Product Structures of Networks and Their Spectra

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Product Structures of Networks and Their Spectra

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Contents

1 Introduction

In the real world, there are many complex networks, such as traffic networks, social networks, biological networks, etc. Such networks are expressed in terms of graph theory. In this paper we are interested in spectral properties of graphs G and digraphs $\overrightarrow{G}$, which are expected to possess important structural information of networks.

The adjacency matrix A is used to represent the adjacency relation of a graph G. The spectrum of a graph G is the table of numbers which are eigenvalues and their multiplicities of A. The eigenvalues of the adjacency matrix A can be obtained by solving the eigenvalue problem $Ax = \lambda x$, or the characteristic equation $\det(\lambda I - A) = 0$ [2, 3, 7, 8, 10].

Though matrix theory helps to understand complex networks easily, it is hard to compute such matrices if the number of vertices is large. Therefore, it is desirable to develop a method for computing spectrum of a large graph G by means of its smaller components. In this sense, certain product structures are within our scope. In this paper, we focus on the Comb product of graphs and Manhattan product of digraphs.

In Chapter 3, we discuss the comb product of graphs, which is a relatively new concept introduced in the context of random walks and quantum physics [1, 4, 11, 12, 13]. Let $G \triangleright P_n$ be the comb product of G and P_n, where P_n is a path graph with n vertices. The spectrum of the comb product of $G \triangleright P_n$ can be computed by using the following theorem:

Theorem 3.3 *Suppose the spectrum of a graph G is given by*

$$\mathrm{Spec}\, G = \begin{pmatrix} \cdots & \mu_k & \cdots \\ \cdots & m_k & \cdots \end{pmatrix}.$$

Then the spectrum of $G \triangleright P_n$ is

$$\mathrm{Spec}\, G \triangleright P_n = \begin{pmatrix} \cdots & \lambda_1(\mu_k) & \cdots & \lambda_n(\mu_k) & \cdots \\ \cdots & m_k & \cdots & m_k & \cdots \end{pmatrix},$$

where $\lambda_1(\mu) < \ldots < \lambda_n(\mu)$ are the solutions of

$$\mu = \frac{\varphi_n(\lambda)}{\varphi_{n-1}(\lambda)},$$

where $\varphi_n(\lambda)$ is the characterisitic polynomial of P_n. (In fact, $\varphi_n(\lambda)$ is essentially the Chebyshev polynomials of the second kind.)

We next study the Manhattan product graph, the idea of which was originated from the Manhattan street network [5, 9]. The Manhattan street network is made up of one-way alternate streets which can be found in the cities of New York and Barcelona. The spectra of this network has been studied in [6]. According to the result presented in [6], we know that the geometric multiplicity of the eigenvalue zero have more than half multiplicity in case of two-dimensional.

In Chapter 4, we study the spectra of Manhattan product of digraphs and the properties of zero eigenvalue. We derive a sufficient condition for $K_{V_1,V_2}\sharp G$ to have zero-eigenvalue in terms of connection matrix of a bipartite digraph K_{V_1,V_2}.

Theorem 4.2 *Let K_{V_1,V_2} be the complete bipartite digraph and $A = [a_{uv}]_{u\in V_1, v\in V_2}$ the matrix defined by $a_{uv} = 1$ for $u \to v$. Then spectrum of $G = K_{V_1,V_2}\sharp C_2$ contains 0 as an eigenvalue if and only if*

$$\mathrm{Spec}\,(A^t A) \ni 1 \ \textit{or}\ \mathrm{Spec}\,((J-A)(J-A)^t) \ni 1,$$

where J is the matrix with all entries being one.

For the full spectrum of $C_2\sharp P_n$ we obtain the following:

Theorem 4.6 *The spectrum of $C_2\sharp P_n$ is given by*

$$\mathrm{Spec}\ C_2\sharp P_n = \begin{pmatrix} 0 & 2\cos\frac{k\pi}{n+2} \\ n-1 & 1 \end{pmatrix}, \quad k = 1, 2, \dots, n+1.$$

The spectrum of $C_2\sharp P_n$ is obtained by solving the characteristic equation. To calculate this spectrum, we apply the Chebyshev polynomials of second kind. From the spectrum of $C_2\sharp P_n$, we can compute explicitly the asymptotic spectral distribution of $G_2\sharp P_n$, as $n \to \infty$.

Theorem 4.7 *The asymptotic spectral distribution of $C_2\sharp P_n$ as $n \to \infty$ is given by*

$$\frac{1}{2}\delta_0 + \frac{1}{2}\rho(x)dx,$$

where

$$\rho(x) = \begin{cases} \dfrac{1}{\pi\sqrt{4-x^2}}, & -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

We still have potential areas for open problem. In Chapter 4, we studied zero eigenvalue property of $K_{V_1,V_2}\sharp G$. We would like to know all the eigenvalues of $K_{V_1,V_2}\sharp G$. We calculated spectrum of $C_2\sharp P_n$ mainly using by the Chebyshev polynomials of second kind. Extending this method, we expect to get the spectrum of $C_{2m}\sharp P_n$, where C_{2m} is a directed cycle with $2m$ vertices.

2 Preliminaries

2.1 Graphs and Digraphs

In this section, we recall basic properties of networks under study. With this aim we begin with some backgrounds on graphs and digraphs and their spectra.

Definition 1. A *graph* $G = (V, E)$ is a pair of sets, where V is a set of vertices and E is a set of unordered pairs of vertices of V. We say that v_i and v_j are adjacent and write $v_i \sim v_j$ if an edge $\{v_i, v_j\}$ belongs to E.

Definition 2. A *directed graph* (or *digraph*) $\overrightarrow{G} = (V, E)$ is a pair of sets, where V is a set of vertices and E is a set of ordered pairs of vertices. We say that v_i and v_j are adjacent from v_i to v_j and write $v_i \to v_j$ if an arc (v_i, v_j) belongs to E.

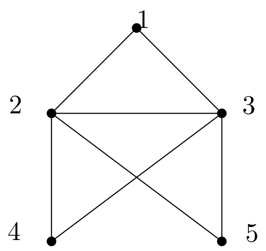
In general, we consider a graph as an undirected graph and a graph G will be identified with a symmetric digraph $\overrightarrow{G}$ in a natural manner.

Definition 3. The adjacency matrix $A = [a_{ij}]$ of a graph G is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j \ , \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4. The adjacency matrix $A = [a_{ij}]$ of a digraph $\overrightarrow{G}$ is defined by

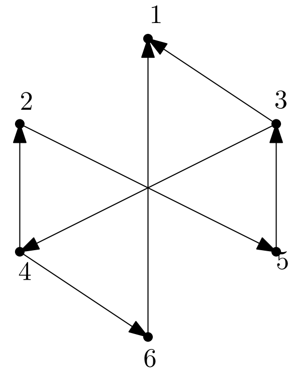
$$a_{ij} = \begin{cases} 1, & \text{if } v_i \to v_j \ , \\ 0, & \text{otherwise.} \end{cases}$$



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$V = \{1, 2, 3, 4, 5\}, \quad E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\}$

Figure 1: Graph and its adjacency matrix



$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \{1, 2, 3, 4, 5, 6\}, \quad E = \{(2, 5), (3, 1), (3, 4), (4, 2), (4, 6), (5, 3)\}$$

Figure 2: Digraph and its adjacency matrix

2.2 Characteristic Polynomials

Let $G = (V, E)$ be a graph with $|V| = n$. The characteristic polynomial of G, denoted by $\varphi_G(\lambda)$, is defined by

$$\varphi_G(\lambda) = \det(\lambda I - A),$$

where A is the adjacency matrix of G. $\varphi_G(\lambda)$ can be written in the form:

$$\varphi_G(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \ldots + c_n.$$

Proposition 2.1. The coefficients of the characteristic polynomial of a graph G satisfy:

(1) $c_1 = 0$;

(2) $-c_2$ is the number of edges of G;

(3) $-c_3$ is twice the number of triangles in G.

Proof. We follow the argument in [2]. For each $i \in \{1, 2, \ldots, n\}$, the number of $(-1)^i c_i$ is the sum of those principal minors of A which have i rows and columns.
(1) Since the diagonal elements of A are all zero, we have $c_1 = 0$.
(2) A principal minor with two rows and columns, and which has a non-zero entry, must be of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

There is one such minor for each pair of adjacent vertices of G, and each has value -1. Hence $(-1)^2 c_2 = -|E|$, giving the result.
(3) There are essentially three possibilities for non-trivial principal minors with three rows and columns:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and, the only non-zero one is the last. This principal minor corresponds to three mutually adjacent vertices in G, and so we have the required description of c_3. □

Example 2.2.

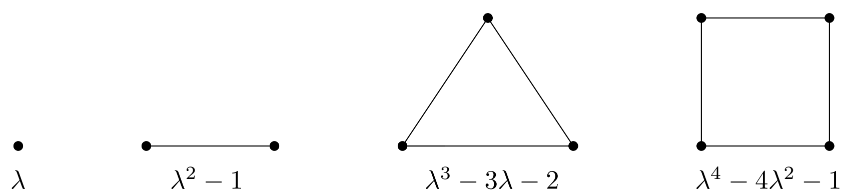


Figure 3: Examples of characteristic polynomials

2.3 Spectrum

Let A be the adjacency matrix of a graph G. We can obtain the eigenvalues of A by solving the eigenvalue problem $Ax = \lambda x$, or the characteristic equation $\det(\lambda I - A) = 0$.

Definition 5. Let A be the adjacency matrix of G. The spectrum of a graph G, Spec G, is the table of numbers which are eigenvalues and the multiplicities of the eigenvalues of A. If the distinct eigenvalues of A are $\lambda_1 > \lambda_2 > \ldots > \lambda_s$, and their multiplicities are $m(\lambda_1)$, $m(\lambda_2)$, ... , $m(\lambda_s)$, respectively, we write

$$\text{Spec } G = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_s \\ m(\lambda_1) & m(\lambda_2) & \ldots & m(\lambda_s) \end{pmatrix}.$$

The spectrum of a graph $\overrightarrow{G}$, Spec $\overrightarrow{G}$, is defined in a similar way.

Since the adjacency matrix of G is symmetric, the multiplicity of an eigenvalue is obtained from the characteristic polynomial, i.e., the algebraic multiplicity, which coincides with the geometric multiplicity. In case of a digraph, since the adjacency matrix of $\overrightarrow{G}$ is not necessarily symmetric, the multiplicity of a digraph is defined to be the dimension of the associated eigenspace, i.e., the geometric multiplicity. The geometric multiplicity of an eigenvalue λ of a matrix A does not exceed its algebraic multiplicity.

We present here some results on spectra of simple graphs.

Definition 6. A graph is called *complete* if all vertices are adjacent to others. A complete graph with n vertices is denoted by K_n.

Theorem 2.3. *The spectrum of K_n is given by*

$$\mathrm{Spec}\, K_n = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

Proof. The adjacency matrix of K_n is written in the form:

$$A = \begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 \\ 1 & \dots & 1 & 1 & 0 \end{bmatrix}.$$

We observe that

$$\begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 \\ 1 & \dots & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} n-1 \\ n-1 \\ \vdots \\ n-1 \\ n-1 \end{bmatrix} = (n-1) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}.$$

This implies that $(n-1)$ is an eigenvalue. On the other hand, we have

$$\begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 \\ 1 & \dots & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence (-1) is an eigenvalue with multiplicity at least $n-1$. Combining the above observation we conclude that the eigenvalues $n-1$ and -1 have multiplicities 1 and $n-1$, respectively. $\square$

Definition 7. A *path* with n vertices is a graph $P_n = (V, E)$ of the form: $V = \{v_1, v_2, \dots, v_n\}$, $E = \{v_i v_j; 1 \le i \le n-1\}$.

Theorem 2.4. *Let $\varphi_{P_n}(\lambda) = \varphi_n(\lambda)$ be the characteristic polynomial of the path P_n. Then it holds that*

$$\begin{aligned} \varphi_1(\lambda) &= \lambda \\ \varphi_2(\lambda) &= \lambda^2 - 1 \\ \varphi_{n+1}(\lambda) &= \lambda\varphi_n(\lambda) - \varphi_{n-1}(\lambda), \qquad n \ge 2. \end{aligned}$$

Proof. The adjacency matrix A of P_n is represented by

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

We see that the characteristic polynomials of P_1 and P_2 are given by

$$\begin{aligned} \varphi_1(\lambda) &= \lambda \\ \varphi_2(\lambda) &= \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1. \end{aligned}$$

For $n \ge 3$ we have

$$\begin{aligned} \varphi_n(\lambda) &= \begin{vmatrix} \lambda & -1 & & & \\ -1 & \lambda & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \lambda & -1 \\ & & & -1 & \lambda \end{vmatrix} \\ &= \lambda\varphi_{n-1}(\lambda) + \begin{vmatrix} -1 & -1 & & & & \\ & \lambda & -1 & & & \\ & -1 & \lambda & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & \lambda & -1 \\ & & & & -1 & \lambda \end{vmatrix} \\ &= \lambda\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda), \end{aligned}$$

which completes the proof. $\square$

The Chebyshev polynomials of the second kind are defined by the trigonometric identity. First we observe that

$$\begin{aligned}
\sin 2\theta &= \sin\theta(2\cos\theta)\\
\sin 3\theta &= \sin\theta(4\cos^2\theta - 1)\\
\sin 4\theta &= \sin\theta(8\cos^3\theta - 4\cos\theta)\\
&\vdots\\
\sin(n+1)\theta &= \sin\theta(\text{polynomial of } \cos\theta).
\end{aligned}$$

Definition 8. For $n \geq 0$, the polynomial $U_n(x)$ defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

is called the Chebyshev polynomial of the second kind of degree n. Then the Chebyshev polynomials of the second kind fulfill the recurrence relation:

$$\begin{aligned}
U_0(\lambda) &= 1\\
U_1(\lambda) &= 2\lambda\\
U_{n+1}(\lambda) &= 2\lambda U_n(\lambda) - U_{n-1}(\lambda). \qquad (2.1)
\end{aligned}$$

Theorem 2.5. *The characteristic polynomial of the path P_n is given by $U_n(\lambda/2)$.*

Proof. Set

$$\tilde{U}_n(\lambda) = U_n\left(\frac{\lambda}{2}\right).$$

Then equation (2.1) gives rise to

$$\tilde{U}_{n+1}(\lambda) = \lambda\tilde{U}_n(\lambda) - \tilde{U}_{n-1}(\lambda), \qquad (2.2)$$

$$\tilde{U}_0(\lambda) = 1,\ \ \tilde{U}_1(\lambda) = \lambda,\ \ \tilde{U}_2(\lambda) = \lambda^2 - 1.$$

Thus $\tilde{U}_n(\lambda)$ and $\varphi_n(\lambda)$ satisfy the same recurrence relation with the same initial condition. Hence

$$\tilde{U}_n(\lambda) = \varphi_n(\lambda),$$

as desired. $\square$

Theorem 2.6. *The spectrum of the path P_n is given by*

$$\operatorname{Spec} P_n = \begin{pmatrix} 2\cos\frac{k\pi}{n+1} \\ 1 \end{pmatrix}, \quad k = 1, 2, \ldots, n.$$

Proof. Let us solve the characteristic equation:

$$\det(\lambda I - A) = \varphi_n(\lambda) = U_n\left(\frac{\lambda}{2}\right) = 0.$$

Note first that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = 0$$

is satisfied for

$$\theta = \frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n.$$

Moreover, for these θ's the values $\cos\theta$ are all distinct. Hence the zeroes of $\varphi_n(\lambda)$ are given by

$$\lambda = 2\cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \dots, n,$$

and the multiplicities are all one. □

Theorem 2.7. *The asymptotic spectral distribution of P_n as $n \to \infty$ is given by $\rho(x)dx$, where*

$$\rho(x) = \begin{cases} \dfrac{1}{\pi\sqrt{4-x^2}}, & -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The probability density $\rho(x)dx$ is called the arcsine law.

Proof. For a continuous function $f(x)$ we set

$$S_n = \frac{1}{n}\sum_{k=1}^{n} f\left(2\cos\frac{k\pi}{n+1}\right).$$

We will compute $\lim_{n\to\infty} S_n$. Let $F(t) = f(2\cos t\pi)$. Then,

$$\begin{aligned} S_n &= \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n+1}\right) \\ &= \frac{n+1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n+1}\right)\frac{1}{n+1} \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} F\left(\frac{k}{n+1}\right)\frac{1}{n+1}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n+1} F\left(\frac{k}{n+1}\right)\frac{1}{n+1}.$$

By the definition of Riemannian integral,

$$\lim_{n\to\infty} S_n = \int_0^1 F(t)dt$$
$$= \int_0^1 f(2\cos t\pi)dt.$$

Let $2\cos t\pi = x$. Then,

$$\lim_{n\to\infty} S_n = \int_{-2}^{2} \frac{f(x)}{-2\pi\sin t\pi} dx$$
$$= \int_{-2}^{2} f(x)\cdot \frac{1}{2\pi\sqrt{1-(\frac{x}{2})^2}}\,dx$$
$$= \int_{-2}^{2} f(x)\cdot \frac{1}{\pi\sqrt{4-x^2}}\,dx$$

Therefore, we see that the asymptotic spectrum distribution of P_n as $n \to \infty$ is given by $\rho(x)dx$. $\square$

2.4 Products of Graphs

Suppose that a graph G is composed of a 'product' of two graphs G_1 and G_2. From the spectra of G_1 and G_2, we can expect to obtain the spectrum of G. In this section, we study the direct sum and direct product of graphs using simple examples.

Definition 9. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$. A *direct sum* is defined to be a graph $G = (V, E)$, where $V = V_1 \cup V_2$, $E = E_1 \cup E_2$. We write $G = G_1 \cup G_2$.

Proposition 2.8. Let the spectra of G_1 and G_2 be given by

$$\text{Spec } G_1 = \begin{pmatrix} \lambda_1' & \dots & \lambda_s' \\ m(\lambda_1') & \dots & m(\lambda_s') \end{pmatrix},$$

$$\text{Spec } G_2 = \begin{pmatrix} \lambda_1^{''} & \dots & \lambda_t^{''} \\ m(\lambda_1^{''}) & \dots & m(\lambda_t^{''}) \end{pmatrix},$$

respectively. Then the spectrum of G is given by

$$\text{Spec } G = \begin{pmatrix} \lambda_1^{'} & \dots & \lambda_s^{'} & \lambda_1^{''} & \dots & \lambda_t^{''} \\ m(\lambda_1^{'}) & \dots & m(\lambda_s^{'}) & m(\lambda_1^{''}) & \dots & m(\lambda_t^{''}) \end{pmatrix}.$$

Proof. Let A_1 and A_2 be the adjacency matrices of graphs G_1 and G_2, respectively. The adjacency matrix A of G is represented by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Then by the definition of characteristic polynomial we have

$$\varphi_G(\lambda) = \begin{vmatrix} \lambda I_1 - A_1 & 0 \\ 0 & \lambda I_2 - A_2 \end{vmatrix} = \det(\lambda I_1 - A_1)\det(\lambda I_2 - A_2).$$

Thus, we can easily know the spectrum of G. $\square$

Definition 10. We set $V = V_1 \times V_2$, where V_1 and V_2 are the sets of vertices of G_1 and G_2, respectively, and

$$E = \left\{ \{(x,y),(x^{'},y^{'})\} \ ; \begin{array}{c} (1)\ x = x^{'}, y \sim y \\ \text{or}\ \ (2)\ x \sim x,\ y = y^{'} \end{array} \right\}$$

Then $G = (V, E)$ is called the *direct product* of G_1 and G_2, and is denoted by $G = G_1 \times G_2$.

Proposition 2.9. Let the spectrum of G_1 and G_2 be given by

$$\text{Spec } G_1 = \begin{pmatrix} \lambda_1^{'} & \cdots & \lambda_s^{'} \\ m(\lambda_1^{'}) & \cdots & m(\lambda_s^{'}) \end{pmatrix},$$

$$\text{Spec } G_2 = \begin{pmatrix} \lambda_1^{''} & \cdots & \lambda_t^{''} \\ m(\lambda_1^{''}) & \cdots & m(\lambda_t^{''}) \end{pmatrix},$$

respectively. Then the spectrum of G is given by

$$\text{Spec } G = \begin{pmatrix} \lambda_1^{'} + \lambda_1^{''} & \cdots & \lambda_1^{'} + \lambda_t^{''} & \cdots & \lambda_s^{'} + \lambda_1^{''} & \cdots & \lambda_s^{'} + \lambda_t^{''} \\ m(\lambda_1^{'})m(\lambda_1^{''}) & \cdots & m(\lambda_1^{'})m(\lambda_t^{''}) & \cdots & m(\lambda_s^{'})m(\lambda_1^{''}) & \cdots & m(\lambda_s^{'})m(\lambda_t^{''}) \end{pmatrix}.$$

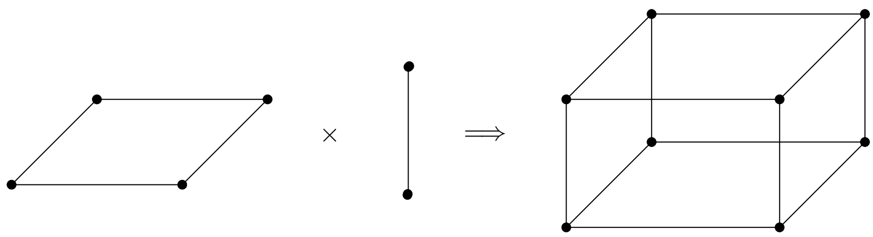
Proof. Let A_1, A_2 be the adjacency matrix of graphs G_1, G_2 respectively. Then the adjacency matrix A of the direct product G is in the following form:

$$A = A_1 \otimes I_2 + I_1 \otimes A_2,$$

where I_i is the identity matrices of the same order as A_i. If u and v are eigenvectors for A_1 and A_2 with eigenvalues of $\lambda^{'}$ and $\lambda^{''}$, respectively, then the vector $u \otimes v$ is an eigenvector of A. Indeed,

$$\begin{aligned} A(u \otimes v) &= (A_1 \otimes I_2 + I_1 \otimes A_2)(u \otimes v) \\ &= A_1 u \otimes v + u \otimes A_2 v \\ &= \lambda^{'} u \otimes v + u \otimes \lambda^{''} v \\ &= (\lambda^{'} + \lambda^{''})(u \otimes v). \end{aligned}$$

Hence we see that $\lambda^{'} + \lambda^{''}$ is an eigenvalue of G. Consequently, the spectrum of G consists of the sum of all possible pairs of eigenvalues of G_1 and G_2. □



C_4 $\qquad$ P_2 $\qquad$ $C_4 \times P_2$

Figure 4: Direct product

Example 2.10. Let G be the direct product of C_4 and P_2. The spectrum of C_4 is

$$\text{Spec } C_4 = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

The spectrum of P_2 is

$$\text{Spec } P_2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The spectrum of $G = C_4 \times P_2$ is

$$\text{Spec } G = \begin{pmatrix} -3 & -1 & 1 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

3 Comb Product Graphs

3.1 Definition and Structures

In this section we discuss the comb product of graphs, which is a relatively new concept introduced in the context of quantum physics. We refer to the mathematical formulation given in [11].

Definition 11. Let G_1 and G_2 be two graphs and assume that the second graph is given a distinguished vertex $o \in V(G_2)$. The *comb product graph* G is defined as a subgraph of $G_1 \times G_2$, obtained by grafting a copy of G_2 at the vertex o into each vertex of G_1. The comb product is denoted by $G = G_1 \triangleright G_2$.

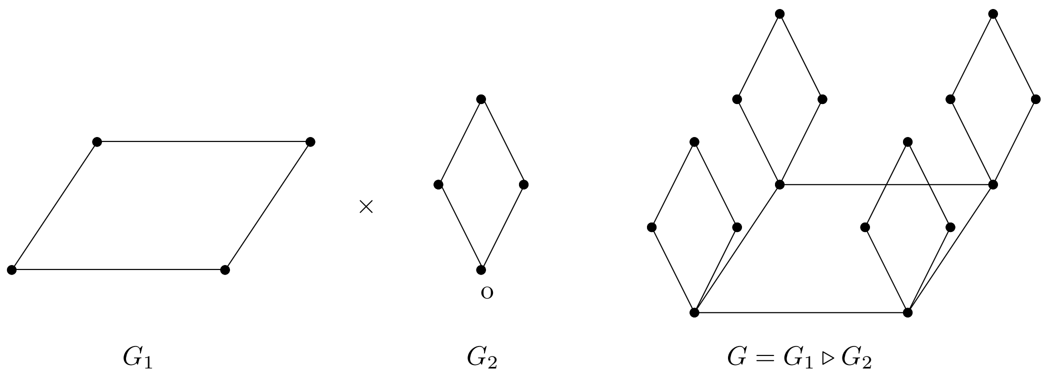


Figure 5: Comb product

3.2 Spectrum of Comb Graphs (A Simple Case)

In this section we derive the spectrum $G \triangleright P_2$, where G is an arbitrary graph and P_2 is the path of two vertices. Note that if the number of vertices of G is n, then that of $G \triangleright P_2$ is $2n$.

Theorem 3.1. *Let $G \triangleright P_2$ be the comb product graph with G being any graph and P_2 a path with two vertices. If the spectrum of G is given by*

$$\text{Spec } G = \begin{pmatrix} \mu_1 & \dots & \mu_s \\ m_1 & \dots & m_s \end{pmatrix},$$

then the spectrum of $G \triangleright P_2$ is

$$\text{Spec } G \triangleright P_2 = \begin{pmatrix} \lambda_-(\mu_1) & \lambda_+(\mu_1) & \dots & \lambda_-(\mu_s) & \lambda_+(\mu_s) \\ m_1 & m_1 & \dots & m_s & m_s \end{pmatrix},$$

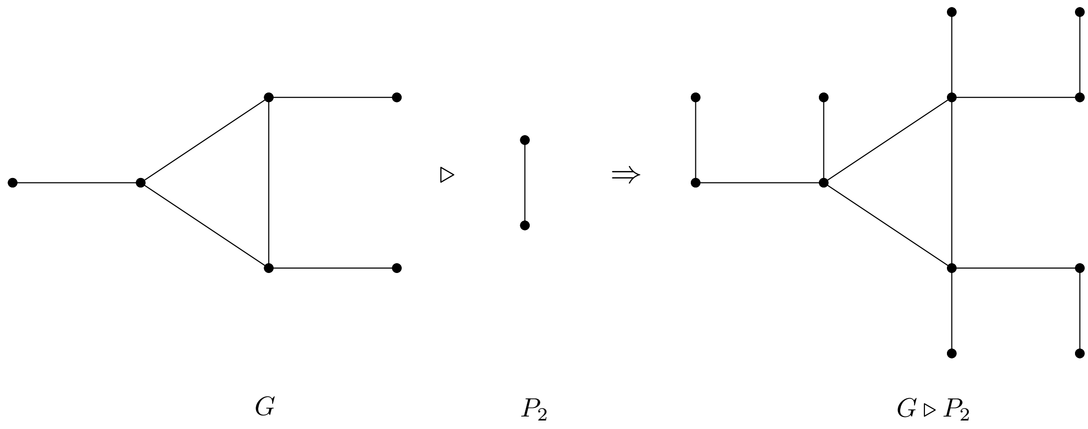


Figure 6: Comb product graph $G \rhd P_2$

where

$$\lambda_{\pm}(\mu) = \frac{\mu \pm \sqrt{\mu^2 + 4}}{2}.$$

Moreover, $\lambda_{\pm}(\mu_1), \ldots, \lambda_{\pm}(\mu_s)$ *are all distinct.*

Proof. Let A be the adjacency matrix of G and $\tilde{\mathbf{A}}$ be the adjacency matrix of $G \rhd P_2$. Then,

$$\tilde{\mathbf{A}} = \begin{bmatrix} A & I \\ I & 0 \end{bmatrix}.$$

We will obtain the eigenvalues of $G \rhd P_2$ by solving the eigenvalue problem:

$$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda \begin{bmatrix} X \\ Y \end{bmatrix},$$

which may be rewritten as

$$\begin{cases} AX + IY = \lambda X, \\ IX = \lambda Y. \end{cases} \tag{3.1}$$

Suppose that μ is an eigenvalue of A satisfying $AX = \mu X, X \neq 0$. Then (3.1) becomes

$$\mu \lambda Y + Y = \lambda^2 Y.$$

Since $Y \neq 0$, we have

$$\lambda^2 - \mu\lambda - 1 = 0.$$

Namely,

$$\lambda = \frac{\mu \pm \sqrt{\mu^2+4}}{2}.$$

We set

$$\lambda_+ = \frac{\mu + \sqrt{\mu^2+4}}{2}, \quad \lambda_- = \frac{\mu - \sqrt{\mu^2+4}}{2}.$$

Since $\lambda_\pm(\mu)$ is a monotone function in μ and $\lambda_- < 0 < \lambda_+$, we see that $\lambda_\pm(\mu_1), \dots, \lambda_\pm(\mu_s)$ are distinct. □

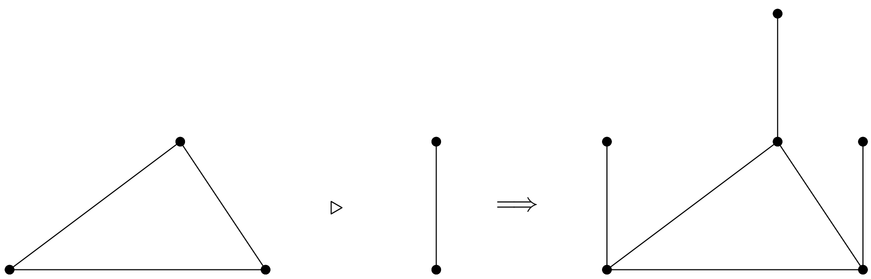
As a result of Theorem 3.1. , we can easily know the spectrum of $G \rhd P_2$ from that of G.

Example 3.2. Here is a simple example. The spectrum of C_3 is

$$\text{Spec } C_3 = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}.$$

We can easily derive the spectrum of $C_3 \rhd P_2$ by using Theorem 3.1.

$$\text{Spec } C_3 \rhd P_2 = \begin{pmatrix} -1-\sqrt{5} & 1-\sqrt{2} & -1+\sqrt{5} & 1+\sqrt{2} \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

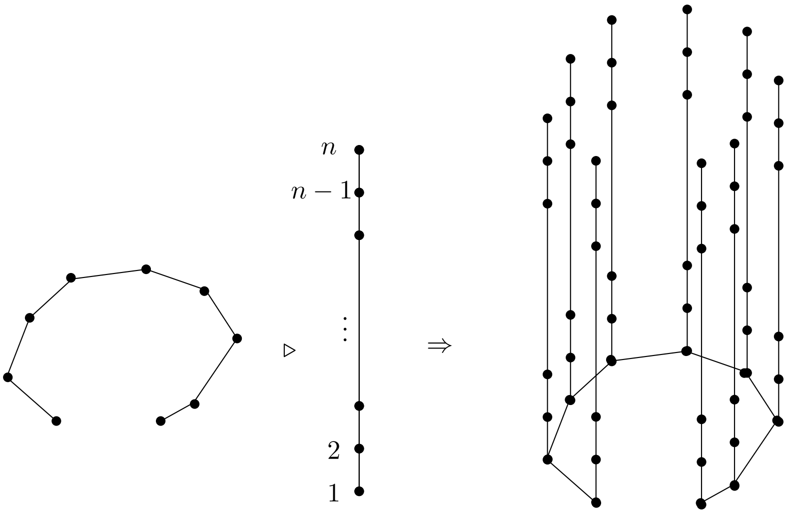


C_3 P_2 $C_3 \rhd p_2$

Figure 7: $C_3 \rhd P_2$

3.3 Spectrum of Comb Graphs (A General Case)

In this section, we discuss a general case of $G \rhd P_n$.



G $\quad P_n \quad$ $G \triangleright P_n$

Figure 8: $G \triangleright P_n$

Theorem 3.3. *Suppose the spectrum of a graph G is given by*

$$\operatorname{Spec} G = \begin{pmatrix} \cdots & \mu_k & \cdots \\ \cdots & m_k & \cdots \end{pmatrix}.$$

Let P_n be the path graph with n vertices. Then the spectrum of $G \triangleright P_n$ is

$$\operatorname{Spec} G \triangleright P_n = \begin{pmatrix} \cdots & \lambda_1(\mu_k) & \cdots & \lambda_n(\mu_k) & \cdots \\ \cdots & m_k & \cdots & m_k & \cdots \end{pmatrix},$$

where $\lambda_1(\mu) < \ldots < \lambda_n(\mu)$ are the solutions of

$$\mu = \frac{\varphi_n(\lambda)}{\varphi_{n-1}(\lambda)},$$

where $\varphi_n(\lambda)$ is the characteristic polynomial of P_n.

Proof. Let A be the adjacency matrix of G. Then the adjacency matrix of $G \triangleright P_n$ is given by

$$\tilde{A} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ I & 0 & & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & & 0 & I \\ 0 & \cdots & 0 & I & A \end{bmatrix}.$$

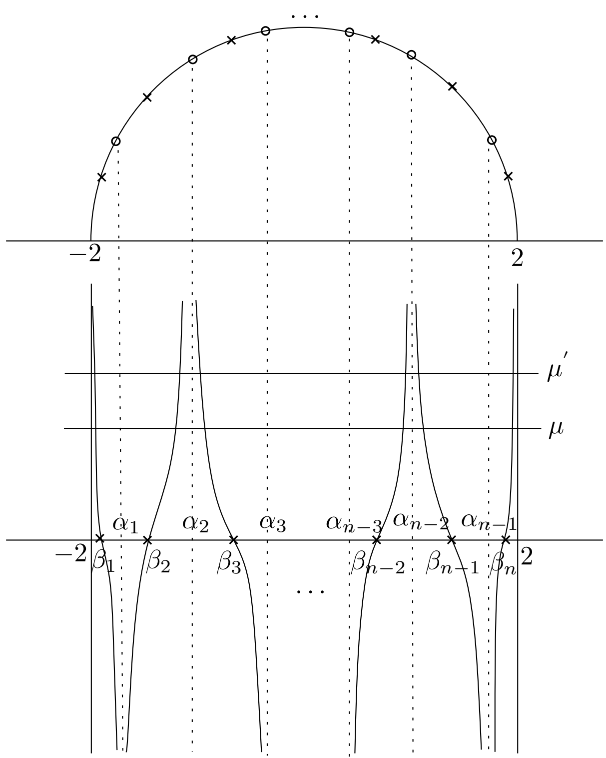


Figure 9: Path

We can obtain the eigenvalues of $G \triangleright P_n$ by solving the eigenvalue problem:

$$\begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ I & 0 & & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & & 0 & I \\ 0 & \cdots & 0 & I & A \end{bmatrix} \begin{bmatrix} X_1 \\ \\ \vdots \\ \\ X_n \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ \\ \vdots \\ \\ X_n \end{bmatrix}.$$

Then,

$$\begin{cases} X_2 = \lambda X_1 \\ X_1 + X_3 = \lambda X_2 \\ \vdots \\ X_{n-2} + X_n = \lambda X_{n-1} \\ X_{n-1} + AX_n = \lambda X_n \end{cases}$$

Let $\varphi_i(\lambda)$ be the characteristic polynomial of the path P_i with i vertices, see

Theorem 2.4. Then the above equations may be rewritten as

$$
\begin{aligned}
X_2 &= \varphi_1(\lambda)X_1 \\
X_3 &= \lambda\varphi_1(\lambda)X_1 - X_1 = (\lambda\varphi_1(\lambda) - \varphi_0(\lambda))X_1 = \varphi_2(\lambda)X_1 \\
&\vdots \\
X_n &= \lambda\varphi_{n-2}(\lambda)X_1 - \varphi_{n-3}(\lambda)X_1 \\
&= (\lambda\varphi_{n-2}(\lambda) - \varphi_{n-3}(\lambda))X_1 = \varphi_{n-1}(\lambda)X_1 \qquad (3.2) \\
AX_n &= \lambda\varphi_{n-1}(\lambda)X_1 - \varphi_{n-2}(\lambda)X_1 \\
&= (\lambda\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda))X_1 = \varphi_n(\lambda)X_1 \qquad (3.3)
\end{aligned}
$$

By solving the equation (3.2) and (3.3), we can obtain

$$(\varphi_{n-1}(\lambda)A - \varphi_n(\lambda))X_1 = 0.$$

Since $X_1 \neq 0$, we have

$$\det(\varphi_{n-1}(\lambda)A - \varphi_n(\lambda)) = 0.$$

Therefore

$$\det\left(A - \frac{\varphi_n(\lambda)}{\varphi_{n-1}(\lambda)}\right) = 0.$$

Thus we have shown that if λ is an eigenvalue of $G \triangleright P_n$, then

$$\mu = \frac{\varphi_n(\lambda)}{\varphi_{n-1}(\lambda)}, \qquad (3.4)$$

is an eigenvalue of A. Namely, to find an eigenvalue of $G \triangleright P_n$ we need to solve the equation (3.4) for a given μ. Now, let $\alpha_1 < \cdots < \alpha_{n-1}$ be the roots of $\varphi_{n-1}(\lambda) = 0$ and $\beta_1 < \cdots < \beta_n$ be the roots of $\varphi_n(\lambda) = 0$. Then it follows that $\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \alpha_{n-1} < \beta_n$ as in Fig 6. And for every i, $x = \alpha_i$ are vertical asymptotes of $\varphi_n(\lambda)$. We also know that $\varphi_n(\lambda)$ is strictly increasing (or decreasing) in each interval (α_{i-1}, α_i), $1 \leq i \leq n$. Let μ be an eigenvalue of A then the equation (3.4) has n distinct solutions, say, $\lambda_1(\mu) < \cdots < \lambda_n(\mu)$. We see from Fig 6 that, $\lambda_1(\mu), \ldots, \lambda_n(\mu), \lambda_1(\mu^{'}), \ldots, \lambda_n(\mu^{'})$ are all distinct for $\mu \neq \mu^{'}$, hence exhaust the eigenvalues of $G \triangleright P_n$. $\square$

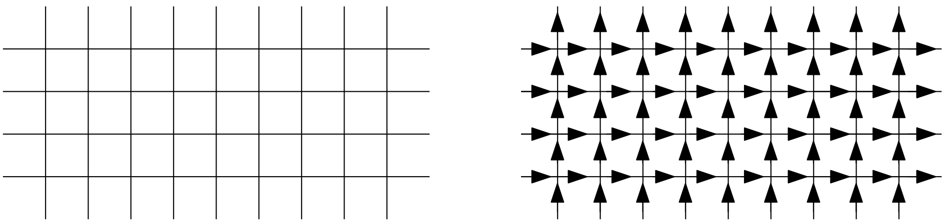
If the number of the vertices of G is m, the total number of vertices of $G \triangleright P_n$ are mn. By using the characteristic equation, it is quite hard to obtain their spectrum. But using the Theorem 3.3., we can easily find out the spectrum.

4 Manhattan Product Digraphs

In this section, we focus on a product of digraphs. Simply by G we denote a digraph.

4.1 Manhattan Street Network

The lattice structure often appears in realistic networks. As a simple example, the 2-dimensional integer lattice Z^2 is composed of the direct product $Z^1 \times Z^1$. Similarly the direct product $\overrightarrow{Z^2} = \overrightarrow{Z^1} \times \overrightarrow{Z^1}$ is a basic digraph. The spectra of Z^2 and $\overrightarrow{Z^2}$ are easily derived from the spectra of Z^1 and $\overrightarrow{Z^1}$, respectively.



$Z^2 = Z^1 \times Z^1$ $\qquad$ $\overrightarrow{Z^2} = \overrightarrow{Z^1} \times \overrightarrow{Z^1}$

Figure 10: Integer lattice

As shown in Fig 11, the Manhattan street network is another lattice structure. This network of one-way alternate streets can be found in the cities of New York and Barcelona. By analysing this network, scientific concepts, such as easy routing, Hamiltonicity and modular structure, has been established. The spectrum of this network has been discussed in [6]. In this study we introduce the Manhattan product motivated by the Manhattan street network.

4.2 Definition and Structures

We introduce, motivated by the Manhattan street network, the Manhattan product of digraphs.

Definition 12. Given a digraph $G = (V, E)$, its converse digraph $G^\vee = (V, E^\vee)$ is obtained from G by reversing all the orientations of the arcs in E; that is $(v_i, v_j) \in E^\vee$ if and only if $(v_j, v_i) \in E$.

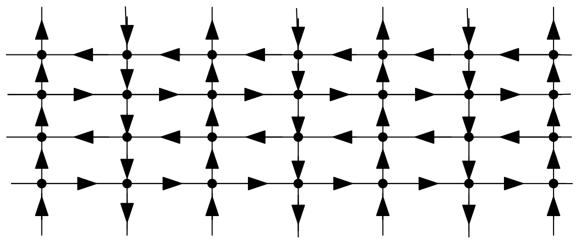


Figure 11: Manhattan street network

Theorem 4.1. *Let G be a digraph and $G^\vee$ the converse digraph of G. Then*

$$\text{Spec } G = \text{Spec } G^\vee.$$

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of G. Then the adjacency matrix of $G^\vee$ is the transposed of A, i.e., $A^t = [a_{ji}]$. Since the characteristic polynomials of A and A^t are the same, we have Spec $G =$ Spec $G^\vee$. □

The Manhattan product G consists of two graphs G_1 and G_2 which are horizontal and vertical digraphs respectively, and G_1 and $G_1^\vee$ are embedded to G_2, $G_2^\vee$ alternately, vice versa.

Definition 13. [5] Let $G_i = (V_i, E_i)$ be bipartite digraphs with independent sets $V_i = V_{i0} \cup V_{i1}$, $N_i = |V_i|$, $i = 1, 2, \ldots, n$. Let π be the characteristic function of $V_{i1} \subset V_i$ for any i; that is,

$$\pi(u) = \begin{cases} 0, \text{ if } u \in V_{i0}, \\ 1, \text{ if } u \in V_{i1}. \end{cases}$$

Then, the *Manhattan product* $M_n = G_1 \sharp G_2 \sharp \cdots \sharp G_n$ is the digraph with vertex set $V(M_n) = V_1 \times V_2 \times \cdots \times V_n$, and each vertex $(u_1, \ldots, u_i, \ldots, u_n)$ is adjacent to vertices $(u_1, \ldots, v_i, \ldots, u_n)$, $1 \le i \le n$, when

(1) $v_i \in \Gamma^+(u_i)$ if $\sum_{j \ne i} \pi(u_j)$ is even,

(2) $v_i \in \Gamma^-(u_i)$ if $\sum_{j \ne i} \pi(u_j)$ is odd,

where $\Gamma^+(u_i)$ be the set of vertices which are adjacent from i, and $\Gamma^-(u_i)$ be the set of vertices which are adjacent to i.

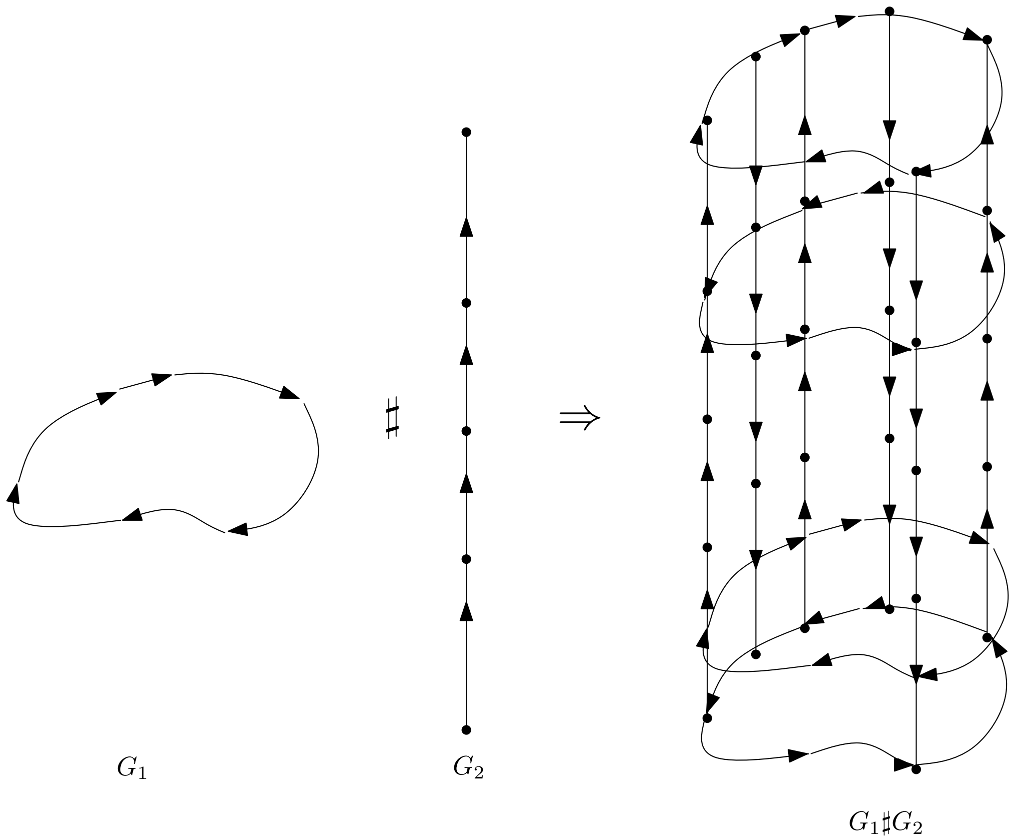


Figure 12: Figure of Manhattan product

4.3 Eigenvalue Properties of $K_{V_1,V_2}\sharp C_2$

One of the interesting properties of the Manhattan product $C_{2m}\sharp C_{2n}$ is that, multiplicity of the zero-eigenvalue is more than half of the total sum of multiplicities [6]. We discuss the eigenvalue properties of $K_{V_1,V_2}\sharp C_2$, in particular, the conditions necessary for having the zero eigenvalue.

Definition 14. A *complete bipartite graph* is a bipartite graph with its vertex set being partitioned into two parts V_1 and V_2, where each edge has one vertex in V_1 and the other in V_2. Thus, every pair of vertices in V_1 and V_2 are adjacent.

Definition 15. A *directed cycle* of length n, denoted by C_n, is a digraph with the vertex set $\{v_1, \ldots, v_n\}$ having arcs (v_i, v_j), $i = 1, \ldots, n-1$, and (v_n, v_1).

We discuss the eigenvalues of $K_{V_1,V_2}\sharp C_2$ as a simple example.

Theorem 4.2. *Let K_{V_1,V_2} be the complete bipartite digraph and $A = [a_{uv}]_{u\in V_1, v\in V_2}$ the matrix defined by $a_{uv} = 1$ for $u \to v$. Then spectrum of $G = K_{V_1,V_2}\sharp C_2$ contains 0 as an eigenvalue if and only if*

$$\mathrm{Spec}\,(A^tA) \ni 1 \ \textit{or}\ \mathrm{Spec}\,((J-A)(J-A)^t) \ni 1,$$

where J is the matrix with all entries being one.

Proof. Let $\tilde{A}$ be the adjacency matrix of $K_{V_1,V_2}\sharp C_2$. Then we have

$$\tilde{A} = \begin{bmatrix} 0 & A & I & 0 \\ (J-A)^t & 0 & 0 & I \\ I & 0 & 0 & J-A \\ 0 & I & A^t & 0 \end{bmatrix},$$

where J is the matrix in which all entries are 1. We consider the eigenvalue problem:

$$\begin{bmatrix} -\lambda & A & I & 0 \\ (J-A)^t & -\lambda & 0 & I \\ I & 0 & -\lambda & J-A \\ 0 & I & A^t & -\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which may be rewritten as

$$\begin{cases} -\lambda X_1 + AX_2 + X_3 = 0 \\ (J-A)^t X_1 - \lambda X_2 + X_4 = 0 \\ X_1 - \lambda X_3 + (J-A)X_4 = 0 \\ X_2 + A^t X_3 - \lambda X_4 = 0. \end{cases}$$

From the first and second equations, we obtain

$$X_3 = \lambda X_1 - AX_2$$

and

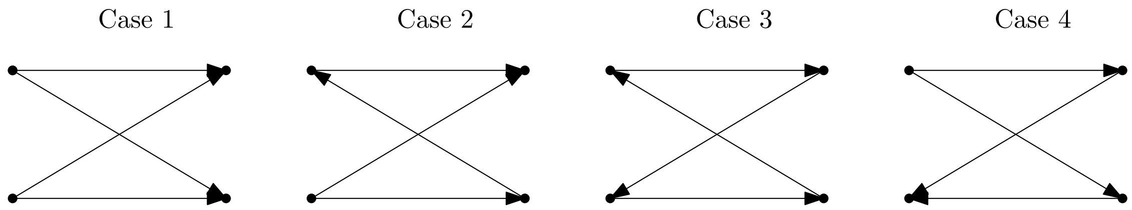
$$X_4 = \lambda X_2 - (J-A)^t X_1,$$

respectively. Substituting X_3 and X_4 into the third and fourth equations, we get the following equations:

$$\begin{cases} ((1-\lambda^2)I - (J-A)(J-A)^t)X_1 + \lambda J X_2 = 0 \\ \lambda J X_1 + (I - A^tA - \lambda^2)X_2 = 0 \end{cases}$$

Therefore we obtain the following equation.

$$\det \begin{bmatrix} (1-\lambda^2)I - (J-A)(J-A)^t & \lambda J \\ \lambda J & (1-\lambda^2)I - A^tA \end{bmatrix} = 0 \qquad (4.1)$$



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Figure 13: $K_{2,2}$ and their adjacency matrix

Thus, G has a zero-eigenvalue if and only if

$$\det \begin{bmatrix} I - (J-A)(J-A)^t & 0 \\ 0 & I - A^t A \end{bmatrix} = 0,$$

namely,

$$\det[I - (J-A)(J-A)^t] = 0 \quad \text{or} \quad \det[I - A^t A] = 0.$$

The above condition is equivalent to

$$\mathrm{Spec}\,((J-A)(J-A)^t) \ni 1 \text{ or } \mathrm{Spec}\,(A^t A) \ni 1,$$

as desired. □

During the above proof, we have established the following.

Corollary 4.3. *The spectrum of $G = K_{V_1,V_2} \sharp C_2$ is determined by the characteristic equation (4.1).*

Example 4.4. We consider Manhattan product of the $K_{2,2} \sharp C_2$. Let $A = [a_{uv}]_{u \in V_1, v \in V_2}$ the matrix defined by $a_{uv} = 1$ for $u \to v$. Then the matrix A of $K_{2,2}$ have 4 cases, as in Fig.13. The characteristic polynomials of $K_{2,2} \sharp C_2$ are given as follows:

$$\begin{aligned} &\text{Case1} : \lambda^8 - 4\lambda^6 + 2\lambda^4 + 4\lambda^2 - 3 \\ &\text{Case2} : \lambda^8 - 4\lambda^6 + 2\lambda^4 \\ &\text{Case3} : \lambda^8 - 4\lambda^6 \\ &\text{Case4} : \lambda^8 - 4\lambda^6 + 2\lambda^4 + 1 \end{aligned}$$

We can easily know that they have 0 eigenvalues in Case 2 and Case 3. From Theorem 4.2, by computing the $\mathrm{Spec}\,(A^t A)$ and $\mathrm{Spec}\,((J-A)(J-A)^t)$ in the four cases, we can check the existence of zero eigenvalues.

$$\begin{aligned}
&(\text{Case1})\,\mathrm{Spec}\,(A^tA) : \det\begin{bmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{bmatrix} = \lambda^4 - 4\lambda \\
&\qquad\quad \mathrm{Spec}\,((J-A)(J-A)^t) : \det\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda^2 \\
&(\text{Case2})\,\mathrm{Spec}\,(A^tA) : \det\begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-1 \end{bmatrix} = \lambda^2 - 3\lambda + 1 \\
&\qquad\quad \mathrm{Spec}\,((J-A)(J-A)^t) : \det\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{bmatrix} = \lambda(\lambda-1) \\
&(\text{Case3})\,\mathrm{Spec}\,(A^tA) : \det\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 \\
&\qquad\quad \mathrm{Spec}\,((J-A)(J-A)^t) : \det\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 \\
&(\text{Case4})\,\mathrm{Spec}\,(A^tA) : \det\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix} = \lambda^2 - 2\lambda \\
&\qquad\quad \mathrm{Spec}\,((J-A)(J-A)^t) : \det\begin{bmatrix} \lambda & 0 \\ 0 & \lambda-2 \end{bmatrix} = \lambda(\lambda-2)
\end{aligned}$$

4.4 Spectrum of $C_2\sharp P_n$

In this section we discuss the spectrum of $G = C_2\sharp P_n$.

Lemma 16. *Let $\varphi_n(\lambda)$ be the characteristic polynomial of $G = C_2\sharp P_n$. Then it holds that*

$$\begin{aligned}
\varphi_1(\lambda) &= \lambda^2 - 1 \\
\varphi_2(\lambda) &= \lambda^4 - 2\lambda^2 \\
\varphi_n(\lambda) &= \lambda^2(\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda)), \;\; n \geq 2
\end{aligned}$$

Here we set $\varphi_0(\lambda) = 1$ tacitly.

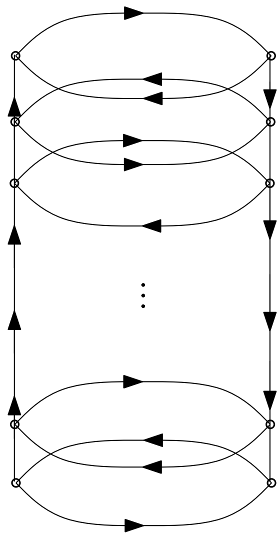


Figure 14: $C_2\sharp P_n$

Proof. The adjacency matrix of $G = C_2\sharp P_n$ is written in the form:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & & & \ddots & & & \\ 0 & 1 & & \ddots & & \ddots & & \\ & & \ddots & & \ddots & & 1 & 0 \\ & & & \ddots & & & 0 & 0 \\ & & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The characteristic equation of G is given by

$$\varphi_n(\lambda) = |\lambda I - A| = \det \left[\begin{array}{cc|cc|cc|cc} \lambda & -1 & -1 & 0 & & & & \\ -1 & \lambda & 0 & 0 & & & & \\ \hline 0 & 0 & \lambda & -1 & \ddots & & & \\ 0 & -1 & -1 & \lambda & & \ddots & & \\ \hline & & \ddots & & \ddots & & -1 & 0 \\ & & & \ddots & & \ddots & 0 & 0 \\ \hline & & & & 0 & 0 & \lambda & -1 \\ & & & & 0 & -1 & -1 & \lambda \end{array}\right].$$

Let $\varphi(\lambda) = \det \Phi_n(\lambda)$. By cofactor expansion with respect to the first column, we obtain

$$\varphi_n(\lambda) = \lambda^2 \cdot \varphi_{n-1}(\lambda) + \det \underbrace{\left[\begin{array}{c|cccc} -1 & -1 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ -1 & & & & \\ 0 & & \Phi_{n-1}(\lambda) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right]}_{\Psi_{n-1}(\lambda)},$$

where

$$\det \Psi_{n-1}(\lambda) = -\varphi_{n-1}(\lambda) + \det \underbrace{\left[\begin{array}{c|cccc} -1 & -1 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ -1 & & & & \\ 0 & & \Phi_{n-2}(\lambda) & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right]}_{\Psi_{n-2}(\lambda)}$$

$$\begin{aligned} &= -\varphi_{n-1}(\lambda) + \det \Psi_{n-2}(\lambda) \\ &= -\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda) + \det \Psi_{n-3}(\lambda) \\ &= -\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda) - \cdots - \varphi_2(\lambda) + \det \Psi_1(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi_n(\lambda) &= \lambda^2 \cdot \varphi_{n-1}(\lambda) - (\varphi_{n-1}(\lambda) + \varphi_{n-2}(\lambda) + \cdots + \varphi_1(\lambda) + \varphi_0(\lambda)) \\ &= \lambda^2 \cdot \varphi_{n-1}(\lambda) - \left(\sum_{i=0}^{n-1} \varphi_i(\lambda)\right), \\ \varphi_0(\lambda) &= 1. \end{aligned}$$

Then,

$$\begin{aligned} \varphi_n(\lambda) - \varphi_{n-1}(\lambda) &= \lambda^2 \cdot \varphi_{n-1}(\lambda) - (\sum_{i=0}^{n-1} \varphi_i(\lambda)) - \lambda^2 \cdot \varphi_{n-2}(\lambda) - (\sum_{i=0}^{n-2} \varphi_i(\lambda)) \\ &= \lambda^2 \cdot \varphi_{n-1}(\lambda) - \varphi_{n-1}(\lambda) - \lambda^2 \cdot \varphi_{n-2}(\lambda). \end{aligned}$$

Consequently we obtain

$$\varphi_n(\lambda) = \lambda^2(\varphi_{n-1}(\lambda) - \varphi_{n-2}(\lambda)), \tag{4.2}$$

as desired. □

Theorem 4.5. *Let $\tilde{U}_n(\lambda) = U_n(\lambda/2)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind. Then the characteristic polynomial $\varphi_n(\lambda)$ of $C_2 \sharp P_n$ is given by*

$$\varphi_n(\lambda) = \lambda^{n-1}\tilde{U}_{n+1}(\lambda), \qquad n \geq 1.$$

Proof. Let

$$\psi_{n+1}(\lambda) = \lambda^{-(n-1)}\varphi_n(\lambda), \qquad n \geq 0.$$

Multiplying $\lambda^{-(n-1)}$ both sides of the equation (4.2) in Lemma 16, we get

$$\begin{aligned} \psi_{n+1}(\lambda) &= \lambda^{-n+3}\ \varphi_{n-1}(\lambda) - \lambda^{-n+3}\ \varphi_{n-2}(\lambda) \\ &= \lambda\psi_n(\lambda) - \psi_{n-1}(\lambda). \end{aligned} \tag{4.3}$$

This coincides with the recurrence relations satisfied by $\tilde{U}(\lambda)$, see (2.2). For $n \geq 1$, it holds that

$$\begin{aligned} &\psi_1(\lambda) = \lambda\varphi_0(\lambda) = \lambda \\ &\psi_2(\lambda) = \varphi_1(\lambda) = \lambda^2 - 1 \\ &\psi_3(\lambda) = \lambda^{-1}\varphi_2(\lambda) = \lambda^3 - 2\lambda \end{aligned}$$

Comparing the equation (4.3) and equation (2.2), we see that

$$\psi_{n+1}(\lambda) = \tilde{U}_{n+1}(\lambda).$$

Therefore,

$$\begin{aligned} \varphi_n(\lambda) &= \lambda^{n-1}\psi_{n+1}(\lambda) \\ &= \lambda^{n-1}\tilde{U}_{n+1}(\lambda). \end{aligned}$$

This completes the proof. □

Theorem 4.6. *The spectrum of $C_2 \sharp P_n$ is given by*

$$\text{Spec } C_2 \sharp P_n = \begin{pmatrix} 0 & 2\cos\frac{k\pi}{n+2} \\ n-1 & 1 \end{pmatrix}, \quad k = 1, 2, \ldots, n+1.$$

Proof. The spectrum of $C_2 \sharp P_n$ is obtained by solving the characteristic equation:

$$\varphi_n(\lambda) = 0.$$

From Theorem 4.6 we see that

$$\varphi_n(\lambda) = \lambda^{n-1}\tilde{U}_{n+1}(\lambda),$$

where

$$\tilde{U}_{n+1}(2\cos\theta) = \frac{\sin(n+2)\theta}{\sin\theta}.$$

Therefore,

$$\lambda = 2\cos\frac{k\pi}{n+2}, \qquad k = 1, 2, \ldots, n+1,$$

are $n+1$ distinct roots of $\varphi_n(\lambda) = 0$ and $\lambda = 0$ is a root with at least $n-1$ multiplicities. Since $\varphi_n(\lambda)$ is a polynomial of degree $2n$, we found all roots of $\varphi_n(\lambda) = 0$. This completes the proof. □

Theorem 4.7. *The asymptotic spectral distribution of $C_2 \sharp P_n$ as $n \to \infty$ is given by*

$$\frac{1}{2}\delta_0 + \frac{1}{2}\rho(x)dx,$$

where

$$\rho(x) = \begin{cases} \dfrac{1}{\pi\sqrt{4-x^2}}, & -2 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\{\lambda_k; k = 1, \ldots, 2n\}$ be the eigenvalues of $C_2 \sharp P_n$. For a continuous function $f(x)$ we set

$$S_n = \frac{1}{2n}\sum_{k=1}^{2n} f(\lambda_k).$$

We will compute $\lim_{n\to\infty} S_n$. Let $F(t) = f(2\cos t\pi)$. Then we have

$$\begin{aligned} S_n &= \frac{n-1}{2n}f(0) + \frac{1}{2n}\sum_{k=1}^{n+1} F\left(\frac{k}{n+2}\right) \\ &= \frac{n-1}{2n}f(0) + \frac{n+2}{2n}\sum_{k=1}^{n+1} F\left(\frac{k}{n+2}\right)\frac{1}{n+2} \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = \frac{1}{2}f(0) + \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n+2} F\left(\frac{k}{n+2}\right)\frac{1}{n+2}$$

By the definition of Riemannian integral,

$$\begin{aligned} \lim_{n\to\infty} S_n &= \frac{1}{2}f(0) + \frac{1}{2}\int_0^1 F(t)dt \\ &= \frac{1}{2}f(0) + \frac{1}{2}\int_0^1 f(2\cos t\pi)dt. \end{aligned}$$

Let $2\cos t\pi = x$. Then,

$$\begin{aligned}\int_0^1 f(2\cos t\pi)dt &= \int_{-2}^{2} \frac{f(x)}{-2\pi \sin t\pi} dx \\ &= \int_{-2}^{2} f(x) \cdot \frac{1}{\pi\sqrt{4-x^2}}\, dx.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=1}^{2n} f(\lambda_k) = \frac{1}{2} f(0) + \frac{1}{2} \int_{-2}^{2} f(x) \cdot \frac{1}{\pi\sqrt{4-x^2}}\, dx.$$

This means that the asymptotic spectrum distribution of $C_2 \sharp P_n$ as $n \to \infty$ is given by

$$\frac{1}{2}\delta_0 + \frac{1}{2}\rho(x)dx$$

as desired. $\square$

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