.9)]. On the other hand, $\tilde{T}_{\phi|Z}$ becomes the identity map. Therefore, if the transference is true, then the inclusion relation $L^{(p,r)}(T) \subset L^{(q,s)}(T)$ holds for $p < q$. But this is impossible.

4 Maximal operators on the Hardy spaces

The difference $r_{j;\varepsilon,k}(x) = \psi_k^\varepsilon(x)(\tilde{T}_jF)(x) - T_j(\psi_k^\varepsilon F)(x)$ plays an important role in Fan [9] and Woźniakowski [22], and we followed the line in the preceding section. The method is also taken in the paper Fan-Wu [10, 11]. Although their interest is directed to the case of $\phi_j = \phi(\lambda_j\cdot)$ $(j = 1, 2, \ldots)$, they took the operators as mappings from Hardy spaces to L^p-spaces and proved the inequality $N_{H^p(T^n),L^p(T^n)}(\{\phi_j|Z^n\}) \le CN_{H^p(R^n),L^p(R^n)}(\{\phi_j\})$ with some constant C for $0 < p \le 1$. The spaces $H^p(R^n)$ and $H^p(T^n)$ are the real Hardy spaces on R^n and $T^n \cong Q = [0, 2\pi)^n$, respectively. The space $H^p(R^n)$ is the totality of all tempered distributions f satisfying $f^\dagger \in L^p(R^n)$, where $f^\dagger(x) = \sup_{0<r<+\infty} |f*v_r(x)|$, $v_r(x) = r^{-n}v(r^{-1}x)$, and v is a member of Schwartz function class $\mathcal{S}(R^n)$ on R^n such that $\int_{R^n} v(x)dx = 1$. It is known that the definition of $H^p(R^n)$ is independent of the choice of v. So we adopt a smooth function v with compact support for the definition of $H^p(R^n)$ and fix it hereafter. The space $H^p(T^n)$ is defined as following. Let Υ_r be the periodization of v_r, that is $\Upsilon_r(x) = \Sigma_{m\in Z^n} v_r(x + 2\pi m)$, and set $F^\dagger(x) = \sup_{0<r<+\infty} |F*\Upsilon_r(x)|$ for a distribution F on T^n, where the asterisk $*$ means the convolution among periodic functions and distributions on T^n. The space $H^p(T^n)$ is defined as the set of all distributions F on T^n such that $F^\dagger \in L^p(Q)$.

The purpose of this section is to give a transferability of the boundedness of the maximal Fourier multiplier operators from H^p to *weak* L^p.

Theorem 4.1 *Let each $\phi_j \in L^\infty(R^n)$ be continuous at every points in Z^n and $0 < p \le 1$. Then*

$$N_{H^p(T^n),wL^p(T^n)}(\{\phi_j|Z^n\}) \le C\{N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\}) + N_{L^2(R^n),wL^2(R^n)}(\{\phi_j\})\},$$

where the constant C is determined by p, n and v, but is independent of $\{\phi_j\}$.

Proof. We may assume that each supp ϕ_j is compact because of the following reason. If we take a $\psi \in \mathcal{S}(R^n)$ such that $\hat{\psi}(0) = 1$ and supp $\hat{\psi} \subset B(0, 1)$, the unit ball in R^n centered at 0, then $\tilde{T}_{\phi_j|Z^n}F = \lim_{\delta\to+0} \tilde{U}_{j,\delta}F$ a.e. for every $j = 1, 2, \ldots$ and periodic function $F \in L^2(Q)$, where we set $\tilde{U}_{j,\delta} = \tilde{T}_{(\phi_j\widehat{\psi_\delta})|Z^n}$. Therefore $\tilde{T}^*F \le \liminf_{v\to+0} \tilde{U}^*_{\delta_v}F$ a.e. for any sequence $\{\delta_v\}$ such that $\delta_v \to +0$ $(v \to +\infty)$, where

$$\tilde{U}^*_\delta = \sup_j |\tilde{U}_{j,\delta}(\,\cdot\,)| = \tilde{T}^*_{\{(\phi_j\widehat{\psi_\delta})|Z^n\}}.$$

This implies

$$\Lambda_{\tilde{T}^*F}(t) \le \liminf_{v\to+\infty} \Lambda_{\tilde{U}^*_{\delta_v}F}(t) \quad (t > 0).$$

On the other hand, $T^*_{\{\phi_j\widehat{\psi_\delta}\}}f = T^*_{\{\phi_j\}}(f*\psi_\delta)$ for $f \in L^2(R^n) \cap H^p(R^n)$, since $T_{\phi_j\widehat{\psi_\delta}}f = T_{\phi_j}(f*\psi_\delta)$ $(j = 1, 2, \ldots)$. And

$$\|f*\psi_\delta\|_{H^p(R^n)} \le C(n, p, \psi)\|f\|_{H^p(R^n)} \quad (0 < \delta \le 1),$$

where the constant $C(n, p, \psi)$ is independent of δ (Stein [21, p. 127, 5.1(c)]). Therefore it is sufficient that the statement for $\{\phi_j\,\widehat{\psi_\delta}\}$ instead of $\{\phi_j\}$ will be proved.

In the following, we assume that all supp ϕ_j are compact.

Following Fan-Wu [10], we introduce such a function that $q_0(\tau) = 1 - 4\tau^2$, if $|\tau| \le 1/2$, and $q_0(\tau) = 0$, if $|\tau| > 1/2$. And set

$$q^\eta(x) = \prod_{v=1}^n q_0(\eta_v x_v) \quad (x = (x_1, \ldots, x_n) \in R^n),$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_n) = \eta(\varepsilon, K)$ is defined by $\eta_v = 1/2\{(1 + 2K)\pi - \varepsilon_v\}$ $(v = 1, 2, \ldots, n)$ with $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0, \pi\}^n$ and $K = 1, 2, \ldots$. Like as the preceding section, we consider the difference

$$r'_{j;\eta}(x; F) = q^\eta(x)(\tilde{T}_jF)(x) - T_j(q^\eta F)(x),$$

where T_j and $\tilde{T}_j$ express T_{ϕ_j} and $\tilde{T}_{\phi_j|Z^n}$, respectively.

We assume temporarily that the 2π-periodic function F is in $C^\infty(R^n)$. Then the similar argument in the last section leads us to the relation

$$|r'_{j;\eta}(x; F)| \le (2\pi)^{-n/2} \sum_{m\in Z^n} |\hat{F}(m)| \int_{R^n} |\phi_j(m) - \phi_j(\dots, m_\nu + \eta_\nu\xi_\nu, \dots)| |\hat{q}(\xi)| d\xi$$

for all x. We denote the right-hand side by $\delta'_{j;\eta}(F)$. By reason of the facts that $\hat{q} \in L^1(R^n)$, each bounded function ϕ_j is continuous at each point $m \in Z^n$, and $\sum_{m\in Z^n} |\hat{F}(m)| < +\infty$, we have $\delta'_{j;\eta}(F) = \delta'_{j;\eta(\varepsilon,K)}(F) \to 0$ ($K \to +\infty$). So, if we set

$$\Delta'_{J;\eta}(F) = \Delta'_{J;\eta(\varepsilon,K)}(F) = \max_{1\le j\le J} \delta'_{j;\eta}(F),$$

then $\lim_{K\to+\infty} \Delta'_{J;\eta}(F) = 0$. Therefore, by making use of the same notation $\tilde{T}^*_J$ as (13) at the outset of the proof of Theorem 3.1, the relation

$$q^\eta(x)(\tilde{T}^*_J F)(x) \le T^*(q^\eta F)(x) + \Delta'_{J;\eta}(F)$$

leads the distributional inequality

$$\lambda_{q^\eta(\tilde{T}^*_J F)}(3^n t/4^n) \le \lambda_{T^*(q^\eta F)}(3^n t/4^n - \Delta'_{J;\eta}(F)) \le \left(\frac{N_{H^p(R^n), wL^p(R^n)}(\{\phi_j\})}{3^n t/4^n - \Delta'_{J;\eta}(F)} \|q^\eta F\|_{H^p(R^n)} \right)^p \tag{23}$$

for sufficiently large K which determines η together with ε. This inequality is meaningful only when $q^\eta F \in H^p(R^n)$ and shall be applied to $H^L_{\varepsilon,\ \eta}$ defined at (28) below.

Now assume $F \in L^2(Q)$. Then, by the same argument for $H^p(R^n)$ in Stein [21, Chapter III], we can decompose F into such a form

$$F = G + H = G + \sum_{l=1}^{+\infty} \gamma_l A_l, \tag{24}$$

where G is a bounded 2π-periodic function such that

$$\|G\|_{L^\infty(Q)} \le C(n, p, v)\|F\|_{H^p(T^n)} \tag{25}$$

and is called exceptional atom in Blank-Fan [4], and each A_l is a 2π-periodic function called regular atom in [4] and having the following properties: For each A_l, if we associate a function $\tilde{a}_l$ on $T^n = \{e^{ix} = (e^{ix_1}, \dots, e^{ix_n}) | 0 \le x_\nu < 2\pi\ (\nu = 1, \dots, n)\}$ defined by $\tilde{a}_l(e^{ix}) = A_l(x)$, then there exist a ball $\tilde{B}(\theta^l, \rho_l)$ on T^n with the center θ^l and the radius ρ_l satisfying that supp $\tilde{a}_l \subset \tilde{B}(\theta^l, \rho_l)$ and $|\tilde{a}_l(\theta)| \le \rho_l^{-n/p}$. Here we can take ρ_l such as $\rho_l < \pi/4$. Furthermore, each $\tilde{a}_l$ has vanishing moments in some sense. To state it more precisely, we introduce the following notations. For a point $\theta \in T$, if θ can be written $\theta = e^{i\tau}$ with some $-\pi/2 \le \tau < \pi/2$, then we set $[\theta] = 0$, and if $\theta = e^{i\tau}$ with a number τ such that $\pi/2 \le \tau < 3\pi/2$, then we put $[\theta] = \pi$. For $\theta = (\theta_1, \dots, \theta_n) \in T^n$, we define $[\theta]$ by $[\theta] = ([\theta_1], \dots, [\theta_n])$. To each $\tilde{a}_l$, we correspond a non-periodic function a_l defined by $a_l(x) = \tilde{a}_l(e^{ix}) = A_l(x)$, if $|x_\nu - [\theta^l_\nu]| < \pi$ ($\nu = 1, \dots, n$), but otherwise $a_l(x) = 0$, where $\theta^l = (\theta^l_1, \dots, \theta^l_n)$ is the center of the ball $\tilde{B}(\theta^l, \rho_l)$ carrying $\tilde{a}_l$. Then

$$\int_{R^n} x^\beta a_l(x)dx = 0 \quad (|\beta| \le d), \tag{26}$$

where we can take a sufficiently large number as d at will. With this decomposition, we have

$$\tilde{T}^*F \le \tilde{T}^*G + \tilde{T}^*H.$$

The term $\tilde{T}^*G$ can be easily handled as following.

$$\begin{aligned} t\Lambda_{\tilde{T}^*G}(t)^{1/p} &\le (2\pi)^{n(1/p-1/2)} t\Lambda_{\tilde{T}^*G}(t)^{1/2} \le (2\pi)^{n(1/p-1/2)} N_{L^2(T^n), wL^2(T^n)}(\{\phi_j | Z^n\})\|G\|_{L^2(T^n)} \\ &\le C(n, p, v) N_{L^2(R^n), wL^2(R^n)}(\{\phi_j\})\|F\|_{H^p(T^n)}. \end{aligned} \tag{27}$$

The last inequality follows from Theorem 3.1 and (25). To manage the term $\tilde{T}^*H$, we classify the regular atoms A_l by the centers $\theta^l \in T^n$ of their corresponding supports $\tilde{B}(\theta^l, \rho_l)$. Let

$$H^L = \sum_{l=1}^{L} \gamma_l A_l = \sum_{\varepsilon\in\{0,\pi\}^n} H^L_\varepsilon \quad \text{and} \quad H^L_\varepsilon = \sum_{1\le l\le L, [\theta^l]=\varepsilon} \gamma_l A_l \quad (\varepsilon \in \{0, \pi\}^n).$$

Here we take an $\varepsilon \in \{0, \pi\}^n$ and fix it for a moment. If we consider the cube $Q(\varepsilon) = \Pi_{\nu=1}^n [\varepsilon_\nu - \pi, \varepsilon_\nu + \pi]$ in R^n, then the non-periodic functions a_l defined above satisfy

$$a_l = A_l \chi_{Q(\varepsilon)}$$

for such l that $[\theta^l] = \varepsilon$, since the radius $\rho_l < \pi/2$ (we have taken ρ_l such as $\rho_l < \pi/4$). It is easy to check that each a_l is an $H^p(\boldsymbol{R}^n)$-atom, and that $h_\varepsilon^L = H_\varepsilon^L \chi_{Q(\varepsilon)} = \sum_{1\le l\le L,[\theta^l]=\varepsilon} \gamma_l a_l$ and $q^{\eta(\varepsilon,K)} H_\varepsilon^L$ are in $H^p(\boldsymbol{R}^n)$. But H_ε^L is not smooth in general. Hence we introduce an infinitely differentiable function ψ satisfying supp $\psi \subset B(0, 1)$ and $\int_{R^n} \psi(x)ds = 1$, and consider the convolution $H_{\varepsilon,\tau}^L = H_\varepsilon^L * \psi_\tau$,

$$H_{\varepsilon,\tau}^L(x) = (H_\varepsilon^L * \psi_\tau)(x) = \int_{R^n} H_\varepsilon^L(x-y)\psi_\tau(y)dy, \tag{28}$$

with $0 < \tau < \pi/4$. Then $H_{\varepsilon,\tau}^L$ is a smooth 2π-periodic function and

$$H_{\varepsilon,\tau}^L(x) = \sum_{m\in Z^n} (h_\varepsilon^L * \psi_\tau)(x - 2\pi m).$$

The support of each $h_\varepsilon^L * \psi_\tau(\cdot - 2\pi m) = \sum_{1\le l\le L,[\theta^l]=\varepsilon} \gamma_l (a_l * \psi_\tau)(\cdot - 2\pi m)$ is completely contained into the interior of $Q(\varepsilon) + 2\pi m$, for which reason we have taken ρ_l and τ such as $\rho_l < \pi/4$ and $\tau < \pi/4$. Furthermore supp $q^\eta = \Pi_{\nu=1}^n [-\{(1+2K)\pi - \varepsilon_\nu\}, (1+2K)\pi - \varepsilon_\nu]$ is just equal to a finite union of such cubes $Q(\varepsilon) + 2\pi m$ and the number of m satisfying $Q(\varepsilon) + 2\pi m \subset$ supp q^η is at most $(1 + 2K)^n$. Therefore

$$q^\eta(x) H_{\varepsilon,\tau}^L(x) = \sum_{\text{finite}} q^\eta(x)(h_\varepsilon^L * \psi_\tau)(x - 2\pi m)$$

and the number of the terms in the right-hand side does not exceed $(1 + 2K)^n$. In addition to them we are able to write

$$q^\eta(x)(h_\varepsilon^L * \psi_\tau)(x - 2\pi m) = \sum_{1\le l\le L,[\theta^l]=\varepsilon} \left(1 + \frac{\tau}{\rho_l}\right)^{n/p} \gamma_l a_l^{(m,\eta,\tau)}(x),$$

where

$$a_l^{(m,\eta,\tau)}(x) = \left(\frac{\rho_l}{\rho_l + \tau}\right)^{n/p} q^\eta(x)(a_l * \psi_\tau)(x - 2\pi m).$$

We see that each $a_l^{(m,\eta,\tau)}$ is an $H^p(\boldsymbol{R}^n)$-atom, if the integer d in (26) is taken sufficiently large, and hence $q^\eta H_{\varepsilon,\tau}^L \in H^p(\boldsymbol{R}^n)$. Furthermore

$$\|q^\eta H_{\varepsilon,\tau}^L\|_{H^p(R^n)}^p \le C(n,p,v)(1+2K)^n \sum_{1\le l\le L,[\theta^l]=\varepsilon} \left(1 + \frac{\tau}{\rho_l}\right)^n |\gamma_l|^p. \tag{29}$$

Taking account of these preparations, we apply the inequality (23) to $H_{\varepsilon,\tau}^L$. Then

$$\lambda_{q^\eta(\tilde{T}_J^* H_{\varepsilon,\tau}^L)}(3^n t/4^n) \le \left(\frac{N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\})}{3^n t/4^n - \Delta'_{J;\eta}(H_{\varepsilon,\tau}^L)} \|q^\eta H_{\varepsilon,\tau}^L\|_{H^p(R^n)}\right)^p. \tag{30}$$

If we set $M(\varepsilon, K) = \{m \in Z^n | Q(\varepsilon) + 2\pi m \subset \Pi_{\nu=1}^n [-1/4\eta_\nu, 1/4\eta_\nu]\}$, then $q^\eta(x) \ge (3/4)^n$ $(x \in Q(\varepsilon) + 2\pi m, m \in M(\varepsilon, K))$ and the number of m in $M(\varepsilon, K)$ is greater than or equal to $(K - 3/2)^n$. Therefore

$$\lambda_{q^\eta(\tilde{T}_J^* H_{\varepsilon,\tau}^L)}(3^n t/4^n) \ge (K - 3/2)^n \Lambda_{\tilde{T}_J^* H_{\varepsilon,\tau}^L}(t).$$

By making use of this relation in (30) and (29), then

$$\Lambda_{\tilde{T}_J^* H_{\varepsilon,\tau}^L}(t) \le C(n,p,v)\left(\frac{2K+1}{K-3/2}\right)^n \left(\frac{N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\})}{3^n t/4^n - \Delta'_{J;\eta}(H_{\varepsilon,\tau}^L)}\right)^p \sum_{1\le l\le L,[\theta^l]=\varepsilon} \left(1 + \frac{\tau}{\rho_l}\right)^n |\gamma_l|^p.$$

In this relation, if we push $K \to +\infty$ and then $\tau \to +0$, then we have

$$\Lambda_{\tilde{T}_J^* H_\varepsilon^L(t)} \le C(n,p,v)\left(\frac{N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\})}{t}\right)^p \sum_{1\le l\le L,[\theta^l]=\varepsilon} |\gamma_l|^p.$$

Since $\tilde{T}_J^* H^L \le \sum_{\varepsilon\in\{0,\pi\}^n} \tilde{T}_J^* H_\varepsilon^L$, we have

$$\Lambda_{\tilde{T}_J^* H^L}(2^n t) \le C(n,p,v)\left(\frac{N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\})}{t}\right)^p \sum_{l=1}^L |\gamma_l|^p. \tag{31}$$

When $L \to +\infty$, $H^L \to H$ in the sense of distribution on $\boldsymbol{T}^n$. Therefore $\hat{H}(m) = \lim_{L\to+\infty} \widehat{H^L}(m)$ $(m \in Z^n)$. Since $\phi_j(m) = 0$ except for a finite number of $m \in Z^n$ by our assumption for $\{\phi_j\}$ supposed at the begining,

$$(\tilde{T}_{\phi_j|Z^n} H)(x) = \sum_{\text{finite}} \phi_j(m)\hat{H}(m)\, e^{im\cdot x} = \lim_{L\to+\infty} (\tilde{T}_{\phi_j|Z^n} H^L)(x).$$

Therefore $(\tilde{T}_J^* H)(x) = \lim_{L\to+\infty} (\tilde{T}_J^* H^L)(x)$. Letting $L \to +\infty$ in (31) and lastly $J \to +\infty$, then we have

$$\Lambda_{\tilde{T}^* H}(2^n t) \le C(n,p,v)\left(\frac{N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\})}{t} \|F\|_{H^p(T^n)}\right)^p. \tag{32}$$

Combining the inequalities (32) and (27), we get

$$\Lambda_{\tilde{T}^*F}((1+2^n)t) \le C(n,p,v)([N_{L^2(R^n),wL^2(R^n)}(\{\phi_j\}) + N_{H^p(R^n),wL^p(R^n)}(\{\phi_j\})]t^{-1}\|F\|_{H^p(T^n)})^p$$

and complete the proof.

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