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An Explicit Formula of Subblock Occurrences for the p-Adic Expansion

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Let $b(n, w)$ be the number of occurrences of subblock w in the p-adic expansion of $n \in N$ and set $B(N, w) = \Sigma_{n=0}^{N} b(n, w)$ for $N \in N$. Properties of the value of $B(N, w)$ were investigated by Prodinger [8] (for $p = 2$) and by Kirschenhofer [3] (for a general p). In this paper we give a simple representation of $B(N, w)$ by means of previous result [5] on the explicit formula of generalized power and exponential sums of digital sums.

KEYWORDS: digital sum problem, singular function, multinomial measure

1 Introduction

Let p be a positive integer greater than 1 and denote the p-adic expansion of $n \in N$ by $n = \Sigma_{i \geq 0} \alpha_i(n) p^i$, where $\alpha_i(n) \in \{0, 1, \cdots, p-1\}$. We set $s(n, l)_{(p)} = \Sigma_{i \geq 0} \mathbf{1}_{\{\alpha_i(n) = l\}}$ for $l = 1, 2, \cdots, p-1$, and $s(n) = \Sigma_{l=1}^{p-1} ls(n, l)_{(p)}$. We define the power sum and the exponential sum of $s(n)$ by

$$S_k(N) = \sum_{n=0}^{N-1} s(n)^k, \quad k \in N,$$

$$F(\xi, N) = \sum_{n=0}^{N-1} e^{\xi s(n)}, \quad \xi \in \boldsymbol{R}$$

for $N \in N$. The problems concerned with $S_k(N)$, $F(\xi, N)$ are called digital sum problems and investigated by many authors. For historical survey, see [10], [7]. Trollope [11] obtained an explicit formula of $S_1(N)$ for $p = 2$ and Delange [2] gave its elegant proof by use of the Takagi function. Coquet [1] studied an explicit formula of $S_k(N)$ for $k \geq 2$, $p = 2$ and obtained an explicit one. For an explicit formula of $F(\xi, N)$, Stein [9] gave a one. In [4], we have obtained explicit formulas of $S_k(N)$ and an explicit formula of $F(\xi, N)$ by use of a probabilistic method.

In [5], we have introduced a generalization of $S_k(N)$ and $F(\xi, N)$, which contain information per digit and obtained explicit formulas of them. We will apply these results to counting the number of occurrences of subblocks of digits.

Let $w = (a_{d-1}, a_{d-2}, \cdots, a_0)$, $a_i \in \{0, 1, \cdots, p-1\}$, $i = 0, 1, \cdots, d-1$, be a subblock of digits (a word) with length $d > 1$. Set $q = p^d$ and $\tilde{w} = a_{d-1}p^{d-1} + a_{d-2}p^{d-2} + \cdots + a_0$, that is, $\tilde{w}$ is a numeric number of w. For a given word w and $N \in N$, we set

(1)
$$B(N, w) = \sum_{n=0}^{N-1} b(n, w),$$

where

(2)
$$b(n, w) = \sum_{i \geq d} \mathbf{1}_{\{(\alpha_{i-1}(n), \alpha_{i-2}(n), \cdots, \alpha_{i-d}(n)) = w\}}.$$

The following theorem was obtained by Prodinger [8] for $p = 2$ and by Kirschenhofer [3] for a general p.

Theorem 1.1 *Let $w = (a_{d-1}, a_{d-2}, \cdots, a_0)$ be a word with length d and $a_{d-1} \neq 0$. Then we have*

$$\frac{1}{N} B(N, w) = \frac{\log_p N - (d-1)}{q} + H_w(\log_p N) + \frac{E_w(\log_p N)}{N}.$$

Here H_w is a continuous periodic function of period 1 with $H_w(0) = 0$ and E_w is a bounded function such that

$$-(1 - p^{1-d}) \frac{p^{-d}(p^d - \tilde{w} - 1)}{p-1} \leq E_w \leq (1 - p^{1-d}) \frac{p^{-d}\tilde{w}}{p-1}.$$

In this paper, we give a simple representation of H_w and E_w. If $w = (a_{d-1}, a_{d-2}, \cdots, a_0)$ with $a_{d-1} = 0$, the

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above definition of $b(n, w)$ also counts the subblocks such that $(\alpha_{i-1}(n), \alpha_{i-2}(n), \cdots, \alpha_{i-d}(n)) = w$ with $i > [\log_p n] + 1$. So we introduce the following definitions:

$$\bar{b}(n, w) = \sum_{[\log_p n]+1 \geq i \geq d} \mathbf{1}_{\{(\alpha_{i-1}(n), \alpha_{i-2}(n), \cdots, \alpha_{i-d}(n)) = w\}} \tag{3}$$

and

$$\bar{B}(N, w) = \sum_{n=0}^{N-1} \bar{b}(n, w), \tag{4}$$

which does not count such blocks. We also give an explicit formula of $\bar{B}(N, w)$.

2 Preliminaries

For $n = \Sigma_{i \geq 0} \alpha_i(n) p^i$ and $l = 1, 2, \cdots, p-1$, set

$$s(n, l)_{(p)} = \sum_{i \geq 0} \mathbf{1}_{\{\alpha_i(n) = l\}}$$

and

$$s(n)_{(p)} = (s(n, 1)_{(p)}, s(n, 2)_{(p)}, \cdots, s(n, p-1)_{(p)}).$$

We define

$$S_k(N)_{(p)} = \sum_{n=0}^{N-1} s(n)_{(p)}^k = \sum_{n=0}^{N-1} \prod_{l=1}^{p-1} s(n, l)_{(p)}^{k_l}, \quad \boldsymbol{k} = (k_1, \cdots, k_{p-1}) \in \boldsymbol{N}^{p-1}, \tag{5}$$

and

$$F(\xi, N)_{(p)} = \sum_{n=0}^{N-1} e^{\langle \xi, s(n)_{(p)} \rangle} = \sum_{n=0}^{N-1} e^{\Sigma_{l=1}^{p-1} \xi_l s(n, l)_{(p)}}, \quad \xi = (\xi_1, \cdots, \xi_{p-1}) \in \boldsymbol{R}^{p-1}. \tag{6}$$

For later convenience, we set $\xi_0 = 0$.

Let $I = I_{0,0} = [0, 1]$ and $I_{n,j} = [j/p^n, (j+1)/p^n), j = 0, 1, \cdots, p^n - 2, I_{n,p^n-1} = [(p^n - 1)/p^n, 1]$ for $n = 1, 2, 3, \cdots$ Let $\boldsymbol{r} = (r_0, r_1, \cdots, r_{p-2})$ be a vector such that $0 < r_l < 1$ for $l = 0, 1, \cdots, p-2$ and $\Sigma_{l=0}^{p-2} r_l < 1$ and set $r_{p-1} = 1 - \Sigma_{l=0}^{p-2} r_l$. The probability measure μ_r on I defined by

$$\mu_r(I_{n+1, pj+l}) = r_l \mu_r(I_{n,j})$$

for $n = 0, 1, 2, \cdots, j = 0, 1, \cdots, p^n - 1, l = 0, 1, \cdots, p-1$, is said to be a multinomial measure. We denote the distribution function of μ_r by $L(\boldsymbol{r}, \cdot)$:

$$L(\boldsymbol{r}, x) = \mu_r([0, x]), \quad x \in I.$$

For $N \in \boldsymbol{N}$, set $t = \log_p N$, and denote its integer part by $[t]$ and its decimal part by $\{t\}$. We now set

$$a(\boldsymbol{m}, x, \xi_0) = \frac{\partial^{|\boldsymbol{m}|}}{\partial \xi_1^{m_1} \cdots \partial \xi_{p-1}^{m_{p-1}}} \left(\frac{1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}}{1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}}} \right)^x \Bigg|_{\xi = \xi_0}, \tag{7}$$

where $\boldsymbol{m} = (m_1, \cdots, m_{p-1}) \in \boldsymbol{Z}_+^{p-1}$, $|\boldsymbol{m}| = m_1 + \cdots + m_{p-1}$, $x \in \boldsymbol{R}$ and $\xi_0 = (\xi_{0,1}, \cdots, \xi_{0,p-1})$ and denote $(\partial^l / \partial x^l) a(\boldsymbol{m}, x, \xi_0)$ by $a^{(l)}(\boldsymbol{m}, x, \xi_0)$.

Theorem 2.1 ([5]) *We have*

$$\frac{\partial^{|\boldsymbol{k}|}}{\partial \xi_1^{k_1} \cdots \partial \xi_{p-1}^{k_{p-1}}} F(\xi, N)_{(p)} \Bigg|_{\xi = \xi_0} = (1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}})^t \times \sum_{l=0}^{|\boldsymbol{k}|} H_{\boldsymbol{k},l}(t, \xi_0) \left(\frac{t}{1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}}} \right)^l.$$

Here $H_{\boldsymbol{k},l}(x, \xi)$ is a continuous periodic function of period 1 *with respect to x defined by*

$$H_{\boldsymbol{k},l}(x, \xi_0) = (1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}})^l \times \sum_{j_1=0}^{(|\boldsymbol{k}|-l) \wedge k_1} \cdots \sum_{j_{p-1}=0}^{(|\boldsymbol{k}| - (j_1 + \cdots + j_{p-2}) - l) \wedge k_{p-1}} \times \binom{k_1}{j_1} \cdots \binom{k_{p-1}}{j_{p-1}} (1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}})^{1 - \{x\}}$$

$$\times \frac{1}{l!} a^{(l)}(k-j, 1-\{x\}, \xi_0) \frac{\partial^{|j|}}{\partial \xi_1^{j_1} \cdots \partial \xi_{p-1}^{j_{p-1}}} L\left(r, \frac{1}{p^{1-\{x\}}}\right)\Bigg|_{\xi=\xi_0}$$

with $r_l = e^{\xi_l}/(1 + e^{\xi_1} + e^{\xi_2} + \cdots + e^{\xi_{p-1}})$, $l = 0, 1, \cdots, p-1$. *In particular, we have*

$$S_k(N)_{(p)} = N \sum_{l=0}^{|k|} H_{k,l}(t, \mathbf{0}) \left(\frac{t}{p}\right)^l.$$

Corollary 2.1 *For the unit vector* $e_j = (0, \cdots, 0, \overset{(j)}{1}, 0, \cdots, 0)$, *we have*

$$S_{e_j}(N)_{(p)} = N\left(\frac{[t]+1}{p} + p^{1-\{t\}} \frac{\partial}{\partial \xi_j} L\left(r, \frac{1}{p^{1-\{t\}}}\right)\Bigg|_{\xi=0}\right). \tag{8}$$

3 Results

Theorem 3.1 *Let* $w = (a_{d-1}, a_{d-2}, \cdots, a_0)$ *be a word with length* d *and* $a_i \neq 0$ *for some* i. *Then we have*

$$\frac{1}{N} B(N, w) = \frac{\log_p N - (d-1)}{q} + H_w(\log_p N) + \frac{E_w(\log_p N)}{N}.$$

Here H_w *is a continuous periodic function of period* 1 *with respect to* t *defined by*

$$H_w(t) = -\frac{1}{q}\left(\sum_{i=0}^{d-1}\left\{\frac{t-i}{d}\right\} - \frac{d-1}{2}\right) + \sum_{i=0}^{d-1} q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right)\Bigg|_{\xi=0} + \frac{d}{q}$$

for $\xi = (\xi_1, \xi_2, \cdots, \xi_{q-1}) \in \boldsymbol{R}^{q-1}$, $r = (r_0, r_1, \cdots, r_{q-2})$ *with* $r_l = e^{\xi_l}/(1 + e^{\xi_1} + e^{\xi_2} + \cdots + e^{\xi_{q-1}})$, $l = 0, 1, \cdots, q-1$ *and*

$$E_w(t) = \frac{1}{q}\sum_{i=0}^{d-1}\left(\left(N - p^i\left[\frac{N}{p^i}\right]\right) - p^i \mathbf{1}_{\{p^d((N/p^i)-[N/p^i])>\tilde{w}\}}\right).$$

Theorem 3.2 *We have*

$$\frac{1}{N}\bar{B}(N, w) = \bar{H}_w(\log_p N) + \frac{\bar{E}_w(\log_p N)}{N}.$$

Here $\bar{H}_w$ *is a continuous periodic function of period* 1 *with respect to* t *defined by*

$$\bar{H}_w(t) = H_w(t) - \sum_{i=0}^{d-1}\left(\frac{q^{-\{(t-i)/d\}}}{q-1} + q^{-\{(t-i)/d\}}\{q^{\{(t-i)/d\}}\}\mathbf{1}_{\{[q^{\{(t-i)/d\}}]=\tilde{w}\}} + q^{-\{(t-i)/d\}}\mathbf{1}_{\{[q^{\{(t-i)/d\}}]>\tilde{w}\}}\right)$$

and

$$\bar{E}_w(t) = E_w(t) + \sum_{i=0}^{d-1} p^i\{q^{(t-i)/d}\}\mathbf{1}_{\{[q^{\{(t-i)/d\}}]=\tilde{w}\}} + \frac{1}{p-1}.$$

Remark 3.1 Delange [2] calculated the Fourier series of the periodic part of S_1 using the expansion of the Takagi function $T(x)$, that is,

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi(2^{n-1}x) \tag{9}$$

for $x \in [0, 1]$ where $\psi(x) = |2x - 2[x + 1/2]|$. Noticing that H_w has an expansion similar to (9), Prodinger [8] (for $p = 2$) and Kirschenhofer [3] (for a general p) gave a representation of the Fourier series of H_w by use of the z-function of Hurwitz. We have already shown that $(\partial/\partial\xi_i)L(r, x)|_{\xi=0}$ has an expansion similar to (9) in [6, Section 5]. So we can also calculate the Fourier series of H_w via our representation by means of a technique of Delange.

4 Proof of Theorems

In the following we use the notations $t_q = \log_q N = t/d$, $t(i) = \log_q(1 + [N/p^i])$ and $\overline{t(i)} = (\log_q [N/p^i])\mathbf{1}_{\{N \geq p^i\}}$ for $N \in \boldsymbol{N}$ and $i = 0, 1, \cdots, d-1$. We use the convention $\infty \cdot 0 = 0$.

Lemma 4.1 1) *We have*

(i) $[\overline{t(i)}] = \left[\frac{t-i}{d}\right]$,

(ii) $\left(1 + \left[\frac{N}{p^i}\right]\right) q^{1-\{t(i)\}}\mathbf{1}_{\{N\geq p^i\}} = \left[\frac{N}{p^i}\right] q^{[t(i)]-[\overline{t(i)}]+1-\{\overline{t(i)}\}}$,

(iii) $\left[\frac{N}{p^i}\right] q^{1-\{\overline{t(i)}\}} = \left(\frac{N}{p^i} q^{1-\{(t-i)/d\}}\right) \mathbf{1}_{\{N \geq p^i\}}$.

2) *One of the following statements holds:*

(i) $[\widehat{t(i)}] = [\overline{t(i)}]$ *for any* i.

(ii) $[\widehat{t(i)}] = [\overline{t(i)}] + 1$ *for some* i. *In this case, we have* $\{\widehat{t(i)}\} = 0$, $1 + [N/p^i] = q^{1+[(t-i)/d]}$ *and* $[\widehat{t(j)}] = [\overline{t(j)}]$ *for any* $j \neq i$.

Proof. 1) (i) It suffices to show the equality for $N \geq p^i$. Since $N = \Sigma_{k=0}^{[t]} \alpha_{[t]-k} p^{[t]-k}$, $\alpha_{[t]-k} \in \{0, 1, \cdots, p-1\}$ with $k = 0, 1, \cdots, [t]$ and $\alpha_{[t]} \neq 0$, we have

$$\left[\frac{N}{p^i}\right] = \sum_{k=0}^{[t]-i} \alpha_{[t]-k} p^{[t]-i-k}.$$

Then

$$\log_q \left[\frac{N}{p^i}\right] = \log_q \left(p^{[t]-i} \sum_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k}\right) = \frac{[t] - i + \log_p \Sigma_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k}}{d}.$$

As $0 \leq \log_p \Sigma_{k=0}^{[t]-i} (\alpha_{[t]-k})/p^k < 1$, we obtain

$$[\overline{t(i)}] = \left[\frac{[t] - i + \log_p \Sigma_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k}}{d}\right] = \left[\frac{[t] - i}{d}\right] = \left[\frac{t - i}{d}\right].$$

1) (ii), (iii) are obviously from definitions of $\widehat{t(i)}$, $\overline{t(i)}$ and 1) (i).

2) As $\log_q (1 + [N/p^i]) = \log_q (p^{[t]-i}(\Sigma_{k=0}^{[t]-i} (\alpha_{[t]-k})/p^k + 1/(p^{[t]-i})))$, we have

$$[\widehat{t(i)}] = \left[\frac{[t] - i + \log_p \left(\Sigma_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k} + \frac{1}{p^{[t]-i}}\right)}{d}\right].$$

If $[\widehat{t(i)}] = [\overline{t(i)}] + 1$ for some i, we know that $\log_p (\Sigma_{k=0}^{[t]-i} (\alpha_{[t]-k})/p^k + 1/(p^{[t]-i})) = 1$, and $[t] - i + 1$ divides by d. Then we have $\alpha_k = p - 1$ for $k = [t], [t] - 1, \cdots, i$ and $\{\widehat{t(i)}\} = 0$. Moreover, as

$$\left[\frac{[t] - i + 1}{d}\right] = \left[\frac{[t] - i}{d}\right] + 1 = \frac{[t] - i + 1}{d},$$

we obtain

$$1 + \left[\frac{N}{p^i}\right] = p^{[t]-i+1} = q^{1+[([t]-i)/d]} = q^{1+[(t-i)/d]}.$$

Since $[t] - j + 1$ does not divide by d for $j \neq i$, we derive the last assertion of (ii).

Proof of Theorem 3.1. Since

$$b(n, w) = \sum_{i=0}^{d-1} s\left(\left[\frac{n}{p^i}\right], \tilde{w}\right)_{(q)},$$

we have

$$B(N, w) = \sum_{i=0}^{d-1} \sum_{n=0}^{N-1} s\left(\left[\frac{n}{p^i}\right], \tilde{w}\right)_{(q)}. \tag{10}$$

We now calculate the right-hand side of (10). By Corollary 2.1, we can easily derive

$$S_{e_{\tilde{w}}}(N)_{(q)} = N\left(\frac{[t_q] + 1}{q} + q^{1-\{t_q\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L\left(r, \frac{1}{q^{1-\{t_q\}}}\right)\Big|_{\xi=0}\right). \tag{11}$$

By use of (11) and Lemma 4.1, we obtain for $N \geq p^i$

$$\begin{aligned}\sum_{n=0}^{N-1} s\left(\left[\frac{n}{p^i}\right], \tilde{w}\right)_{(q)} &= p^i S_{e_{\tilde{w}}}\left(1 + \left[\frac{N}{p^i}\right]\right)_{(q)} - \left(p^i - N + p^i\left[\frac{N}{p^i}\right]\right) s\left(\left[\frac{N}{p^i}\right], \tilde{w}\right)_{(q)} \\ &= p^i S_{e_{\tilde{w}}}\left(1 + \left[\frac{N}{p^i}\right]\right)_{(q)} - \left(p^i - N + p^i\left[\frac{N}{p^i}\right]\right)\left(S_{e_{\tilde{w}}}\left(1 + \left[\frac{N}{p^i}\right]\right)_{(q)} - S_{e_{\tilde{w}}}\left(\left[\frac{N}{p^i}\right]\right)_{(q)}\right)\end{aligned}$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right) S_{e_{\bar w}}\left(1 + \left[\frac{N}{p^i}\right]\right)_{(q)} + \left(p^i - N + p^i\left[\frac{N}{p^i}\right]\right) S_{e_{\bar w}}\left(\left[\frac{N}{p^i}\right]\right)_{(q)}$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right)\left(\frac{[\widehat{t(i)}] + 1}{q} + q^{1-\{\widehat{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}}\right)\Bigg|_{\xi=0}\right)$$

$$+ \left(p^i - N + p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right]\left(\frac{[\overline{t(i)}] + 1}{q} + q^{1-\{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\Bigg|_{\xi=0}\right)$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right)\frac{[\widehat{t(i)}] + 1}{q} + \left(p^i - N + p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right]\frac{[\overline{t(i)}] + 1}{q}$$

$$+ \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right) q^{1-\{\widehat{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}}\right)\Bigg|_{\xi=0}$$

$$+ \left(p^i - N + p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right] q^{1-\{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\Bigg|_{\xi=0}$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right)\frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} + \frac{N}{q}\left(\frac{t-i}{d} - \left\{\frac{t-i}{d}\right\} + 1\right)$$

$$+ \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right]\left(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}}\right)\Bigg|_{\xi=0}\right.$$

$$\left. - q^{1-\{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}}\right)\Bigg|_{\xi=0}\right)$$

$$+ Nq^{1-\{(t-i)/d\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right)\Bigg|_{\xi=0}$$

$$- Nq^{1-\{(t-i)/d\}}\frac{\partial}{\partial \xi_{\bar w}}\left(L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right) - L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\right)\Bigg|_{\xi=0}$$

$$+ \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right] q^{1-\{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}}\left(L\left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}}\right) - L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\right)\Bigg|_{\xi=0}.$$

For $N < p^i$, since $[N/p^i] = [\widehat{t(i)}] = [\overline{t(i)}] = 0$ and $L(r, 1/q^{1-\{(t-i)/d\}})|_{\xi=0} = -1/q^2$, same equality holds.

By the definition of L (for details, see [6, Section 2]), we have

$$Nq^{1-\{(t-i)/d\}}\frac{\partial}{\partial \xi_{\bar w}}\left(L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right) - L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\right)\Bigg|_{\xi=0}$$

$$= Nq^{1-\{(t-i)/d\}}\frac{\partial}{\partial \xi_{\bar w}}\left((r_0 + r_1 + \cdots + r_{k(i)})\left(L\left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}}\right) - L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\right)\right)\Bigg|_{\xi=0}$$

$$= p^i\frac{\partial}{\partial \xi_{\bar w}}(r_0 + r_1 + \cdots + r_{k(i)})\Bigg|_{\xi=0}$$

$$+ \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right] q^{1-\{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}}\left(L\left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}}\right) - L\left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right)\right)\Bigg|_{\xi=0}$$

where $k_{(i)} = p^d((N/p^i) - [N/p^i]) - 1$. Therefore we have

$$B(N, w) = \sum_{i=0}^{d-1}\left(\frac{N}{q}\left(\frac{t-i}{d} - \left\{\frac{t-i}{d}\right\} + 1\right) + \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right)\frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q}\right.$$

$$+ \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right]\left(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{\overline{t(i)}\}}\frac{\partial}{\partial \xi_{\bar w}} L\left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}}\right)\Bigg|_{\xi=0}\right.$$

$$- q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L \left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}} \right) \Bigg|_{\xi=0} \Bigg)$$

$$+ N q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}} \right) \Bigg|_{\xi=0}$$

$$- p^i \frac{\partial}{\partial \xi_{\tilde{w}}} (r_0 + r_1 + \cdots + r_{k(i)}) \Bigg|_{\xi=0} \Bigg).$$

We now break the proof up into two cases.

Case (i) Let $[\widehat{t(i)}] = [\overline{t(i)}]$ for any i. Then, since

$$\frac{1}{q^{1-\{\widehat{t(i)}\}}} = \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[\overline{t(i)}]}} = \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}},$$

we have

$$\left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right) \frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} + \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right] \Bigg(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L \left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}} \right) \Bigg|_{\xi=0}$$

$$- q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L \left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}} \right) \Bigg|_{\xi=0} \Bigg) = 0$$

for any i.

Case (ii) $[\widehat{t(i)}] = [\overline{t(i)}] + 1$ for some i. Then, by lemma 4.1, we have

$$\left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(1 + \left[\frac{N}{p^i}\right]\right) \frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} + \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left[\frac{N}{p^i}\right] \Bigg(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L \left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}} \right) \Bigg|_{\xi=0}$$

$$- q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L \left(r, \frac{1 + \left[\frac{N}{p^i}\right]}{q^{1+[(t-i)/d]}} \right) \Bigg|_{\xi=0} \Bigg)$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(\left(1 + \left[\frac{N}{p^i}\right]\right)\frac{1}{q} + \left[\frac{N}{p^i}\right] q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} \left(qL\left(r, \frac{1}{q}\right) - L(r, 1)\right)\Bigg|_{\xi=0}\right)$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(\left(1 + \left[\frac{N}{p^i}\right]\right)\frac{1}{q} + \left[\frac{N}{p^i}\right] q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\tilde{w}}} (qr_0 - 1)\Bigg|_{\xi=0}\right)$$

$$= \left(N - p^i\left[\frac{N}{p^i}\right]\right)\left(\left(1 + \left[\frac{N}{p^i}\right]\right)\frac{1}{q} - \left[\frac{N}{p^i}\right] q^{-\{\overline{t(i)}\}}\right) = 0$$

for i.

In any case, we have

$$B(N, w) = \sum_{i=0}^{d-1} \Bigg(\frac{N}{q}\left(\frac{t-i}{d} - \left\{\frac{t-i}{d}\right\} + 1\right) + N q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right)\Bigg|_{\xi=0}$$

$$- p^i \frac{\partial}{\partial \xi_{\tilde{w}}} (r_0 + r_1 + \cdots + r_{k(i)})\Bigg|_{\xi=0} \Bigg)$$

$$= N\left(\frac{t-(d-1)}{q} - \frac{1}{q}\left(\sum_{i=0}^{d-1}\left\{\frac{t-i}{d}\right\} - \frac{d-1}{2}\right) + \sum_{i=0}^{d-1} q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right)\Bigg|_{\xi=0} + \frac{d}{q}\right)$$

$$- \sum_{i=0}^{d-1} p^i \frac{\partial}{\partial \xi_{\tilde{w}}} (r_0 + r_1 + \cdots + r_{k(i)})\Bigg|_{\xi=0}.$$

As

$$\left.\frac{\partial}{\partial \xi_{\tilde{w}}}(r_0 + r_1 + \cdots + r_{k(i)})\right|_{\xi=0} = \left.\frac{\partial}{\partial \xi_{\tilde{w}}}\left(\frac{1 + e^{\xi_1} + \cdots + e^{\xi_{k(i)}}}{1 + e^{\xi_1} + \cdots + e^{\xi_q}}\right)\right|_{\xi=0}$$
$$= -\frac{1}{q}\left(\frac{N}{p^i} - \left[\frac{N}{p^i}\right]\mathbf{1}_{\{p^d((N/p^i)-[N/p^i])>\tilde{w}\}}\right),$$

we obtain our formula.

Proof of Theorem 3.2. Let

$$c(n, w) = \sum_{i>[\log_p n]+1} \mathbf{1}_{\{(\alpha_{i-1}(n), \alpha_{i-2}(n), \cdots, \alpha_{i-d}(n))=w\}},$$

and set

$$C(N, w) = \sum_{i=0}^{N-1} c(n, w).$$

Then, by an easy calculation, we have

$$C\left(\left[\frac{N}{p^i}\right], w\right) = \sum_{k=0}^{[\overline{t(i)}]-1} q^k + \left(\left[\frac{N}{p^i}\right] - q^{[\overline{t(i)}]}\left[\frac{\left[\frac{N}{p^i}\right]}{q^{[\overline{t(i)}]}}\right]\right)\mathbf{1}_{\{[q^{\{(t-i)/d\}}]=\tilde{w}\}} + q^{[\overline{t(i)}]}\mathbf{1}_{\{[q^{\{(t-i)/d\}}]>\tilde{w}\}}.$$

Therefore, we have

$$C(N, w) = \sum_{i=0}^{d-1} p^i C\left(\left[\frac{N}{p^i}\right], w\right)$$
$$= \sum_{i=0}^{d-1} p^i\left(\sum_{k=0}^{[\overline{t(i)}]-1} q^k + \left(\left[\frac{N}{p^i}\right] - q^{[\overline{t(i)}]}\left[\frac{\left[\frac{N}{p^i}\right]}{q^{[\overline{t(i)}]}}\right]\right)\mathbf{1}_{\{[q^{\{(t-i)/d\}}]=\tilde{w}\}} + q^{[\overline{t(i)}]}\mathbf{1}_{\{[q^{\{(t-i)/d\}}]>\tilde{w}\}}\right).$$

As $[[N/p^i]/q^{[\overline{t(i)}]}] = [(N/p^i)/q^{[(t-i)/d]}] = [q^{\{(t-i)/d\}}]$, we have

$$p^i\left(\left[\frac{N}{p^i}\right] - q^{[\overline{t(i)}]}\left[\left[\frac{N}{p^i}\right]\Big/q^{[\overline{t(i)}]}\right]\right) = p^i([q^{(t-i)/d}] - q^{[(t-i)/d]}[q^{\{(t-i)/d\}}])$$
$$= Nq^{-\{(t-i)/d\}}\{q^{\{(t-i)/d\}}\} - p^i\{q^{(t-i)/d}\}.$$

Moreover, as

$$\sum_{i=0}^{d-1}\sum_{k=0}^{[\overline{t(i)}]-1} q^k p^i = \frac{N}{q-1}\sum_{i=0}^{d-1} q^{-\{(t-i)/d\}} - \frac{1}{p-1},$$

we obtain our formula.

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