# Performance analysis of a time diversity ARQ in land mobile radio



| | |
|---|---|
| 著者 | 安達　文幸 |
| journal or publication title | IEEE Transactions on Communications |
| volume | 37 |
| number | 2 |
| page range | 177-183 |
| year | 1989 |
| URL | http://hdl.handle.net/10097/46448 |

doi: 10.1109/26.20086

# Correspondence

## Performance Analysis of a Time Diversity ARQ in Land Mobile Radio

F. ADACHI, S. ITO, AND K. OHNO

*Abstract*—**A time diversity automatic repeat-request (ARQ) scheme with the finite number of transmissions is investigated for a digital FM mobile radio with frequency demodulation (FD). It processes all the retransmissions of a single data block using postdetection diversity combining. The analysis of the signal energy per bit required for a given bit error rate (BER) and the spectral efficiency in a cellular mobile radio system are presented. The results obtained from the numerical calculations show that this ARQ scheme offers a performance superior to both the basic ARQ scheme and the time diversity scheme.**

### I. Introduction

Digital transmission is of growing interest in the field of mobile radio where the channels are characterized by rapid multipath Rayleigh fading superimposed on slow shadow fading [1]–[3, ch. 2]. Rayleigh fading is caused by interference between multipath waves from scatterers surrounding the mobile station, while shadowing is introduced by gross variation in the terrain between the base and mobile stations. Since signal transmission performance is severely degraded due to fading, some auxiliary techniques are required. There are two promising techniques: diversity reception and error control. Among the various diversity schemes [3, chs. 5 and 6], time diversity is most attractive, because it does not require multiple antennas at the receiver. However, its main drawback is that it transmits the same data blocks even in the absence of errors. As an error control technique, the automatic repeat-request (ARQ) scheme [4] is currently being studied for mobile radio use [5]. The simplest implementation of ARQ schemes (referred to as the basic ARQ scheme) uses only error detection, and detection errors lead to a repeat request and discarding of information in erroneous blocks. Benelli [6] has described an efficient ARQ scheme with memory and soft error detectors, which makes use of all the retransmissions, even those that contain errors. The main drawback with ARQ schemes when applied to mobile radio is an increase, due to fading, in the number of transmissions needed before receiving a block correctly [5].

Recently, Adachi and Itoh [7] have proposed a time diversity ARQ scheme achieved by combining the basic ARQ and time diversity schemes. It makes use of all retransmissions to obtain a new reliable block by postdetection diversity combining to enhance performance in mobile radio channels with fading. This scheme can be viewed as a modified time diversity scheme using an adaptive number of diversity branches. A considerable reduction in the average number of transmissions has been shown in comparison to the basic ARQ scheme in a Rayleigh fading environment. However, when the average signal-to-noise ratio (SNR) is reduced due to shadowing, a large number of transmissions may occur. This can be avoided by employing a finite number of transmissions for the sake of transmission performance.

In this paper, a performance analysis of the time diversity ARQ scheme with a finite number of transmissions is presented for a binary digital FM mobile radio using frequency demodulation (FD). The performance of radio communications systems can be compared based on signal energy per bit required to achieve a given reliability (bit error rate (BER) is often used for this purpose). In this paper, the required signal energy per bit taking into account multiple retransmission is considered for performance comparison. After describing the time diversity ARQ scheme in Section II, how a finite number of transmissions affects the signal energy per bit required for a given BER in a Rayleigh fading environment is investigated in Section III to find the minimum required signal energy per bit. Section IV presents spectral efficiency in a cellular mobile radio system. The performance of this ARQ scheme is compared to the basic ARQ and time diversity schemes.

### II. Time Diversity ARQ Scheme

A block diagram of a time diversity ARQ communication system is shown in Fig. 1. At the transmitter, a message is split into groups of length $k$ bits. Each group coming out of the source is then coded into an $n$ bit block through a code of type $(n, k)$. Suppose that the block to be transmitted is denoted by $\boldsymbol{x} = (x_1, x_2 \cdots x_n)$ where $x_m = 0, 1\,(m = 1, 2 \cdots n)$. Then the input to the binary digital FM modulator is $\boldsymbol{a} = (a_1, a_2 \cdots a_n)$ where $a_m = 1/-1$ for $x_m = 1/0$. The transmitted binary digital FM signal at the angular frequency $\omega_c$ can be represented as Re $[A \exp j(\omega_c t + \Phi_s(t))]$ where $A$ is the amplitude and $\Phi_s(t)$ is the modulating phase, the time derivative of which is expressed as $\Phi_s'(t) = {}^{d}/_{dt}\Phi_s(t) = (2\pi \Delta f)a_m$ for $(m - 0.5)T \leqq t < (m + 0.5)T$, with $T$ being the bit duration and $\Delta f$ the frequency deviation. Signal transmission between mobile and base stations takes place over the Rayleigh fading channel. Assuming that the receiver predetection filter bandlimits the additive white Gaussian noise (AWGN) but is wide enough not to cause intersymbol interference, the input to the demodulator can be written as Re $[z(t) \exp(j\omega_c t)]$ where $z(t) = z_s(t) \exp\{j\Phi_s(t)\} + z_n(t)$ where $z_s(t)$ and $z_n(t)$ are the independent zero-mean complex Gaussian processes for the signal and the bandlimited AWGN, respectively.

Suppose that the $i$th transmission of $\boldsymbol{a}$ has been made. Let the demodulator output vector, i.e., the instantaneous angular frequency, corresponding to the $i$th reception of $\boldsymbol{a}$ be denoted by $\boldsymbol{\Psi}_i' = (\Psi_{i1}', \Psi_{i2}' \cdots \Psi_{in}')$. The $\Psi_{im}'$ can be represented as

$$\Psi_{im}' = \mathrm{Im}\left\{\frac{z_{im}(-z_{im}')^*}{|z_{im}|^2}\right\}, \quad m = 1, 2, \cdots n \tag{1}$$

where $z_{im}$ and $z_{im}'$ are the values of $z(t)$ and of its time derivative at the sampling instant $t_{im}$. If there is no AWGN and no fading, $\Psi_{im}' = (2\pi\Delta f)a_m$. The time diversity ARQ scheme combines all the successive $i$ vectors, $\boldsymbol{\Psi}_1', \boldsymbol{\Psi}_2', \cdots$, and $\boldsymbol{\Psi}_i'$, using postdetection diversity combining [8] to form a new reliable received vector. Let us first consider postdetection maximal-ratio combining (MRC). The $\Psi_{i1}', \Psi_{i2}', \cdots$, and $\Psi_{in}'$ are weighted in proportion to the demodulator input envelopes squared; $\Psi_{im}'$ is multiplied with $R_{im}^2 (= |z_{im}|^2)$ to obtain $w_{im} = \Psi_{im}' \times R_{im}^2$. Then, the MRC combiner updates

Paper approved by the Editor for Mobile Communications of the IEEE Communications Society. Manuscript received June 4, 1987; revised October 7, 1987.

The authors are with NTT Radio Communications Systems Laboratories, Yokosuka 238, Japan.

IEEE Log Number 8825338.

0090-6778/89/0200-0177$01.00 

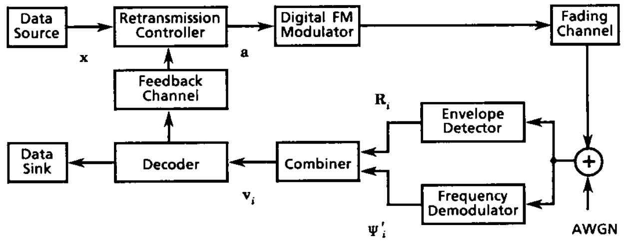


Fig. 1. Block diagram of a time diversity ARQ communication system.

the combiner output vector $\boldsymbol{v}_{i-1} = (v_{(i-1)1}, v_{(i-1)2}, \cdots v_{(i-1)n})$ that has been obtained at the $(i-1)$th reception using $\boldsymbol{w}_i = (w_{i1}, w_{i2}, \cdots w_{in})$, i.e.,

$$v_{0m}=0$$
$$v_{im}=v_{(i-1)m}+w_{im} \tag{2}$$

to obtain

$$v_{\rm im}=\mathrm{Im}\left\{\sum_{l=1}^{i} z_{lm}(-z'_{lm})^*\right\}. \tag{3}$$

Therefore, the time diversity ARQ scheme using MRC makes the contribution of $\Psi'_{lm}(l=1, 2 \cdots i)$ to the combiner output $v_{im}$ small for a small $R_{lm}$, since $\Psi'_{lm}$ is noisy and unreliable.

Another combining method easy to implement and hence attractive for mobile radio use may be selection combining (SC). Now the envelope vector $\boldsymbol{r}_{i-1} = (r_{(i-1)1}, r_{(i-1)2} \cdots r_{(i-1)n})$ that has been obtained at the $(i-1)$th reception is introduced. The SC combiner compares $R_{\rm im}$ with $r_{(i-1)m}$ to update both the envelope vector and the combiner output vector as follows:

$$r_{0m}=0$$
$$r_{\rm im}=R_{\rm im} \text{ if } R_{\rm im} \geqq r_{(i-1)m} \text{ and } r_{(i-1)m} \text{ if } R_{\rm im}<r_{(i-1)m}$$
$$v_{0m}=0$$
$$v_{\rm im}=\Psi'_{\rm im} \text{ if } R_{\rm im} \geqq r_{(i-1)m} \text{ and } v_{(i-1)m} \text{ if } R_{\rm im}<r_{(i-1)m}. \tag{4}$$

Equation (4) results in

$$r_{\rm im}=R_{lm}=\max\,[R_{1m}, R_{2m}, \cdots, R_{im}]$$
$$v_{\rm im}=\Psi'_{lm}=\mathrm{Im}\left\{\frac{z_{lm}(-z'_{lm})^*}{|z_{lm}|^2}\right\}. \tag{5}$$

This shows that after the $i$th reception, the time diversity ARQ scheme using SC simply selects the demodulator output having the maximum demodulator input envelope.

Then, a vector $\boldsymbol{u}_i$ is constructed, the $m$th element of which is $u_{\rm im}=1$ if $v_{\rm im} \geqq 0$, or 0 if $v_{\rm im}<0$. The decoder determines whether $\boldsymbol{u}_i$ is a codeword. If it is a codeword, it is accepted as the transmitted codeword $\boldsymbol{x}$; otherwise, the $(i+1)$th transmission is requested.

## III. Required Signal Energy Per Bit

For simplicity, we assume that no error correcting code is used and that error detection code used will detect all errors.[1] Furthermore, the feedback channel is assumed to be noise free. When the errors are detected in the resultant vector $\boldsymbol{u}_i$ after the $i$th reception, then $(i+1)$th transmission is requested. With $M$ finite number of transmissions, the block (word) error rate (BKER) is equal to the repeat-request probability (RRP) of the $(M+1)$th transmission; the resultant bit error rate (BER) can then be determined from BKER $= 1-(1-\text{BER})^n$ since we are assuming random bit errors. The average number of transmissions is the sum of the RRP's from the first transmission to the $M$th transmission. The required signal energy per bit-to-noise ratio $E_s/N_0$ is defined as

$$E_s/N_0=\Gamma_0 \times \text{Average number of transmissions} \times \text{BT} \tag{6}$$

where $\Gamma_0$ is the average SNR required for a given BER, $N_0$ the one-sided noise power spectral density and BT the bandwidth-time product of the receiver predetection filter.

We first investigate the average number of transmissions and BER performance. Then, the required signal energy per bit is calculated.

### A. Bound Estimation for Average Number of Transmissions and BER Performance

The RRP of the $(i+1)$th transmission is given by Pr $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots$, and $\boldsymbol{u}_i$ are all incorrect$\}$. Vector $\boldsymbol{u}_l$ is constructed from combiner output vector $\boldsymbol{v}_l$ which is formed by the time diversity combiner after $l$th reception $(l=1, 2 \cdots i)$. The vector $\boldsymbol{u}_l$ is incorrect if one or more bit errors are produced. In a Rayleigh fading channel, bit errors in $\boldsymbol{u}_l$ are not random because of burst errors that are produced when the signal fades. To calculate the probability of getting at least one bit error, the distribution of the number of bit errors in $\boldsymbol{u}_l$ must be known. Computer simulations can be used to obtain this distribution [9]. However, the assumption of random bit errors in $\boldsymbol{u}_l$ is made throughout this paper because time diversity combining can reduce signal variations and tends to make the occurrence of bit errors random.[2] However, it can be understood from (2) and (4) the $m$th elements of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots$, and $\boldsymbol{v}_i$ are not independent variables and hence bit errors in $u_{1m}, u_{2m}, \cdots$, and $u_{\rm im}$ are not random. Therefore, the events $\{\boldsymbol{u}_1$ is incorrect$\}$, $\{\boldsymbol{u}_2$ is incorrect$\}$, $\cdots$, and $\{\boldsymbol{u}_i$ is incorrect$\}$ are not independent. Because of this, obtaining the RRP is rather difficult. However, it is possible to derive the upper and lower bounds. Realizing that Pr $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots$, and $\boldsymbol{u}_i$ are all incorrect$\}$ = Pr $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots$, and $\boldsymbol{u}_{i-1}$ are all incorrect $|\boldsymbol{u}_i$ is incorrect$\}$ $\times$ Pr $\{\boldsymbol{u}_i$ is incorrect$\}$, the upper bound is given by Pr $\{\boldsymbol{u}_i$ is incorrect$\}$ $= P_i$. The lower bound can be obtained by assuming that $\{\boldsymbol{u}_1$ is incorrect$\}$, $\{\boldsymbol{u}_2$ is incorrect$\}$, $\cdots$, and $\{\boldsymbol{u}_i$ is incorrect$\}$ are all independent events and is given by $P_1 \cdot P_2 \cdots P_i$.

Using the upper and lower bounds for the RRP, the average number $N(M)$ of transmissions and the BKER $P_B(M)$ can be

[1] This is a reasonable assumption in practice because the undetected error probability can be made very small.

[2] For the first transmission, no diversity combining processing is involved and bit errors may occur in burst. However, in a fast Rayleigh fading environment, signals vary within a single block received and occurence of bit errors even in $\boldsymbol{u}_1$ can be assumed random for the calculation of the probability that $\boldsymbol{u}_1$ is incorrect.

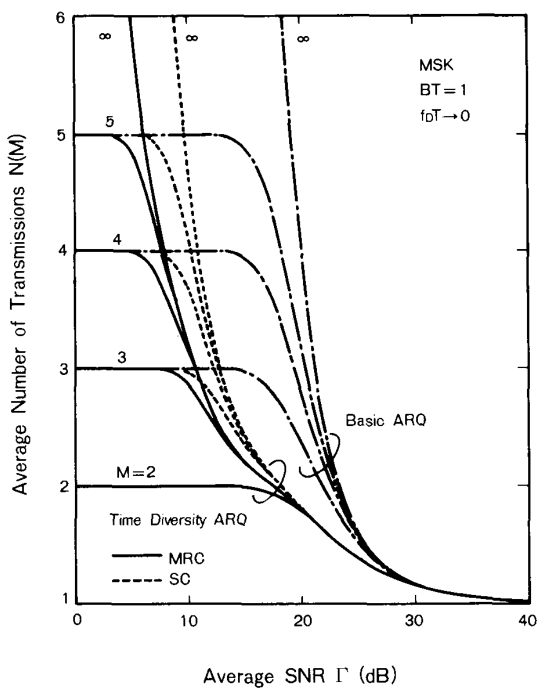



Fig. 2. Average number of transmissions (upper bound) for $n = 256$ in Rayleigh fading.

bounded by

$$1+\sum_{i=1}^{M-1} P_i \geqq N(M) \geqq 1+\sum_{i=1}^{M-1}\prod_{l=1}^{i} P_l$$

$$P_M \geqq P_B(M) \geqq \prod_{i=1}^{M} P_i \qquad (7)$$

for the time diversity ARQ scheme. For the conventional ARQ scheme, Pr $\{u_l$ is incorrect$\}$ = Pr $\{u_1$ is incorrect$\}$ = $P_1$ because the vector $u_l$ is obtained simply from the $l$th reception. Hence, the RRP of the $(i+1)$th transmission is simply given by $P_1^i$. Therefore, when the basic ARQ scheme is used, $N(M) = (1 - P_1^M)/(1 - P_1)$ and $P_B(M) = P_1^M$.

Since we have assumed random bit errors in the vector $u_i$, $P_i = 1 - (1 - Pe_i)^n$ where $Pe_i$ is the BER after successive $i$ transmissions.[3] The bit errors are produced when $a_m v_{im} < 0$. Equations (3) and (5) are the same expressions for the $i$ branch postdetection diversity combiner output using MRC and SC, respectively [8]. We obtain [see Appendix]

$$Pe_i=\frac{1}{2}-\frac{1}{2}\frac{a_m\rho_s}{\sqrt{[1-\rho_c^2]}}\sum_{l=0}^{i-1}\frac{(2l-1)!!}{(2l)!!}\left\{\frac{1-|\rho|^2}{1-\rho_c^2}\right\}^l, \quad \text{MRC}$$

$$=\frac{1}{2}-\frac{1}{2}\sum_{l=1}^{i}\binom{i}{l}(-1)^{l+1}\frac{a_m\rho_s}{\sqrt{[\rho_s^2+l(1-|\rho|^2)]}}, \quad \text{SC} \qquad (8)$$

where

$$\rho=\rho_c+j\rho_s=j\frac{\Gamma\Phi_s'}{\sqrt{[\Gamma+1]}\sqrt{[\Gamma\{\Phi_s'^2-\xi_s''(0)\}-\xi_n''(0)]}}, \qquad (9)$$

$\Gamma$ is the average SNR and $\xi_s(\tau)$ is the autocorrelation function of $z_s(t)$ and $\xi_n(\tau)$ that of $z_n(t)$.

We assume that many multipath waves having the same amplitude and independent random phases arrive at the receiving antenna from all directions with uniform probability and that a rectangular bandpass filter with bandwidth $B$ is used for the receiver predetection filter, $\xi_s(\tau) = J_0(2\pi f_D\tau)$ where $f_D$ is the maximum Doppler frequency given by vehicle speed/carrier wavelength [1] and $\xi_n(\tau) = \sin(\pi B\tau)/(\pi B\tau)$. Equation (9) becomes

$$\rho=ja_m\frac{\beta\Gamma}{\sqrt{[\Gamma+1]}\sqrt{\left[\Gamma(\beta^2+2(f_DT)^2)+\frac{(BT)^2}{3}\right]}} \qquad (10)$$

where $\beta = 2\Delta fT$ is the modulation index. Substituting (10) into (8) and averaging with respect to $a_m = \pm 1$, $Pe_i$ can be obtained and then $P_i$ is calculated. For the following calculations, we assume MSK transmission ($\beta = 0.5$) with BT = 1. For a 16 kbit/s transmission with a 900 MHz carrier frequency and a 100 km/h vehicle speed, the maximum Doppler frequency-bit duration product $f_DT = 5.2 \times 10^{-3}$. In such a case, the effect of the random FM noise [3, ch. 1] is negligible. Hence, we assume $f_DT \to 0$. Fig. 2 presents the calculated upper bound for the average number of transmissions because the bound estimation is found to be very tight (e.g., $N(\infty)$ of time diversity ARQ scheme using MRC is found to be between 3.1 and 3.2 at $\Gamma = 10$ dB). The calculated upper and lower bounds for BER performance are shown in Fig. 3. As $M$ increases, the average SNR required for a given BER decreases for both ARQ schemes. However, the advantage of the time diversity ARQ scheme over the basic scheme is evident. When the SC is used instead of the MRC for diversity combining, the required value is somewhat larger.

It is worthwhile to compare the BER performance to that achievable by the $M$ branch time diversity scheme. The BER

[3] If there are burst errors, $P_i$ tends to be smaller than $1 - (1 - P_{ei})^n$. Hence, this expression gives an upper bound.

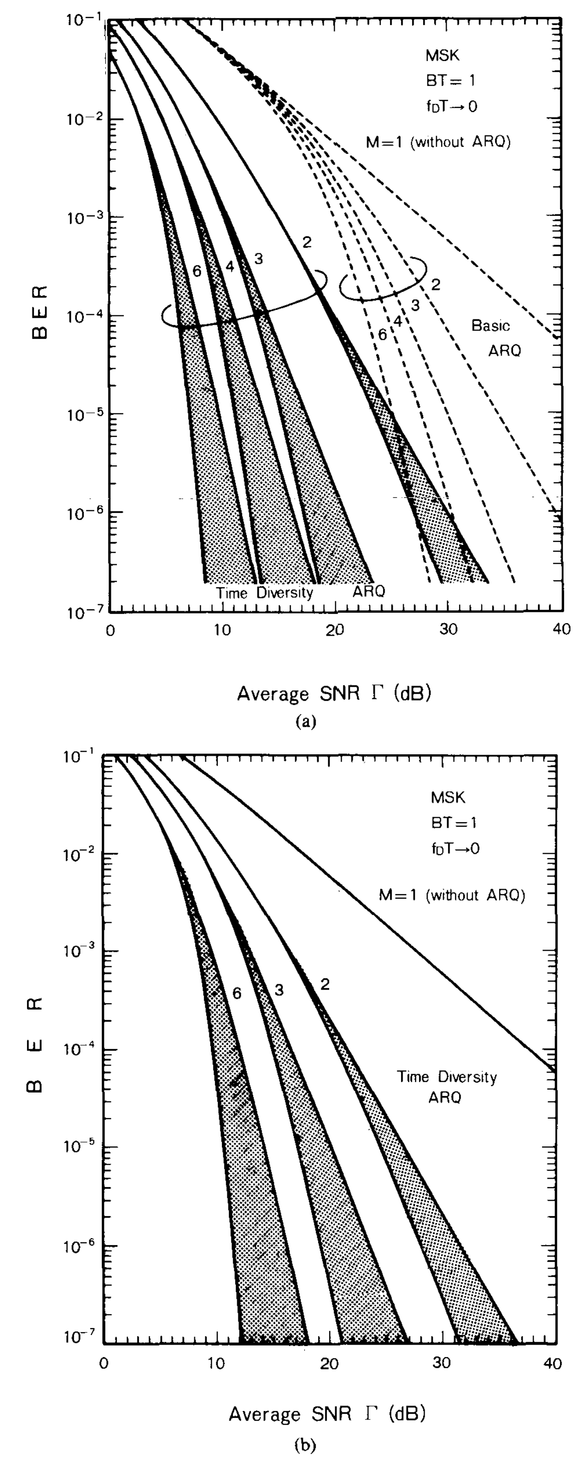



Fig. 3. BER performance for $n = 256$ in Rayleigh fading. Right and left edges of the shaded region show the upper and lower bounds for the time diversity ARQ scheme using (a) MRC and (b) SC.

achieved by the time diversity scheme is given by (8) by replacing $i$ with $M$. Hence, it is identical with the upper bound for the BER of the time diversity ARQ scheme with $M$ finite number of transmissions. It can, therefore, be seen from Figs. 2 and 3 that the time diversity ARQ scheme requires a smaller average SNR with a lower average number of transmissions than the time diversity scheme.

### B. Calculated Signal Energy Per Bit

The calculated results are shown in Fig. 4. The results for the time diversity ARQ scheme are presented by the shaded area, the upper (lower) curves of which are obtained using the upper bounds (lower bounds) for the average number of transmissions and the BER performance. The time diversity ARQ scheme using MRC offers the best performance; the minimum required $E_s/N_0$ is achieved around $M = 18$ and is about 12 dB for BER $= 10^{-4}$ and $n = 256$, while the basic ARQ scheme requires a much larger value (26 dB achieved at $M = 8$). As the BER becomes smaller or the block length becomes larger, larger minimum $E_s/N_0$ is required. However, the minimum for the time diversity ARQ scheme is relatively insensitive to those variations (note that the performance of the time diversity scheme is independent from the block length because no error detection and no repeat request are used). The optimum $M$ yielding the minimum $E_s/N_0$ exists because of the following. On the one hand, the increase in $M$ decreases the required average SNR and thus reduces $E_s/N_0$. On the other hand, for a too large $M$, a slow decrease in the required average SNR is approached while the average number of transmissions increases. Consequently, $E_s/N_0$ becomes larger. The optimum $M$ for the time diversity ARQ scheme using MRC is largest and tends to become larger. However, it is worthwhile noting that since an $E_s/N_0$ close to the minimum can be attained with a small $M$, one never needs to realize these values of $E_s/N_0$ because of a large $M$. For example, for BER $= 10^{-4}$ and $n = 256$, the time diversity ARQ scheme with $M = 5$ requires a value only about 2 dB larger than the minimum when MRC is used.

## IV. Spectral Efficiency

In a cellular land mobile radio system, the radio channels are divided into several channel sets and the same channel sets are reused in different cells, spatially separated from each other in order to realize efficient utilization of the limited radio spectrum. We consider the cell fringe to be where the worst cochannel interference is produced in order to simplify the calculation. Only one interferer is assumed since interference from other cells is less at the worst point than that from the nearest cell.

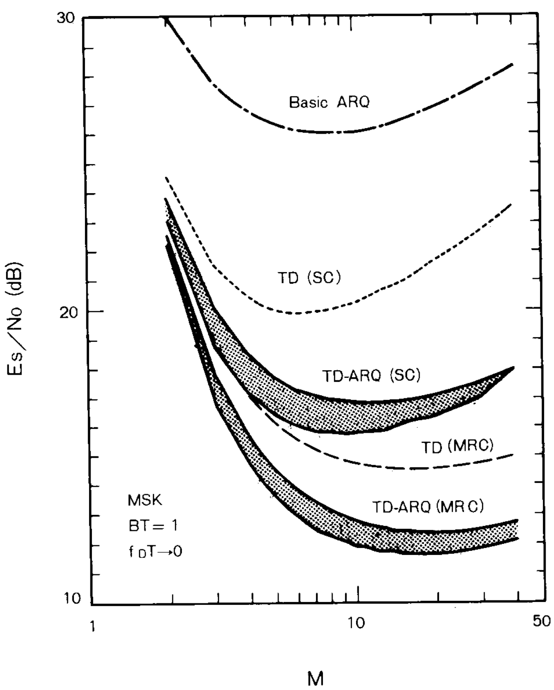
Assuming uniformly distributed traffic over the whole area of interest, spectral efficiency can be defined as

$$\eta = \frac{tc}{WS} = \frac{t}{f_s KS}\ (\text{erl/Hz}\cdot m^2) \tag{11}$$

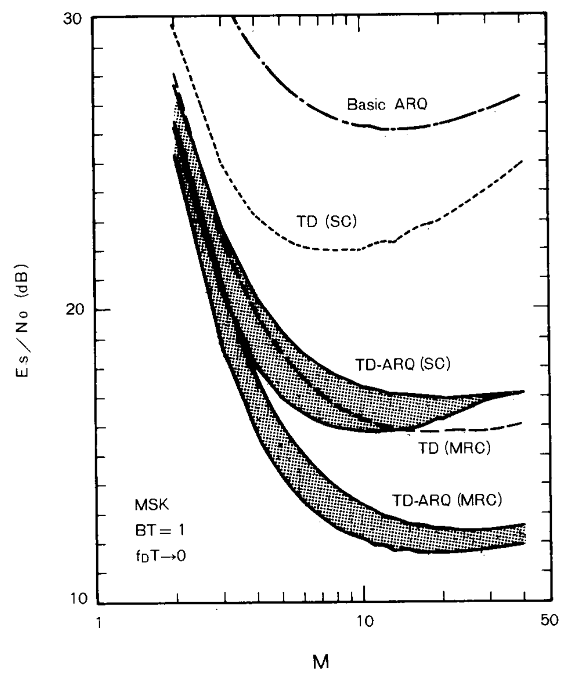
where $t$ is channel traffic capacity (erl/channel), $c$ is the number of channels per cell, $W$ is the whole radio bandwidth $(H_z)$, $S$ is the area of a cell $(m^2)$, $f_s$ is the radio channel spacing $(H_z)$, and $K$ is the number of channel sets. In the ARQ communications system, $t$ is in proportion to the inverse of the average number of transmissions. Letting the distance between the two nearest cochannel cells be denoted by $D$ and the cell radius by $R$, $K$ is given by $K = (D/R)^2/3$ for the hexagonal cell layout [3, ch. 7]. First, we show how the reuse distance $D/R$ is determined and then calculate the spectral efficiency.

### A. Reuse Distance

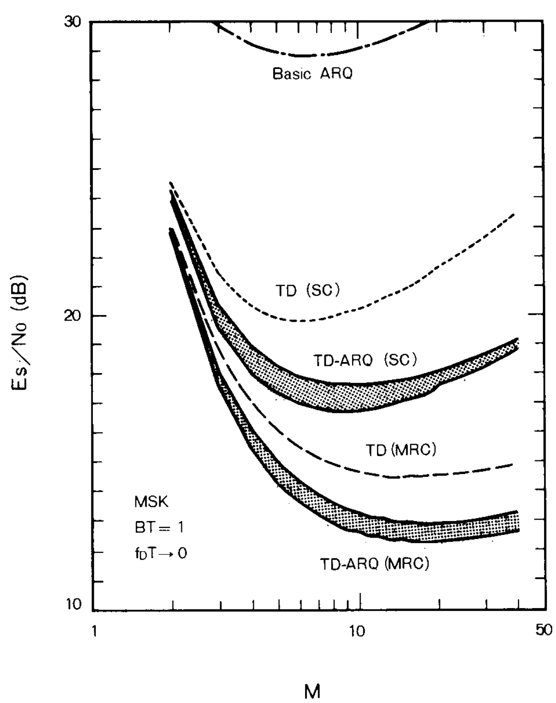
We have assumed Rayleigh fading with constant average power in the previous analysis. However, the average signal and cochannel interference powers measured over a distance of several tens of carrier wavelengths are not constant, but slowly vary around their area average values due to shadow-




(a)




(b)




(c)

Fig. 4. Required $E_s/N_0$ in Rayleigh fading. TD-ARQ = time diversity ARQ scheme. (a) $n = 256$, BER $= 10^{-4}$. (b) $n = 256$, BER $= 10^{-5}$. (c) $n = 512$, BER $= 10^{-4}$.

ing. Their variations can be assumed independent and if expressed on dB scale, follow a Gaussian distribution with the identical standard deviation $\sigma$, typcially between 5 and 7 dB for 900 MHz band in urban areas [2], [3, ch. 2]. Therefore, the resultant average signal-to-interference ratio (SIR) $\Lambda$, on dB scale, also varies around its area average value following a Gaussian distribution with a standard deviation of $\sqrt{2}\sigma$.

The quality of the channels is usually measured by the BER. The probability that the average SIR $\Lambda$ will fade below a specific value $\Lambda_0$ determined in the Rayleigh fading environment (without shadowing) is called the outage probability due to shadowing. For a given outage probability $Q$, the required area average SIR $\Lambda_m$ can be determined from $Q = 1/2$ erfc $[(10 \log_{10} \Lambda_m/\Lambda_0)/2\sigma]$. Since the area average values of the received signal and cochannel powers are proportionate to an inverse $\alpha$th power of the distance between the mobile and base stations where $\alpha$ is 3–4 for 900 MHz band [2], [3, ch. 2], the reuse distance $D/R$ can be obtained from

$$D/R = 1 + \Lambda_m^{1/\alpha}. \tag{12}$$

### B. Calculated Spectral Efficiency

We assume that interference is the predominant cause of errors. The cochannel interference is frequency-modulated with the same modulation index $\beta$ as for the signal. The fading on the interference is independent, but has identical statistical properties with that of signal fading and thus $\rho$ is given by

$$\rho = j\,\frac{\beta(a_m\Lambda + b_i)}{(\Lambda+1)\sqrt{[\beta^2 + 2(f_D T)^2]}} \tag{13}$$

where $b_i = \pm 1$ (with equal probability) is a binary data of cochannel interference. The average number $N(M)$ of transmissions and the BER performance can be obtained as the upper and lower bounds in the same way described in Section III ($\beta = 0.5$ and $f_D T \to 0$). From the BER performance, the required average SIR $\Lambda_0$ is obtained. Then, using the values of $\Lambda_m$ and $N(M)$, the spectral efficiency can be calculated from

$$\eta \propto \eta_0 = N^{-1}(M)(1 + \Lambda_m^{1/\alpha})^{-2}. \tag{14}$$

The spectral efficiency was calculated for $Q = 10$ percent under the condition $\sigma = 6$ dB and $\alpha = 3.5$. The results are shown in Fig. 5 for BER $= 10^{-4}$ and $n = 256$. The results for the time diversity ARQ scheme are represented by the shaded region between the two curves, which are calculated using the upper/lower bounds of $N(M)$ and the BER. The optimum value of $M$ yielding the maximum spectral efficiency is relatively small in comparison to that minimizing $E_s/N_0$. The time diversity ARQ scheme using MRC offers the largest efficiency; the maximum is achieved at $M = 4$ and is about 2 times larger than the basic scheme if the optimum $M$ is used for each scheme. The maximum efficiency decreases as the BER becomes smaller or the block length becomes larger. However, it was found that the maximum for the time diversity ARQ scheme is relatively insensitive to those variations (note that the performance of the time diversity scheme is independent of block length).

## V. Conclusion

The performance of the time diversity ARQ scheme with $M$ finite number of transmissions has been investigated. With the optimum $M$ being used for each ARQ scheme, the time diversity ARQ scheme using MRC has about a 14 dB advantage in required signal energy per bit in a Rayleigh fading environment. It is about 2 times more spectral efficient than the basic ARQ scheme for BER $= 10^{-4}$ and $n = 256$ when outage probability $Q = 10$ percent at the cell fringe if the standard deviation of shadowing $\sigma = 6$ dB and the propagation exponent $\alpha = 3.5$. It has also been proved superior to the time diversity scheme.

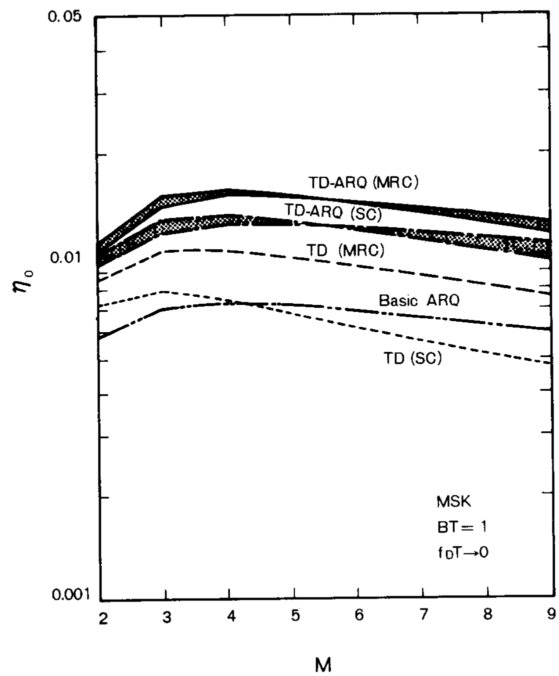



Fig. 5. Spectral efficiency. $n = 256$, BER $= 10^{-4}$.

This paper assumed random bit errors in the vector $u_l$ for the calculation of repeat request probability. Analysis of the effects of burst errors is an area for further study. The application of the analysis presented in this paper to other modulation/demodulation schemes, i.e, digital FM with differential and coherent demodulations, is straightforward.

## Appendix

Obtaining the expression (8) for the BER is summarized below. We have applied the analysis presented in [8] with some modification. In a Rayleigh fading environment, both $z_{lm}$ and $z_{lm}'$ are zero-mean complex Gaussian variables ($l = 1, 2 \cdots i$). Let us first consider the MRC case [see (3)]. With all $z_{lm}$ being given, $v_{\rm im}$ becomes a Gaussian variable with mean $(\sigma_2/\sigma_1)\rho_s R_m^2$ and variance $\sigma_2^2(1 - |\rho|^2)R_m^2$ where $\sigma_1^2 = 1/2 \langle|z_{lm}|^2\rangle$ (average received signal power), $\sigma_2^2 = 1/2 \langle|z_{lm}'|^2\rangle$, and $\rho = 1/2 \langle z_{lm}(-z_{lm}')^*\rangle/\sigma_1\sigma_2$ for all $l$ and $m$ and $R_m^2 = R_{1m}^2 + R_{2m}^2 + \cdots + R_{\rm im}^2$. Bit error is produced when $a_m v_{\rm im} < 0$. Therefore, the conditional BER with given $R_m$ is obtained by

$$P_{ei}(R_m) = \frac{1}{2}\, erfc\left(\frac{a_m\rho_s}{\sqrt{[1-|\rho|^2]}}\,\frac{R_m}{\sqrt{2}\sigma_1}\right) \tag{A1}$$

where

$$\rho = \rho_c + j\rho_s = j\,\frac{\Gamma\Phi_s'}{\sqrt{[\Gamma+1]}\sqrt{[\Gamma\{\Phi_s'^2 - \xi_s''(0)\} - \xi_n''(0)]}} \tag{A2}$$

and $\Gamma$ is the average SNR and $\xi_s(\tau)$ and $\xi_n(\tau)$ are the autocorrelation functions of $z_s(t)$ and $z_n(t)$, respectively. Next, consider the SC case [see (5)]. With all $z_{lm}$ being given, $v_{\rm im}$ is a Gaussian variable with mean $(\sigma_2/\sigma_1)\rho_s$ and variance $\sigma_2^2(1 - |\rho|^2)R_m^2$ where $R_m = \max\,[R_{1m}, R_{2m}, \cdots R_{\rm im}]$. Hence, the conditional BER can be represented by (A1) with $R_m = \max\,[R_{1m}, R_{2m}, \cdots R_{\rm im}]$.

Since each retransmission takes place after a sufficient time interval, $R_{1m}$, $R_{2m}$, $\cdots$, and $R_{\rm im}$ can be assumed to be independent Rayleigh envelopes. The probability density function (pdf) of $R_m$ is given by

$$p(R_m)=\frac{R_m}{\sigma_1^2}\frac{(R_m^2/2\sigma_1^2)^{i-1}}{(i-1)!}\exp\left(-\frac{R_m^2}{2\sigma_1^2}\right),\quad \text{MRC}$$
$$=i\frac{R_m}{\sigma_1^2}\exp\left(-\frac{R_m^2}{2\sigma_1^2}\right)\cdot\left[1-\exp\left(-\frac{R_m^2}{2\sigma_1^2}\right)\right]^{i-1},\quad \text{SC.} \tag{A3}$$

Averaging (A1) with (A3) leads to the expression given by (8).

ACKNOWLEDGMENT

The authors are grateful to the reviewers for their helpful criticism and comments.

REFERENCES

[1] R. H. Clarke, "A statistical theory of mobile radio reception," *Bell Syst. Tech. J.*, vol. 47, pp. 957-1000, July 1968.
[2] Y. Okumura *et al.*, "Field strength and its variability in UHF and VHF landmobile radio service," *Rev. Elec. Commun.*, vol. 16, pp. 825-873, Sept.-Oct. 1968.
[3] W. C. Jakes, Jr., Ed., *Microwave Mobile Communications.* New York: Wiley, 1974.
[4] S. Lin, D. J. Costello, and M. J. Miller, "Automatic-repeat request error control schemes," *IEEE Commun. Mag.*, vol. 22, pp. 5-17, Dec. 1984.
[5] R. A. Comroe and D. J. Costello, Jr., "ARQ schemes for data transmission in mobile radio systems," *IEEE Trans. Vehic. Technol.*, vol. VT-33, pp. 88-97, Aug. 1984.
[6] G. Benelli, "An ARQ scheme with memory and soft error detection," *IEEE Trans. Commun.*, vol. COM-33, pp. 285-288, Mar. 1985.
[7] F. Adachi and S. Itoh, "Efficient ARQ with time diversity reception technique—A time diversity ARQ," *Electron. Lett.*, vol. 22, pp. 1257-1258, Nov. 1986.
[8] F. Adachi and J. D. Parsons, "Postdetection diversity using modified phase correction loop," *IEE Proc. F., Commun., Radar, Signal Process.*, vol. 134, pp. 27-34, Feb. 1987.
[9] C. S. K. Leung and A. Lam, "Forward error correction for an ARQ scheme," *IEEE Trans. Commun.*, vol. COM-29, pp. 1514-1519, Oct. 1981.

# Transmission Efficiency in Photon Counting Channels

GIOVANNI CANCELLIERI

*Abstract*—**The ultimate theoretical limit of transmission efficiency over an ideal photon counting channel is investigated on the basis of Shannon theorem for discrete channels. A binary transmission, with different *a priori* symbol probabilities, is demonstrated to be more efficient than an *M*-ary orthogonal PPM transmission.**

## I. INTRODUCTION

$M$-ary orthogonal PPM has been demonstrated to be a very efficient transmission technique over an ideal photon counting channel [1]. As an ideal photon counting channel, we assume that based on the intensity modulation of the optical source, without background thermal noise and without dark current in the photodetector, which operates a direct detection of the incident light. This transmission obeys to pure Poisson statistics. Considering a two-level intensity modulation, with the low level at zero (on-off intensity modulation), like that we could obtain by means of an external shutter placed across the optical beam, subdivision of the symbol time duration into $M$ equal adjacent time slots enables us to obtain an $M$-ary orthogonal set of symbols, each characterized by the high transmission level in only one time slot of the subdivision. In order to reach the maximum efficiency, it is convenient to set the receiver decision threshold between zero and one detected photon [1]. This way, letting $N$ the average number of photons carried by each $M$-ary symbol, we have only an erasure probability affecting any symbol, which coincides with the probability $\exp(-N)$ that no photons are detected.

The average information per symbol in a transmission of this type [2] tends to $\lg_2 M$. This implies an efficiency, expressed in bit/photon, which increases indefinitely with increasing $M$. As a drawback, we have an enormous spread of bandwidth, according to the factor $M/\lg_2 M$. This can be reduced by the use of a suitable code. For instance, with a Reed-Solomon code, having $n$ total $M$-ary symbol per word, $k$ of which are significant and $(n-k)$ redundant, we have the constraint $n = M - 1$, and the code is able to correct any pattern of $(n-k)$ erasures in a word, leading to a remarkable reduction in bandwidth spreading. The efficiency of 1 bit/photon, affected by a bit error rate of about $10^{-6}$, can be reached with $M = 32$ instead of $2^{21}$, which means a reduction in the bandwidth expansion factor $\beta$ from $2^{21}/21 \simeq 10^6$ to 31/16 32/5 $\simeq$ 12 [3].

This is probably one of the most efficient practical codes to be employed in a transmission of this type. However, it could be of some interest to investigate the theoretical maximum limit of efficiency, fixed a given bandwidth expansion, we can reach over an ideal photon counting channel. The purpose of the present paper is to evaluate such a limit on the basis of Shannon theorem for discrete channels [4]. In particular, instead of starting directly from an $M$-ary orthogonal PPM, we consider the transmission of binary symbols, associated to the low and to the high level of the optical source intensity modulation, with *a priori* probabilities $P_0$ and $P_1$, which appears as a more general transmission technique.

Although $N$, which represents the average number of received photons when the high level is transmitted, is in principle a continuous variable, we will restrict ourselves to assume only positive integer values for it. This choice is justified by the following considerations. When $N$ is large (of the order of ten or even larger), discrete variations by one unit in its value are sufficiently dense to describe accurately the effects of such a parameter on the main transmission figures, e.g., the efficiency or the bandwidth expansion factor. When $N$ is small (and the results of the present analysis will show that efficiency increases as $N$ decreases), we have to keep well in mind that the model assumed tends to fail, in practical cases, as $N$ tends to zero because of the possible presence of background thermal noise photons. These photons, even if extremely rare, would change completely the scheme of the channel, by introducing not negligible probabilities of wrong transitions from transmitted to received symbols [5]. In this sense, the value $N = 1$ is to be considered as the minimum limit of $N$, for practical transmissions.

## II. EFFICIENCY AGAINST BANDWIDTH EXPANSION

The amount of information per symbol carried on average in a binary channel (symbols 0 and 1), according to Shannon [4],

Paper approved by the Editor for Synchronization and Optical Detection of the IEEE Communications Society.
The author is with the Dipartimento di Elettronica e Automatica, Università di Ancona, Via Brecce Bianche, 60131 Ancona, Italy.
IEEE Log Number 8825331.

0090-6778/89/0200-0183$01.00 

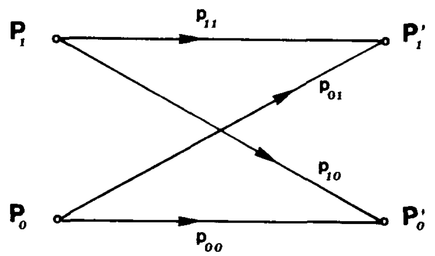



Fig. 1. Scheme of a general binary channel.

can be written as

$$I = H(Y) - H(Y|X) \tag{1}$$

where $H(Y)$ is the entropy

$$H(Y) = \sum_{j=0}^{1} P_j' \lg_2 \left(\frac{1}{P_j'}\right) \tag{2}$$

and $H(Y|X)$ is the conditional entropy

$$H(Y|X) = \sum_{i=0}^{1} P_i \sum_{j=0}^{1} P_{ij} \lg_2 \left(\frac{1}{p_{ij}}\right). \tag{3}$$

In the above expressions $P_j$, $P_j'$ represent the input and output probabilities of the two symbols, and $p_{ij}$ their transition probabilities, according to the scheme of Fig. 1.

For an ideal photon counting channel, with symbol 0 transmitted when the shutter is closed (no photons in the channel), and symbol 1 transmitted when the shutter is open ($N$ photons received on average), we have

$$p_{10} = \exp(-N), \quad p_{11} = 1 - \exp(-N),$$
$$p_{01} = 0, \quad p_{00} = 1. \tag{4}$$

Furthermore, we can write

$$P_0 = 1 - P_1, \quad P_1' = p_{11} P_1, \quad P_0' = P_0 + p_{10} P_1.$$

Substitution into (1), considering (2) and (3), yields a simple function of the two variables $N$, $P_1$,

$$\begin{aligned} I(N, P_1) = {} & P_1[1-\exp(-N)] \lg_2 \left\{\frac{1}{P_1[1-\exp(-N)]}\right\} \\ & + [1 - P_1 + P_1 \exp(-N)] \\ & \cdot \lg_2 \left[\frac{1}{1 - P_1 + P_1 \exp(-N)}\right] \\ & - P_1 \left\{\exp(-N) \lg_2 \left[\frac{1}{\exp(-N)}\right]\right. \\ & \left. + [1-\exp(-N)] \lg_2 \left[\frac{1}{1-\exp(-N)}\right]\right\}, \end{aligned} \tag{5}$$

which represents the maximum average information per symbol, according to Shannon theorem for discrete channels. This limit can be reached by means of an ideal code. In order to maintain a fixed information rate, we have to consider a bandwidth expansion factor $\beta$ whose minimum value coincides with $1/I(N, P_1)$, i.e.

$$\beta > \frac{1}{I(N, P_1)}. \tag{6}$$

The transmission efficiency, expressed in bit/photon, on the other hand, turns out to be

$$\rho = \frac{I(N, P_1)}{P_1 N}, \tag{7}$$

in fact, at the denominator we would add another term, $(1 - P_1)$ times the photons carried by symbol 0, which however are zero. In conclusion, letting $\beta_0$ the minimum of $\beta$, combination of (6) and (7) yields

$$\rho(N, P_1) = \frac{1}{\beta_0(N, P_1) P_1 N}. \tag{8}$$

The main objective of an optimized transmission system is to maximize $\rho$, maintaining $\beta_0$ as small as possible. In this respect, it is convenient to express $P_1$ as a function of $\beta_0$ and $N$, through inversion of the equation

$$\beta_0 = \frac{1}{I(N, P_1)}. \tag{9}$$

In Fig. 2 some curves of $\beta_0$ against $P_1$ are plotted with $N$ as a parameter. Each curve exhibits a minimum. The branches on the left-hand side of such minimum are expected to be those associated with the higher efficiency. In fact the input probability of symbol 1, which is affected by possible errors, should be minimized. In order to obtain $P_1 = P_1(\beta_0, N)$, therefore, we will invert only these branches. Next, we can rewrite (8) as

$$\rho(\beta_0, N) = \frac{1}{\beta_0 P_1(\beta_0, N) N}. \tag{10}$$

Fig. 3 shows the efficiency $\rho$ against the minimum bandwidth expansion factor $\beta_0$, with $N$ as a parameter, as obtained from (10). Each curve tends asymptotically to $\infty$, as expected, on the analogy of the known property of $M$-ary orthogonal PPM. This increase is however very slow, therefore, the bandwidth expansion factor becomes rapidly too large. For very small values of $\beta_0$, a higher $N$ is more convenient, whereas for $\beta_0$ larger than about 3 the lower the $N$ the better the efficiency. The most efficient transmission, characterized by integer $N$, is therefore that having $N = 1$.

Each curve starts from an initial point, not exactly coincident with $\beta_0 = 1$. These starting points can be calculated analytically, in correspondence with the values of $P_1$ which maximize $I(P_1, N)$ (minimize $\beta_0$), for a given $N$. So, differentiation of (5) with respect to $P_1$ yields an expression, which, equated to zero, enables us to state that such values turn out to be

$$\bar{P}_1(N) = \frac{1}{1-\exp(-N)} \frac{1}{1+2^{a(N)}}.$$

where

$$a(N) = \frac{1}{1-\exp(-N)} \left\{\exp(-N) \lg_2 \left[\frac{1}{\exp(-N)}\right] + [1-\exp(-N)] \lg_2 \left[\frac{1}{1-\exp(-N)}\right]\right\}.$$

For example, we obtain $\bar{P}_1(1) = 0.413$, $\bar{P}_1(2) = 0.448, \cdots$, $\bar{P}_1(5) = 0.493$, while, for higher values of $N$, $\bar{P}_1$ is closer and closer to 0.5. Correspondingly, we have $\beta_0|_{N=1} = 2.29$, $\beta_0|_{N=2} = 1.42, \cdots, \beta_0|_{N=5} = 1.03$, as starting points of the curves in Fig. 3.

As regards the curve characterized by $N = 1$, we can observe how, in its first part, efficiency doubling is possible

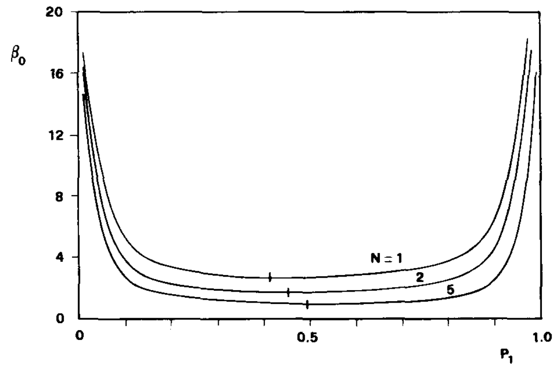



Fig. 2. Bandwidth expansion factor $\beta_0$ against $P_1$, with $N$ as a parameter.

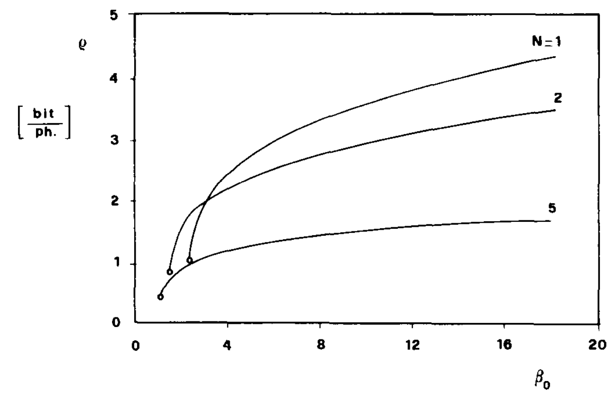



Fig. 3. Transmission efficiency $\rho$ against the bandwidth expansion factor $\beta_0$, with $N$ as a parameter.

by means of bandwidth doubling, for instance passing from $\beta_0 = 2.5$ ($\rho \simeq 1.5$ bit/photon) to $\beta_0 = 5$ ($\rho \simeq 3$ bit/photon). This is a very effective use of bandwidth expansion. More on the right, even this curve tends to saturate, to the point where increasing $\beta_0$ over 10 appears useless.

## III. Comparison to an $M$-ary Orthogonal PPM

The transmission described in the previous section is based on simple binary symbols with different *a priori* probabilities $P_0$ and $P_1$. It can be compared to that based on an $M$-ary orthogonal PPM by setting

$$P_1 = \frac{1}{M}, \quad P_0 = \frac{M-1}{M}.$$

The maximum average information per binary symbol, in our transmission, from (5), can be obtained simply by letting $P_1 = 1/M$, i.e.

$$\begin{aligned} I\left(N, \frac{1}{M}\right) &= \frac{1}{M}[1-\exp(-N)] \lg_2 \left[\frac{M}{1-\exp(-N)}\right] \\ &+ \frac{M-1+\exp(-N)}{M} \lg_2 \left[\frac{M}{M-1+\exp(-N)}\right] \\ &- \frac{1}{M}\left\{\exp(-N) \lg_2 \left[\frac{1}{\exp(-N)}\right]\right. \\ &\left. + [1-\exp(-N)] \lg_2 \left[\frac{1}{1-\exp(-N)}\right]\right\}. \end{aligned} \tag{11}$$

On the other hand, the maximum average information per $M$-ary symbol, in an orthogonal PPM, considering the erasures, turns out to be [3]

$$I'_M = \lg_2 M[1-\exp(-N)] \tag{12}$$

so that each elementary time slot, in the PPM word, on average, carries the maximum information

$$I'\left(N, \frac{1}{M}\right) = \frac{1}{M} I'_M = \frac{1}{M} \lg_2 M[1-\exp(-N)]. \tag{13}$$

From (7), we can observe that the transmission efficiency in the two cases results in

$$\rho = M\frac{I}{N}, \quad \rho' = M\frac{I'}{N}, \tag{14}$$

with $I$ and $I'$ respectively given by (11) and (13). At the same time, the minimum bandwidth expansion factor becomes, respectively,

$$\beta_0 = \frac{1}{I}, \quad \beta_0' = \frac{1}{I'}. \tag{15}$$

In this sense, a greater average information per binary symbol or per time slot implies a higher efficiency, and, at the same time, a smaller bandwidth expansion factor. Since $I(N, 1/M)$ is always greater than $I'(N, 1/M)$, as (11) and (13) easily show, we can conclude that the binary transmission discussed in the previous section does exhibit a better performance than traditional PPM, at least in principle.

This is due to the weaker constraints which characterize this binary transmission. In fact, the two techniques appear are very similar because they both are based on short pulses (transmission of the high level), separated by long time intervals in which no photons are sent on the channel, and the receiver has to detect the position and not the amplitude of pulses. Nevertheless, for example, in PPM we certainly have at least one pulse over a time interval whose duration is of the order of $2M$ time slots, whereas this constraint is not present in the binary transmission, although, on average, we have the same number of pulses per unit time, in the two situations. Weaker constraints entail a greater uncertainty in what is received, that is a larger amount of information transfer, and hence a better transmission efficiency.

In order to evaluate quantitatively the advantage of the binary transmission over the $M$-ary PPM transmission, we can compare $\rho$ and $\rho'$ as functions of $M$, with $N$ as a parameter. This comparison is reported in Fig. 4. Nevertheless, a more accurate evaluation can be performed on the basis of an equal bandwidth expansion factor. To this purpose, we have to express $M$ as a function of $\beta_0'$ and $N$, for a PPM transmission. From (13) and the second of (15), we have

$$\beta_0' = \frac{M}{\lg_2 M[1-\exp(-N)\}}. \tag{16}$$

In Fig. 5 some curves of $\beta_0'$ against $M$, with $N$ as a parameter, are shown. They are the same function of $M$, characterized by a minimum at $M = 3$, weighted by the factor $1/[1 - \exp(-N)]$. The branches on the right-hand side are here expected to be those characterized by the higher efficiency because $M$ should be maximized in order to reduce, on average, the effects of erasures. Inversion of these branches leads to $M = M(\beta_0', N)$, finally substitution into (14) enables us to plot $\rho'$ as a function of $\beta_0'$, with $N$ as a parameter, for an $M$-ary transmission. Also in this case, when $\beta_0'$ is larger than about 3, the choice $N = 1$ is that characterized by the maximum efficiency.

In Fig. 6 the curves $\rho'(\beta_0')$ with $N = 1$ and $N = 2$ (dashed

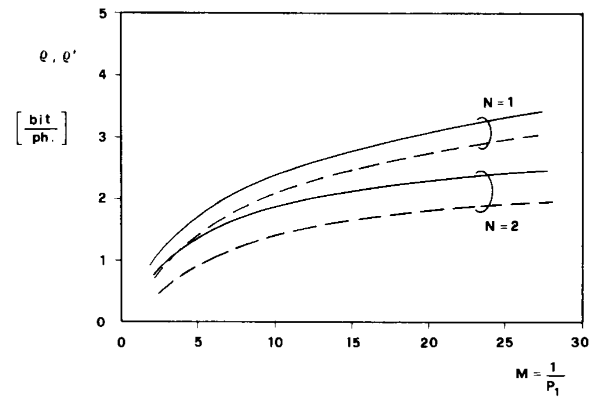



Fig. 4. Comparison between $\rho$ (continuous lines) and $\rho'$ (dashed lines) against $M = 1/P_1$ with $N = 1$ and $N = 2$.

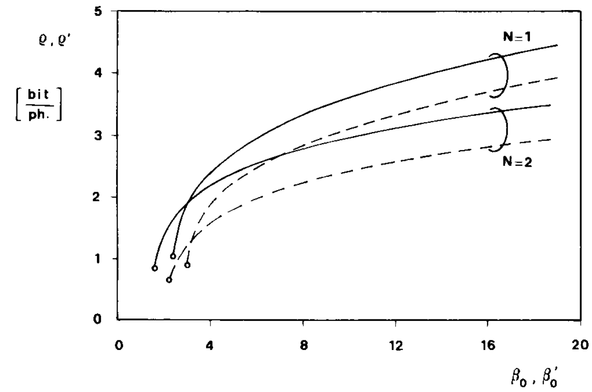



Fig. 6. Comparison between $\rho$ (continuous lines) and $\rho'$ (dashed lines) against the bandwidth expansion factor for $N = 1$ and $N = 2$.

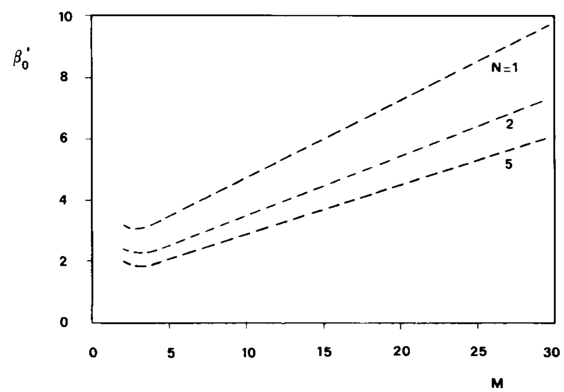



Fig. 5. Bandwidth expansion factor $\beta_0'$ against $M$, with $N$ as a parameter, for an $M$-ary orthogonal PPM.

lines) are compared to the corresponding curves $\rho(\beta_0)$ of the binary transmission (continuous line) already shown in Fig. 3. For a given $\beta_0 = \beta_0'$, larger than about 4, $\rho$ exceeds $\rho'$ by a quantity practically constant, of the order of 0.5 bit/photon. It is not easy to explain mathematically this net advantage of the binary transmission over the PPM transmission, (which, however, has been numerically verified up to rather high values of the bandwidth expansion factor), in fact, the limit $M \to \infty$ yields correctly $I = I' = 0$. In this sense, the theoretical limit of maximum efficiency, independently of any practical constraint on bandwidth expansion, is to be fixed at $M$ very large, but not infinite, for both the transmission techniques.

## IV. Comparison Based on Error Probabilities

The main results of Sections II and III can be summarized as follows.

1) Except when the bandwidth expansion factor is very small, it is convenient to take the average number of photons per pulse $N$ as low as possible (the best choice for an integer $N$ is 1).

2) Surprisingly, a simple binary transmission, with different *a priori* probabilities of the two transmitted symbols, appears to be more efficient than an orthogonal PPM transmission, whose $M$-ary symbols are equiprobable.

The transmission efficiency here considered is that one could reach, without errors, by means of an ideal transmission system. A legitimate question is how the above conclusions can be extended to the case of real transmission systems, for which a finite probability of error always occurs. In this situation, the transmission efficiency is to be calculated as

$$\rho=\frac{1}{N}\left[\lg_2\left(\frac{1}{P_1}\right)+\frac{1-P_1}{P_1}\lg_2\left(\frac{1}{1-P_1}\right)\right], \quad \rho'=\frac{1}{N}\lg_2 M,$$

respectively, for a binary and for a PPM $M$-ary transmission. Correspondingly, the average bit error probabilities are

$$P_E \simeq \frac{1}{2}P_1 \exp(-N), \quad P_E' \simeq \frac{1}{2}\exp(-N).$$

These approximate expressions tends to be exact with increasing $1/P_1$ or $M$. Finally, the bandwidth expansion factors turn out to be

$$\beta=\frac{1}{P_1\lg_2\left(\frac{1}{P_1}\right)+(1-P_1)\lg_2\left(\frac{1}{1-P_1}\right)}, \quad \beta'=\frac{M}{\lg_2 M}.$$

Also this comparison shows an advantage of the binary transmission. In fact, setting $M = 1/P_1$, we have $\rho > \rho'$, $\beta < \beta'$, and $P_E/P_E' \simeq P_1$. Thus, the higher the efficiency, which increases as $P_1$ decreases, the lower the $P_E$ with respect to $P_E'$. Furthermore, as regards observation *i* listed above, a too small value of $N$ cannot be accomplished by the PPM transmission, because $P_E'$ becomes too large. On the contrary, when $P_1$ is of the order of $10^{-3}$, even $N$ of few units can be supported by the binary transmission.

Yet, as already stressed, very efficient practical codes can be applied to the PPM transmission, whereas this is not easy for the binary transmission, especially owing to the different a priori probabilities of its two symbols. Nevertheless, we can compare a coded PPM transmission, like that suggested in [3], with a binary uncoded transmission, like that here proposed. The residual average bit error probability of an $M$-ary PPM transmission, applying a Reed–Solomon code, with $M = 32$, $n = 31$, $k = 16$, can be estimated as

$$P_E' \simeq \frac{1}{2}\sum_{j=16}^{31}\binom{31}{j}\exp(-Nj)[1-\exp(-N)]^{31-j}.$$

The bandwidth expansion factor of this transmission is $\beta' = 31/16\ 32/5 = 12.4$, whereas its efficiency turns out to be $\rho' = 16/31\ 5/N \simeq 2.5/N$. Approximately the same bandwidth expansion factor is obtained by a binary transmission characterized by $P_1 = 10^{-2}$.

In Fig. 7, $P_E$ and $P_E'$ are compared in this situation, considering various integer values for $N$. Interpolation curves

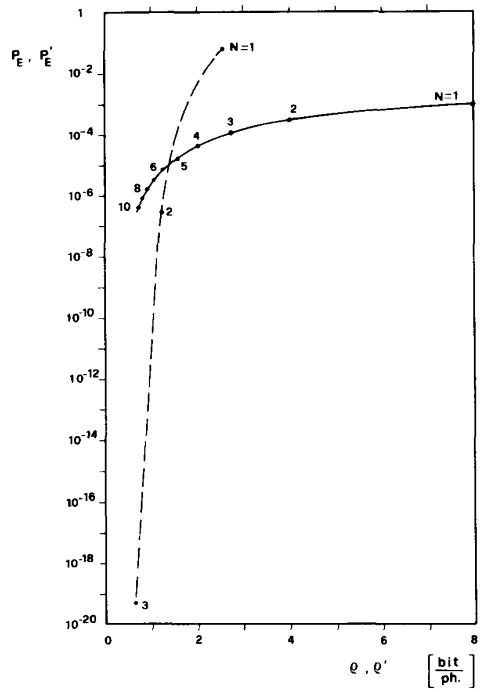



Fig. 7. Bit error probabilities against the transmission efficiency, for binary (continuous line) and PPM transmission (dashed line). The two techniques exhibit approximately the same bandwidth expansion factor ($\beta = \beta' \simeq 12.4$).

are depicted in continuous and in dashed lines, respectively. These curves exhibit an intersection at $\rho = \rho' \simeq 1.55$ bit/photon. For higher values of the efficiency, the binary transmission appears, even in this case, to be preferable. On the other hand, for lower values of the efficiency, the PPM transmission achieves bit error probabilities extremely low. $\rho$ and $\rho'$ increase as $N$ decreases, as expected. When $N = 1$, which is the smallest integer values for $N$, the PPM transmission gives $\rho' \simeq 2.5$ bit/photon, whereas the binary transmission approaches an efficiency $\rho$ more than three times larger.

## V. Conclusions

The objective of this paper was to investigate the maximum transmission efficiency one can reach over an ideal photon counting channel, having fixed the bandwidth expansion factor. First, the ideal situation, represented by Shannon theorem for discrete channels, has been analyzed. A low average number of photons per pulse is demonstrated to be preferable. A binary transmission, with different *a priori* probabilities of the two transmitted symbols, exhibits a higher efficiency than that of an orthogonal PPM transmission, whose $M$-ary symbols are equiprobable, for an equal bandwidth expansion.

Then practical transmissions have been considered. The PPM technique can be very efficiently coded, and, in some situations, is characterized by a bit error probability lower than that of the uncoded binary technique. However, uncoded binary transmission remains extremely attractive for the achievement of ultra-high transmission efficiencies.

## References


[1] J. R. Pierce, "Optical channels: Practical limits with photon counting," *IEEE Trans. Commun.*, vol. COM-26, pp. 1819–1821, 1978.

[2] J. R. Pierce, E. C. Posner, and E. R. Rodemich, "The capacity of the photon counting channel," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 61–77, 1981.

[3] R. J. McEliece, "Practical codes for photon communication," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 393–398, 1981.

[4] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication*. Urbana, IL: The Univ. Illinois Press, 1959.

[5] M. Charbit and C. Benjaballah, "Probability of error and capacity of PPM photon counting channel," *IEEE Trans. Commun.*, vol. COM-34, pp. 600–605, 1986.


# Comments on "Fundamental Conditions Governing TDM Switching Assignments in Terrestrial and Satellite Networks"

SOUNG C. LIEW

*Abstract*—**The problem in the above paper[1] is formulated in terms of a max-flow network problem. The main theorem in the paper can be proved quite simply using the max-flow–min-cut theorem once the proper way of looking at the problem is identified.**

An alternative, and perhaps simpler proof of Theorem I of the above paper[1] is presented. This proof uses the max-flow network formulation. Since max-flow is a well-studied combinatorial optimization problem, it may shed some light on designing good algorithms for the problem concerned.

Unless otherwise defined, the notation here is adopted from the paper. The assumptions are listed here for reference

$$\sum_{i=1}^{M}\sum_{j=1}^{M} t_{ij} = NC \tag{1}$$

$$\sum_{j=1}^{M} t_{ij} \le C, \quad i=1 \text{ to } M \tag{2}$$

$$\sum_{i=\Omega_1(g)}^{\Omega_2(g)}\sum_{j=1}^{M} t_{ij} = K_g C, \quad g=1 \text{ to } G \tag{3}$$

$$\sum_{i=1}^{M} t_{ij} \le C, \quad j=1 \text{ to } M \tag{4}$$

$$\sum_{i=1}^{M}\sum_{j=\Omega_1'(g')}^{\Omega_2'(g')} t_{ij} = K'_{g'} C, \quad g'=1 \text{ to } G'. \tag{5}$$

In addition, (2) and (4) are satisfied with equality only if $P_i$ and $P'_j$ are equal to 1, respectively.

As suggested, it suffices to show how to find a 0-1 matrix $T' = (t'_{ij}) \le T = (t_{ij})$ such that

$$\sum_{i=1}^{M}\sum_{j=1}^{M} t'_{ij} = N \tag{6}$$

Paper approved by the Editor for Satellite Communications and Coding of the IEEE Communications Society. Manuscript received November 1, 1987; revised April 17, 1988.

The author is with Bellcore, Morristown, NJ 07960-1910.

IEEE Log Number 8825340.

[1] K. Y. Eng and A. S. Acampora, *IEEE Trans. Commun.*, vol. COM-35, pp. 755–761, July 1987.