Random FM noise with selection combining

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Random FM Noise with Selection Combining

F. ADACHI AND J. D. PARSONS

Abstract—**Random FM noise using two branch selection combining with correlated Rayleigh fading signals is analyzed. General expressions are derived for the cumulative distribution function (cdf) and mean square value of the random FM noise, which can be applied to any type of diversity such as space, polarization, etc. Calculated results show that if two horizontally spaced antennas parallel with the direction of vehicle motion are used at a mobile station, random FM noise can be significantly reduced for small antenna spacings.**

I. INTRODUCTION

In land mobile radio systems, the envelope and phase of the received signal vary in a random manner because of multipath fading. If an FM signal having a bandwidth less than the coherence bandwidth of the multipath channel is transmitted, then at the receiver the random FM noise produced by the received signal phase variation is superimposed on the desired modulation [1] and thus places an upper bound on the baseband signal-to-noise power ratio achievable at the output of a frequency discriminator [2]. Diversity reception [3] can be used to combat the effects of random FM noise and Davis [1] has shown theoretically that the use of selection combining can substantially reduce the random FM noise for independent Rayleigh fading signals. However, in practical diversity systems, fading signals received at diversity antennas may be partially correlated and it is therefore important to investigate how the fading correlation affects the random FM noise with diversity. This paper extends Davis' results to the case of two branch selection combining with correlated Rayleigh fading signals.

II. ANALYSIS

We assume that the transmitted signal is unmodulated and that two correlated Rayleigh fading signals, Re $[z_1(t) \exp (j\omega_c t)]$ and Re $[z_2(t) \exp (j\omega_c t)]$, are received ($\omega_c$ is the carrier angular frequency) and $z_i(t) = R_i(t) \exp (j\theta_i^{(t)})$. It is also assumed that the power spectra of $z_1(t)$ and $z_2(t)$ are identical and that both spectra are symmetrical.

The random FM noise of ith signal ($i = 1, 2$) is

$$\dot{\theta}_i(t) = -\text{Im} \left\{ \frac{z_i(t)\dot{z}_i(t)^*}{|z_i(t)|^2} \right\} \tag{1}$$

where the asterisk and dot denote complex conjugate and time derivative, respectively. Selection combining produces an output $\dot{\theta}(t) = \dot{\theta}_1(t)$ for $R_1(t) \geqq R_2(t)$ and $\dot{\theta}_2(t)$ for $R_1(t) < R_2(t)$ where $R_i(t) = |z_i(t)|$ is the envelope.

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In our case, $z_1 = z_1(t)$, $z_2 = z_2(t)$, $\dot{z}_1 = \dot{z}_1(t)$ and $\dot{z}_2 = \dot{z}_2(t)$ are mutually correlated zero mean complex Gaussian variables. It can be observed from (1) that the random FM noise $\dot{\theta}_i$, with both z_1 and z_2 being given, becomes a Gaussian variable. The conditional probability density function (pdf) of $\dot{\theta}_i$ can be expressed as

$$p(\dot{\theta}_i | z_1, z_2) = \frac{1}{\sqrt{2\pi}\dot{\sigma}_i} \exp \left[-\frac{(\dot{\theta}_i - m_i)^2}{2\dot{\sigma}_i^2} \right]. \tag{2}$$

The values of m_i and $\dot{\sigma}_i^2$ can be obtained as follows. Let $\dot{z}$ and z be the column matrices of the $\dot{z}_i$ and z_i, respectively, and $\phi = (\dot{z}, z)^T$ be the partitioned column matrix of $\dot{z}$ and z where $(\cdot)^T$ is a transposed matrix. Then, the covariance matrix of ϕ is given by

$$\frac{1}{2} \langle \phi^* \phi^T \rangle = \sigma^2 \begin{pmatrix} -\ddot{\rho}(0) & -\dot{\rho}(0) \\ -\dot{\rho}(0)^{T*} & \rho(0) \end{pmatrix} \tag{3}$$

where $\rho(\tau)$ is the correlation matrix with components $\rho_{ij}(\tau) = \langle z_i(t)^* z_j(t + \tau) \rangle / 2\sigma^2$, and $\sigma^2 = \langle |z_i(t)|^2 \rangle / 2$ is the average received signal power at each branch. The matrices $\dot{\rho}(0)$ and $\ddot{\rho}(0)$ have the components $\dot{\rho}_{ij}(0) = (d/d\tau)\, \rho_{ij}(\tau)|_{\tau=0}$ and $\ddot{\rho}_{ij}(0) = (d^2/d\tau^2) \rho_{ij}(\tau)|_{\tau=0}$, respectively. Applying the matrix theory described in [5, pp. 495–496], the mean value M and the covariance matrix Λ of $\dot{z}$, with z being given, can be obtained as

$$\begin{aligned} M &= -\{\dot{\rho}(0)\rho(0)^{-1}\}^* z \\ \Lambda &= \sigma^2 \{ -\ddot{\rho}(0) - \dot{\rho}(0)\rho(0)^{-1}\dot{\rho}(0)^{T*} \} \end{aligned} \tag{4}$$

where $(\cdot)^{-1}$ is an inverse matrix. Hence, using the fact that $\dot{\rho}_{11}(0) = \dot{\rho}_{22}(0) = 0$, we have

$$m_1 = -\text{Im} \left\{ \frac{z_1 \mu_1^*}{|z_1|^2} \right\} = \frac{1}{1 - |\rho_{12}|^2} \cdot \text{Im} \left\{ \dot{\rho}_{12}^* \rho_{12} - \dot{\rho}_{12}^* \frac{R_2}{R_1} e^{j\theta_{12}} \right\}$$

$$\dot{\sigma}_1^2 = \frac{\lambda_{11}}{|z_1|^2} = \frac{\sigma^2}{R_1^2} \left\{ -\ddot{\rho}_{11} - \frac{|\dot{\rho}_{12}|^2}{1 - |\rho_{12}|^2} \right\} \tag{5}$$

where μ_i and λ_{ij} are the components of the matrices M and Λ, respectively, $\theta_{12} = \arg (z_1^* z_2)$, $\rho_{12} = \rho_{12}(0)$, $\dot{\rho}_{12} = \dot{\rho}_{12}(0)$, and $\ddot{\rho}_{11} = \ddot{\rho}_{11}(0)$. m_2 and $\dot{\sigma}_2^2$ can be obtained by interchanging the subscripts 1 and 2 in (5).

The cdf is defined as the probability of the random FM noise exceeding some value. Using the fact that $\rho_{12} = \rho_{21}^*$, $\dot{\rho}_{12} = -\dot{\rho}_{21}^*$, and $\ddot{\rho}_{11} = \ddot{\rho}_{22}$, the cdf with diversity can be evaluated from

$$P(\dot{\theta}) = 2 \int_{-\pi}^{\pi} \int_0^{R_1} \int_0^{\infty} \int_{\dot{\theta}}^{\infty} p(\dot{\theta}_1 | z_1, z_2) \cdot p(R_1, R_2, \theta_{12})\, d\dot{\theta}_1\, dR_1\, dR_2\, d\theta_{12} \tag{6}$$

where $p(R_1, R_2, \theta_{12})$ is the joint pdf of R_1, R_2, and θ_{12} given by [6, ch. 8.5]

$$p(R_1, R_2, \theta_{12}) = \frac{R_1 R_2}{2\pi\sigma^4(1-|\rho_{12}|^2)} \cdot \exp\left[-\frac{R_1^2+R_2^2-2R_1R_2\,\mathrm{Re}(\rho_{12}^* e^{j\theta_{12}})}{2\sigma^2(1-|\rho_{12}|^2)}\right]. \quad (7)$$

By substituting $R_1 = R\cos\psi$, $R_2 = R\sin\psi$ and $x = R/(\sqrt{2}\,\sigma\sqrt{1-|\rho_{12}|^2})$ where $\pi/2 \geqq \psi \geqq 0$ and $x \geqq 0$, and performing the integration with respect to x, we can obtain the following general expression for the cdf.

$$P(\dot\theta) = \frac{1-|\rho_{12}|^2}{2}\int_{-\pi}^{\pi}\int_0^{\pi/4}\frac{\sin(2\psi)}{2\pi} \cdot \frac{2+\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}}{(\alpha^2+\beta^2)^2\left\{1+\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right\}^2}\,d\psi\,d\theta_{12} \quad (8)$$

where

$$\alpha = \sqrt{\frac{1-|\rho_{12}|^2}{-\ddot\rho_{11}-\dfrac{|\dot\rho_{12}|^2}{1-|\rho_{12}|^2}}}\left\{\left(\dot\theta-\frac{\mathrm{Im}(\dot\rho_{12}^*\rho_{12})}{1-|\rho_{12}|^2}\right)\cos\psi + \frac{\mathrm{Im}(\dot\rho_{12}^* e^{j\theta_{12}})}{1-|\rho_{12}|^2}\sin\psi\right\}$$

$$\beta = \sqrt{1-\sin(2\psi)\,\mathrm{Re}(\rho_{12}^* e^{j\theta_{12}})}. \quad (9)$$

For large $\dot\theta$, $\alpha \gg \beta$ and hence

$$P(\dot\theta) \approx \frac{3}{8}\frac{1}{1-|\rho_{12}|^2}\left\{\frac{\sqrt{-\ddot\rho_{11}-\dfrac{|\dot\rho_{12}|^2}{1-|\rho_{12}|^2}}}{\dot\theta}\right\}^4. \quad (10)$$

Recognizing that the conditional mean square value of the random FM noise of the ith branch, with z_1 and z_2 being given, is $m_i^2 + \dot\sigma_i^2$, the following general expression for the mean square value with diversity can be obtained

$$\langle\dot\theta^2\rangle = 2\int_{-\pi}^{\pi}\int_0^{R_1}\int_0^{\infty}(m_1^2+\dot\sigma_1^2)p(R_1, R_2, \theta_{12})\,dR_1\,dR_2\,d\theta_{12}$$
$$= -\ddot\rho_{11}\ln\left(\frac{1+\sqrt{1-|\rho_{12}|^2}}{\sqrt{1-|\rho_{12}|^2}}\right) - \frac{|\dot\rho_{12}|^2}{1-|\rho_{12}|^2}\frac{1}{1+\sqrt{1-|\rho_{12}|^2}} + \left\{\frac{\sqrt{1-|\rho_{12}|^2}}{1+\sqrt{1-|\rho_{12}|^2}}\right\}^2\left\{\frac{\mathrm{Im}(\dot\rho_{12}^*\rho_{12})}{1-|\rho_{12}|^2}\right\}^2. \quad (11)$$

For independent fading signals (ρ_{12} and $\dot\rho_{12} = 0$), (8) and (11) reduce to

$$P(\dot\theta) = \frac{1}{2}\left[1-\frac{2\dfrac{\dot\theta}{\sqrt{-\ddot\rho_{11}}}}{\sqrt{\left(\dfrac{\dot\theta}{\sqrt{-\ddot\rho_{11}}}\right)^2+1}}+\frac{\dfrac{\dot\theta}{\sqrt{-\ddot\rho_{11}}}}{\sqrt{\left(\dfrac{\dot\theta}{\sqrt{-\ddot\rho_{11}}}\right)^2+2}}\right]$$

$$\langle\dot\theta^2\rangle = -\ddot\rho_{11}\ln 2 \quad (12)$$

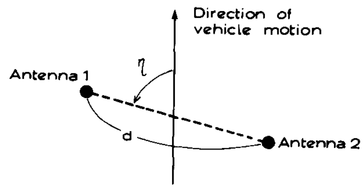

Fig. 1. Antenna configuration at mobile station.

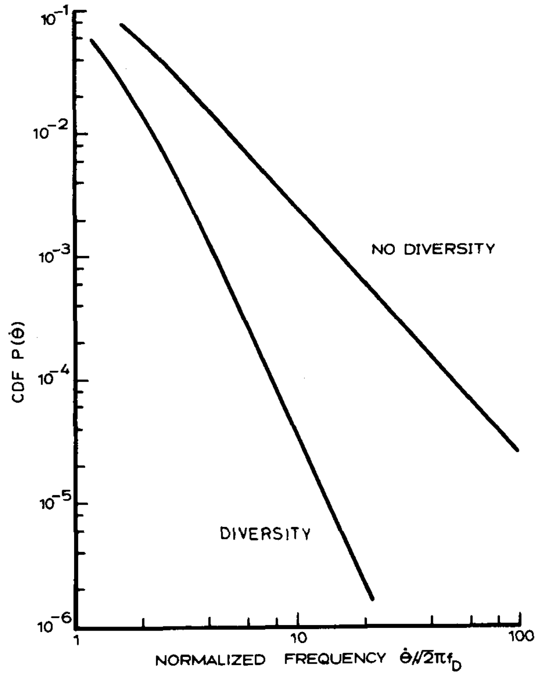

Fig. 2. Cumulative distribution function (cdf) of random FM noise for independent Rayleigh fading.

which are identical to those derived in [1] except for the different notation.

III. Numerical Calculations

The general expressions above can be applied to any types of diversity (space, polarization, angle, frequency, and time). Here, we consider a space diversity system using two horizontally spaced antennas with omnidirectional radiation patterns at a mobile station as shown in Fig. 1. In the figure, d and η denote the antenna spacing and the angle between the antenna axis and the direction of vehicle motion, respectively. Assuming that many incoming multipath waves with the same amplitude and independent random phases arrive from all directions with equal probability, $\rho_{11}(\tau)$ and $\rho_{12}(\tau)$ are given by $J_0(2\pi f_D\tau)$ and $J_0(2\pi\sqrt{(f_D\tau)^2+(d/\lambda)^2-2(f_D\tau)(d/\lambda)\cos\eta})$, respectively. $J_0(\cdot)$ is the zero-order Bessel function and f_D is the maximum Doppler frequency given by vehicle speed v/ carrier wavelength λ [4].

For large d, the fading signals are independent. The cdf of $\dot\theta$ can then be evaluated from (12) with the result shown in Fig. 2. $\dot\theta$ is normalized by the rms angular frequency ($\sqrt{2}\,\pi f_D$) in the fading power spectrum. For comparison, the cdf without diversity [1] is also shown in the same figure. As the antenna spacing is reduced, the fading signals become partially correlated. The cdf at $\dot\theta = 10 \times \sqrt{2}\,\pi f_D$ evaluated from (8) is shown in Fig. 3. The dotted lines in this figure are approximate calculations derived from (10), which are clearly a good

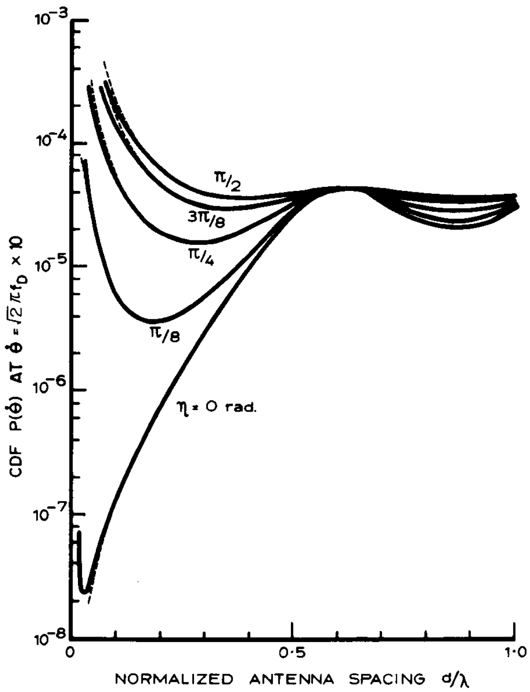

Fig. 3. Cumulative distribution of random FM noise with diversity at $\dot{\theta} = \sqrt{2}\,\pi f_D \times 10$. Dashed lines show approximate results computed from (10).

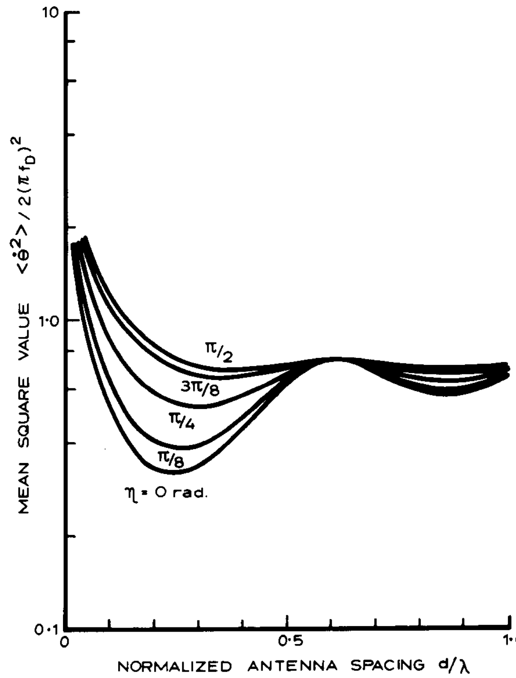

Fig. 4. Mean square value of random FM noise with diversity.

approximation except at very low values of d. Fig. 4 shows the mean square values evaluated from (11). It can be seen from Figs. 3 and 4 that η affects the diversity improvement and that the cdf and mean square value decreases further as d is reduced. However, at a very small value of d there is a sharp increase towards the value obtained without diversity (note that the mean square value without diversity is infinite [1]). In particular, it is worth noting that when the two antennas are parallel with the direction of vehicle motion ($\eta = 0$), the cdf can be substantially reduced even for very low values of d ($<0.1\lambda$).

IV. Discussion

The reason for the reduction in random FM noise with small antenna spacings can be qualitatively explained. Large excursions of random FM noise are produced at the instant when the fading signal envelope is local-minimum, as illustrated in Fig. 5. Let us first consider the case of $\eta = 0$. The received signal on the second antenna is a delayed replica of that on the first with a delay time of $d/v = d/(\lambda f_D)$. It is apparent that for very small antenna spacings the two signals are almost identical and hence the random FM noise is the same as that obtained without diversity. However, when d/λ becomes nearly equal to $f_D \times$ the time duration of the random FM noise, selection combining can remove almost all segments of large random FM noise (see Fig. 5). When the antenna spacing is made much larger, so that the fading signals are independent, some of the large excursions of random FM noise may remain because there is a possibility that the particular fading signal having a local-minimum envelope will be selected by the diversity system. Nevertheless, it is worth noting that even with independent fading, the random FM noise can be reduced well below the value obtained without diversity. We can therefore conclude that when $\eta = 0$ further diversity improvement with respect to random FM noise can be obtained for small antenna spacings. However, as η becomes larger ($\rightarrow \pi/2$), the coherent relationship between the

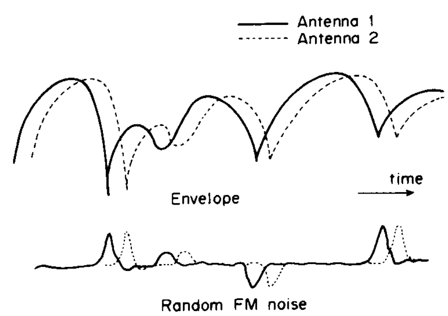

Fig. 5. Envelopes and random FM noises of the received signals on the horizontally closed spaced two antennas with $\eta = 0$.

two signals gradually diminishes and hence the further diversity improvement decreases.

V. Conclusion

General expressions have been derived for the cdf and mean square value of random FM noise using two branch selection combining with correlated Rayleigh fading signals. Calculated results show that the use of closely spaced antennas at a mobile station can reduce the random FM noise below the value obtainable with independent fading signals.

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Probability of Error for *M*-ary PSK and DPSK on a Rayleigh Fading Channel

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Abstract—***M*-ary phase shift keying and differential phase shift keying on a slow fading Rayleigh channel without diversity is investigated. Expressions for the distribution of the phase angle between a vector with Rayleigh amplitude distribution and a noiseless reference and between two vectors both with Rayleigh amplitude distribution perturbed by Gaussian noise are obtained.**

I. Introduction

Exact expressions for the probabilities of bit error and symbol error for *M*-ary PSK and DPSK on a slow fading Rayleigh channel are available in the literature [1], [2]. In this paper, alternative expressions are derived using new expressions for the distribution of the phase angles between two vectors perturbed by Gaussian noise [3].

II. Phase Distribution for DPSK

Consider two vectors equal to each other in both length and direction, perturbed by two uncorrelated noise vectors of equal variance. The distribution function of the phase θ between the two vectors is given by [3]

$$F(\Psi, \rho)=P[-\Pi<\theta \leqslant \Psi] = \frac{-\sin(\Psi)}{4\Pi}\int_{-\Pi/2}^{\Pi/2} \frac{\exp[-\rho(1-\cos\Psi\cos t)]}{1-\cos\Psi\cos t}\,dt \cdot (-\Pi<\Psi<0) \quad (1)$$

where ρ is the instantaneous signal-to-noise ratio

$$\rho=\frac{E_s}{\eta} \quad (2)$$

with E_s the energy per symbol and η the single-sided noise power spectral density.

Now let the length V of the vectors be Rayleigh distributed

$$f(V)=\left(\frac{2V}{\bar{\rho}}\right)\exp\left(\frac{-V^2}{\bar{\rho}}\right)u(V). \quad (3)$$

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Let us call the distribution function of the phase when the amplitude is Rayleigh distributed $F_R(\Omega)$. It is found by averaging $F(\Psi, e)$ in (1) over $f(v)$ given in (3)

$$F_R(\Psi)=\int_0^\infty f(V)F(\Psi, V^2)\,dV = \int_0^\infty \frac{2V}{\bar{\rho}}e^{-V^2/\bar{\rho}}\int_{-\Pi/2}^{\Pi/2}\frac{e^{-V^2(1-\cos\Psi\cos t)}}{1-\cos\Psi\cos t}\,dt\,dV.$$

Exchange the order of integration and evaluate the integral with respect to V

$$F_R(\Psi)=\frac{-\sin\Psi}{4\Pi\bar{\rho}}\int_{-\Pi/2}^{\Pi/2} \cdot \frac{1}{(1-\cos\Psi\cos t)(1/\bar{\rho}+1-\cos\Psi\cos t)}\,dt.$$

Using a partial fraction expansion, the integrals can be evaluated using [4, eq. 2.554 (2)]

$$F_R(\Psi)=\frac{1}{\Pi}\left[\frac{-\sin\Psi}{\sqrt{(1+1/\bar{\rho})^2-(\cos\Psi)^2}} \cdot \tan^{-1}\frac{\sqrt{(1+1/\bar{\rho})^2-(\cos\Psi)^2}}{1+1/\bar{\rho}-\cos\Psi} - \tan^{-1}\frac{-\sin\Psi}{1-\cos\Psi}\right] \quad (-\Pi<\Psi<0). \quad (4)$$

An approximate expression for $F_R(\Psi)$ at high values of SNR can be obtained by using Taylor series expansions in terms of $1/\bar{\rho}$ for all the terms in (4). After discarding higher order terms it is found that

$$F_R(\Psi) \leqslant \frac{2\Pi+2\Psi-\sin 2\Psi}{4\Pi\bar{\rho}\sin^2\Psi} \quad (-\Pi<\Psi<0). \quad (5)$$

The equality applies in the limit when $\bar{\rho}$ is infinite.

III. Phase Distribution for PSK

PSK requires a phase reference. The distribution of the phase between a noiseless reference and a vector perturbed by Gaussian noise is given by [3]

$$F(\Psi, \rho)=\frac{1}{2\Pi}\int_{-\Pi/2}^{\Pi/2+\Psi} e^{-\rho\sin^2\Psi\sec^2\theta}\,d\theta \quad (-\Pi<\Psi<0). \quad (6)$$

When the signal part of the second vector has a Rayleigh amplitude density, the phase distribution is found by averaging $F(\Psi, e)$ in (6) with respect to $F(v)$ *in* (3)

$$F_R(\Psi)=\frac{1}{2\Pi}\int_{-\Pi/2}^{\Pi/2+\Psi}\int_0^\infty e^{-(V^2/\bar{\rho}+V^2\sin^2\Psi\sec^2\theta)}\,dV\,d\theta = \frac{1}{2\Pi}\int_{-\Pi/2}^{\Pi/2+\Psi}\frac{1+\cos 2\theta}{\cos 2\theta+(1+2\bar{\rho}\sin^2\Psi)}\,d\theta = \frac{1}{2}\left\{1+\frac{\Psi}{\Pi}-\frac{(a-1)}{\sqrt{a^2-1}} \cdot \left[\frac{1}{2}+\frac{1}{\Pi}\tan^{-1}\frac{\sqrt{a^2-1}\tan(\Pi/2+\Psi)}{a+1}\right]\right\} \quad (7)$$