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State-Space Formulation of Frequency Transformation for 2-D Digital Filters

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***Abstract*—This letter proposes frequency transformation for two-dimensional (2-D) digital filters in terms of the state-space equations. In order to derive the 2-D state-space formulation of the frequency transformation, we extend Mullis and Roberts' state-space formulation of one-dimensional frequency transformation to 2-D case. The proposed frequency transformation is very suitable for state-space analysis of the transformed filters since the resultant formulation consists of simple algebraic operations such as matrix addition, multiplication, inverse and so on. It can be easily implemented in MATLAB without any complicated algorithm.**

***Index Terms*—2-D digital filter, frequency transformation, state-space equation.**

I. Introduction

FREQUENCY transformation is a very useful and straightforward technique for design of digital filters both in the one-dimensional (1-D) and two-dimensional (2-D) case [1], [2]. Given a prototype filter, which is generally low-pass, we can transform it into other transfer functions of various amplitude responses including low-pass of different cutoff, high-pass, band-pass, and bandstop. This theory is also used in signal processing applications such as variable digital filters [3], [4], which can control the frequency characteristic of a digital filter during its operation.

Though the frequency transformation is represented in terms of the transfer functions, there exists another representation of frequency transformation in the 1-D case [5], which is based on the state-space equations. By using this state-space formulation of 1-D frequency transformation, Mullis and Roberts revealed an important property that the minimum attainable value of roundoff noise in 1-D digital filters is constant under any frequency transformation. Therefore, the state-space formulation of the frequency transformation plays a crucial role in the analysis for the family of filters generated through the frequency transformation. However, there has been no research on 2-D state-space formulation of frequency transformation, except for the discussion restricted to the 2-D digital filters of separable denominator [6].

Though we can obtain a 2-D state-space model of a transformed filter without the state-space formulation of the frequency transformation, the resultant model has unnecessarily larger dimension than the order of the transfer function [7], which causes inconvenience to state-space analysis for the family of filters related by the frequency transformation. In order to overcome this problem, it is necessary to formulate frequency transformation of 2-D filters in terms of the state-space equations. Of course, the state-space formulation of frequency transformation for 2-D filters in itself will be an effective strategy for the analysis of state-space digital filters.

In this letter, we extend Mullis and Roberts' approach [5] to the general 2-D case and propose the state-space formulation of frequency transformation for 2-D nonseparable denominator digital filters. The resultant formulation consists of simple algebraic operations such as matrix addition, multiplication, inverse and so on. Therefore, the formulation is very suitable for state-space analysis of the 2-D filters obtained by frequency transformation. In addition, the proposed formulation can be easily implemented in MATLAB without any complicated algorithm.

II. 2-D State-Space Digital Filters

Consider the transfer function

$$H(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)} \quad (1)$$

of a stable 2-D digital filter of order N_1 in z_1^{-1} and N_2 in z_2^{-1}. The input-output relation of the filter is represented by the following Roesser's state-space equations [8]:

$$\begin{bmatrix} \boldsymbol{x}_h(n_1+1, n_2) \\ \boldsymbol{x}_v(n_1, n_2+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_h(n_1, n_2) \\ \boldsymbol{x}_v(n_1, n_2) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} u(n_1, n_2) \quad (2)$$

$$y(n_1, n_2) = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_h(n_1, n_2) \\ \boldsymbol{x}_v(n_1, n_2) \end{bmatrix} + du(n_1, n_2) \quad (3)$$

where $u(n_1, n_2)$ and $y(n_1, n_2)$ are the scalar input and output of the filter, and $\boldsymbol{x}_h(n_1, n_2)$ and $\boldsymbol{x}_v(n_1, n_2)$ are the $N_1 \times 1$ horizontal state vector and the $N_2 \times 1$ vertical state vector, respectively. This means that the state-space model has the dimension of (N_1, N_2). Matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, $\boldsymbol{A}_4$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and d are real coefficient matrices with appropriate size. The block diagram of the 2-D digital filter is given in Fig. 1. The transfer function $H(z_1, z_2)$ of the 2-D digital filter is given in terms of the coefficient matrices as

$$H(z_1, z_2) = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix} \left[\begin{array}{c|c} z_1 \boldsymbol{I}_{N_1} - \boldsymbol{A}_1 & -\boldsymbol{A}_2 \\ \hline -\boldsymbol{A}_3 & z_2 \boldsymbol{I}_{N_2} - \boldsymbol{A}_4 \end{array} \right]^{-1} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} + d \quad (4)$$

where $\boldsymbol{I}_{N_1}$ and $\boldsymbol{I}_{N_2}$ are the $N_1 \times N_1$ and $N_2 \times N_2$ identity matrices, respectively.

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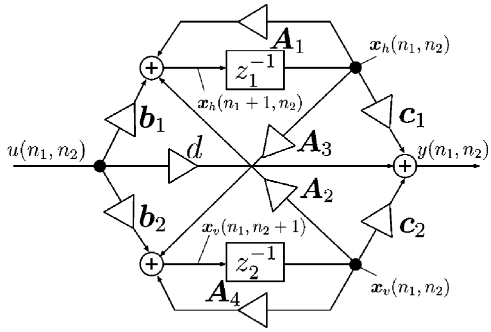

Fig. 1. Block diagram of a 2-D state-space digital filter.

III. Conventional Frequency Transformation and Its Drawbacks

Let $H(z_1, z_2)$ be a prototype 2-D transfer function, which has low-pass characteristic, in general. The frequency transformation in 2-D case is given by [2]

$$H_d(z_1, z_2) = H(z_1, z_2)|_{z_1^{-1} \leftarrow T_1(z_1), z_2^{-1} \leftarrow T_2(z_2)} \tag{5}$$

where $H_d(z_1, z_2)$ is a desired 2-D digital filter obtained by the frequency transformations $z_1^{-1} \leftarrow T_1(z_1)$ and $z_2^{-1} \leftarrow T_2(z_2)$. $T_1(z_1)$ and $T_2(z_2)$ are M_1th order and M_2th order 1-D all-pass filters, respectively. Although there exists another type of frequency transformation which uses the 2-D all-pass transformations $z_1^{-1} \leftarrow T_1(z_1, z_2)$ and $z_2^{-1} \leftarrow T_2(z_1, z_2)$, this type of frequency transformation is not suitable for 2-D filter design because designing $T_1(z_1, z_2)$ and $T_2(z_1, z_2)$ to effect a desired frequency transformation is a difficult problem [2], [9]. Hence, in this letter we restrict ourselves to the transformation based on 1-D all-pass filters.

Through the transformation (5), we can easily design digital filters which have different characteristics from those of $H(z_1, z_2)$. However, the transformation (5) has some drawbacks when we analyze the filters $H_d(z_1, z_2)$ by the state-space equations (2) and (3). Suppose we have an (N_1, N_2)th-order prototype 2-D digital filter $H(z_1, z_2)$ of which coefficient matrices are $(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{c}_1, \boldsymbol{c}_2, d)$. Let the size of $\boldsymbol{A}_1$ and $\boldsymbol{A}_4$ be $N_1 \times N_1$ and $N_2 \times N_2$, respectively. That is, the state-space model of $H(z_1, z_2)$ has the dimension of (N_1, N_2), which is equal to the order of $H(z_1, z_2)$. In order to obtain the state-space model of $H_d(z_1, z_2)$, we must take the following steps.

1) Compute $H(z_1, z_2)$ from the coefficient matrices by using **(4)**.
2) Obtain $H_d(z_1, z_2)$ through the frequency transformation **(5)**.
3) Obtain a state-space representation of $H_d(z_1, z_2)$.

Needless to say, taking these steps is very tedious and troublesome. Moreover, the coefficient matrices of $H_d(z_1, z_2)$ have unnecessarily large size: $H_d(z_1, z_2)$ has the order of (N_1M_1, N_2M_2), but the state-space model obtained through the step 3 has the dimension of either $(N_1M_1, 2N_2M_2)$ or $(2N_1M_1, N_2M_2)$ [7]. This is quite unacceptable in state-space analysis of $H_d(z_1, z_2)$. Therefore, another algebraic formulation of the frequency transformation is necessary for the state-space analysis of the transformed filters. Of course, the algebraic formulation in itself will be an effective strategy for the analysis of state-space digital filters.

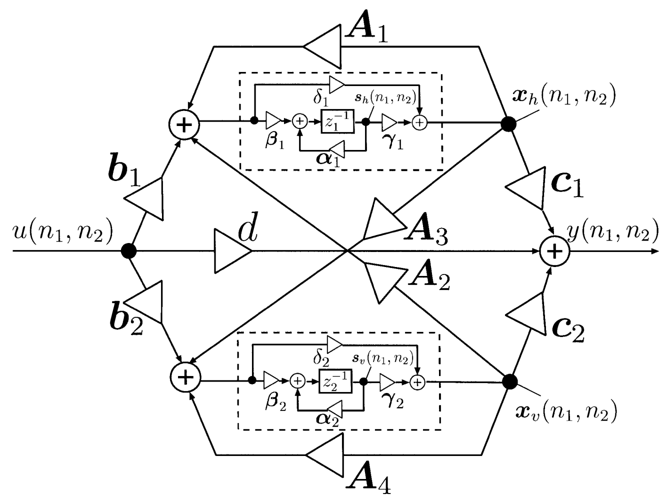

Fig. 2. Frequency transformation in a 2-D state-space digital filter.

IV. State-Space Formulation of Frequency Transformation for 2-D Digital Filters

Here we will derive the state-space equations for the 2-D digital filter $H_d(z_1, z_2)$; that is, we will obtain the coefficient matrices $(\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2, \boldsymbol{\mathcal{A}}_3, \boldsymbol{\mathcal{A}}_4, \boldsymbol{\mathcal{B}}_1, \boldsymbol{\mathcal{B}}_2, \boldsymbol{\mathcal{C}}_1, \boldsymbol{\mathcal{C}}_2, \mathcal{D})$ satisfying

$$\begin{aligned} H_d(z_1, z_2) &= H(z_1, z_2)|_{z_1^{-1} \leftarrow T_1(z_1), z_2^{-1} \leftarrow T_2(z_2)} \\ &= [\boldsymbol{\mathcal{C}}_1 \quad \boldsymbol{\mathcal{C}}_2] \left[\begin{array}{c|c} z_1\boldsymbol{I}_{N_1M_1} - \boldsymbol{\mathcal{A}}_1 & -\boldsymbol{\mathcal{A}}_2 \\ \hline -\boldsymbol{\mathcal{A}}_3 & z_2\boldsymbol{I}_{N_2M_2} - \boldsymbol{\mathcal{A}}_4 \end{array} \right]^{-1} \\ &\quad \begin{bmatrix} \boldsymbol{\mathcal{B}}_1 \\ \boldsymbol{\mathcal{B}}_2 \end{bmatrix} + \mathcal{D}. \end{aligned} \tag{6}$$

Note that $(\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2, \boldsymbol{\mathcal{A}}_3, \boldsymbol{\mathcal{A}}_4, \boldsymbol{\mathcal{B}}_1, \boldsymbol{\mathcal{B}}_2, \boldsymbol{\mathcal{C}}_1, \boldsymbol{\mathcal{C}}_2, \mathcal{D})$ has the dimension of (N_1M_1, N_2M_2), which is equal to the order of $H_d(z_1, z_2)$. Our goal is to describe $(\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2, \boldsymbol{\mathcal{A}}_3, \boldsymbol{\mathcal{A}}_4, \boldsymbol{\mathcal{B}}_1, \boldsymbol{\mathcal{B}}_2, \boldsymbol{\mathcal{C}}_1, \boldsymbol{\mathcal{C}}_2, \mathcal{D})$ in terms of the coefficient matrices of the prototype filter $H(z_1, z_2)$ and the 1-D all-pass filters $T_1(z_1)$ and $T_2(z_2)$. To this end, we let $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1, \delta_1)$ be coefficient matrices of $T_1(z_1)$ and $(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_2, \delta_2)$ be coefficient matrices of $T_2(z_2)$; that is

$$T_i(z_i) = \delta_i + \boldsymbol{\gamma}_i(z_i\boldsymbol{I}_{M_i} - \boldsymbol{\alpha}_i)^{-1}\boldsymbol{\beta}_i, \quad i = 1, 2. \tag{7}$$

As shown in Fig. 2, frequency transformation in (5) can be interpreted as substitution of the all-pass filters $T_1(z_1)$ and $T_2(z_2)$ into the delays of z_1^{-1} and z_2^{-1}, respectively. Replacing each horizontal delay element z_1^{-1} of the prototype filter with the all-pass filter $T_1(z_1)$, we have the following equations with respect to the new horizontal state vector $\boldsymbol{s}_h(n_1,n_2)$ and the old state vectors $\boldsymbol{x}_h(n_1,n_2), \boldsymbol{x}_v(n_1,n_2)$:

$$\begin{aligned}\boldsymbol{s}_h(n_1+1,n_2) &= \boldsymbol{\alpha}_1\boldsymbol{s}_h(n_1,n_2)+\boldsymbol{\beta}_1[\boldsymbol{A}_1\boldsymbol{x}_h(n_1,n_2)\\ &\quad +\boldsymbol{A}_2\boldsymbol{x}_v(n_1,n_2)+\boldsymbol{b}_1u(n_1,n_2)]^t \qquad (8)\\ \boldsymbol{x}_h(n_1,n_2) &= [\boldsymbol{s}_h(n_1,n_2)]^t\boldsymbol{\gamma}_1^t+\delta_1[\boldsymbol{A}_1\boldsymbol{x}_h(n_1,n_2)\\ &\quad +\boldsymbol{A}_2\boldsymbol{x}_v(n_1,n_2)+\boldsymbol{b}_1u(n_1,n_2)] \qquad (9)\end{aligned}$$

where $\boldsymbol{s}_h(n_1,n_2)$ is the $M_1\times N_1$ matrix and the superscript "t" denotes the matrix transpose. Similarly, we can represent the vertical frequency transformation $z_2^{-1}\leftarrow T_2(z_2)$ in terms of the following equations:

$$\begin{aligned}\boldsymbol{s}_v(n_1,n_2+1) &= \boldsymbol{\alpha}_2\boldsymbol{s}_v(n_1,n_2)+\boldsymbol{\beta}_2[\boldsymbol{A}_3\boldsymbol{x}_h(n_1,n_2)\\ &\quad +\boldsymbol{A}_4\boldsymbol{x}_v(n_1,n_2)+\boldsymbol{b}_2u(n_1,n_2)]^t \qquad (10)\\ \boldsymbol{x}_v(n_1,n_2) &= [\boldsymbol{s}_v(n_1,n_2)]^t\boldsymbol{\gamma}_2^t+\delta_2[\boldsymbol{A}_3\boldsymbol{x}_h(n_1,n_2)\\ &\quad +\boldsymbol{A}_4\boldsymbol{x}_v(n_1,n_2)+\boldsymbol{b}_2u(n_1,n_2)] \qquad (11)\end{aligned}$$

where $\boldsymbol{s}_v(n_1,n_2)$ is the new vertical state vector, which has the size of $M_2\times N_2$. However, the transformed filters described in this way have delay-free loops. In order to avoid this problem, we remove the old state vectors $\boldsymbol{x}_h(n_1,n_2)$ and $\boldsymbol{x}_v(n_1,n_2)$ from (3) and (8)–(11) and derive new state equations for the transformed filters. After a number of matrix manipulations, we obtain the following set of equations:

$$\begin{aligned}&\boldsymbol{s}_h(n_1+1,n_2)\\ &=\boldsymbol{\alpha}_1\boldsymbol{s}_h(n_1,n_2)\\ &\quad+\boldsymbol{\beta}_1\boldsymbol{\gamma}_1\boldsymbol{s}_h(n_1,n_2)[\boldsymbol{A}_1\boldsymbol{F}^{-1}+\delta_2\boldsymbol{A}_2\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}]^t\\ &\quad+\boldsymbol{\beta}_1\boldsymbol{\gamma}_2\boldsymbol{s}_v(n_1,n_2)[\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}]^t\\ &\quad+\boldsymbol{\beta}_1u(n_1,n_2)[\boldsymbol{F}^{-1}\boldsymbol{b}_1+\delta_2\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2]^t \qquad (12)\\ &\boldsymbol{s}_v(n_1,n_2+1)\\ &=\boldsymbol{\alpha}_2\boldsymbol{s}_v(n_1,n_2)+\boldsymbol{\beta}_2\boldsymbol{\gamma}_1\boldsymbol{s}_h(n_1,n_2)[\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}]^t\\ &\quad+\boldsymbol{\beta}_2\boldsymbol{\gamma}_2\boldsymbol{s}_v(n_1,n_2)[\boldsymbol{A}_4\boldsymbol{G}^{-1}+\delta_1\boldsymbol{A}_3\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}]^t\\ &\quad+\boldsymbol{\beta}_2u(n_1,n_2)[\boldsymbol{G}^{-1}\boldsymbol{b}_2+\delta_1\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\boldsymbol{b}_1]^t \qquad (13)\\ &y(n_1,n_2)\\ &=\boldsymbol{\gamma}_1\boldsymbol{s}_h(n_1,n_2)[\boldsymbol{c}_1\boldsymbol{F}^{-1}+\delta_2\boldsymbol{c}_2\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}]^t\\ &\quad+\boldsymbol{\gamma}_2\boldsymbol{s}_v(n_1,n_2)[\boldsymbol{c}_2\boldsymbol{G}^{-1}+\delta_1\boldsymbol{c}_1\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}]^t\\ &\quad+\{d+\delta_1\boldsymbol{c}_1\boldsymbol{F}^{-1}[\boldsymbol{b}_1+\delta_2\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2]\\ &\quad+\delta_2\boldsymbol{c}_2\boldsymbol{G}^{-1}[\boldsymbol{b}_2+\delta_1\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\boldsymbol{b}_1]\}u(n_1,n_2) \qquad (14)\end{aligned}$$

where we let

$$\begin{aligned}\boldsymbol{F} &= (\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)-\delta_1\delta_2\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\boldsymbol{A}_3\\ \boldsymbol{G} &= (\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)-\delta_1\delta_2\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\boldsymbol{A}_2. \qquad (15)\end{aligned}$$

Equations (12)–(14) are the state-space equations for the transformed filters $H_d(z_1,z_2)$, but their states $\boldsymbol{s}_h$ and $\boldsymbol{s}_v$ are given in $M_1\times N_1$ and $M_2\times N_2$ matrix forms, respectively. In order to rearrange the states $\boldsymbol{s}_h$ and $\boldsymbol{s}_v$ in vector forms, we use the column string expansion $\mathrm{cs}(\boldsymbol{P})$ for a matrix $\boldsymbol{P}$, which is the vectors formed by stacking the columns of $\boldsymbol{P}$ and has the following property [10]:

$$\mathrm{cs}(\boldsymbol{QPR}^t)=[\boldsymbol{R}\otimes\boldsymbol{Q}]\mathrm{cs}(\boldsymbol{P})$$

where $\otimes$ is the Kronecker product for matrices. Letting $\boldsymbol{X}_h(n_1,n_2)=\mathrm{cs}(\boldsymbol{s}_h(n_1,n_2))$ and $\boldsymbol{X}_v(n_1,n_2)=\mathrm{cs}(\boldsymbol{s}_v(n_1,n_2))$ in (12)–(14), we have the final state-equations

$$\begin{aligned}\begin{bmatrix}\boldsymbol{X}_h(n_1+1,n_2)\\ \boldsymbol{X}_v(n_1,n_2+1)\end{bmatrix} &= \begin{bmatrix}\boldsymbol{\mathcal{A}}_1 & \boldsymbol{\mathcal{A}}_2\\ \boldsymbol{\mathcal{A}}_3 & \boldsymbol{\mathcal{A}}_4\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_h(n_1,n_2)\\ \boldsymbol{X}_v(n_1,n_2)\end{bmatrix}\\ &\quad+\begin{bmatrix}\boldsymbol{\mathcal{B}}_1\\ \boldsymbol{\mathcal{B}}_2\end{bmatrix}u(n_1,n_2) \qquad (16)\\ y(n_1,n_2) &= [\boldsymbol{\mathcal{C}}_1 \quad \boldsymbol{\mathcal{C}}_2]\begin{bmatrix}\boldsymbol{X}_h(n_1,n_2)\\ \boldsymbol{X}_v(n_1,n_2)\end{bmatrix}\\ &\quad+\mathcal{D}u(n_1,n_2) \qquad (17)\end{aligned}$$

where $\boldsymbol{\mathcal{A}}_1,\boldsymbol{\mathcal{A}}_2,\boldsymbol{\mathcal{A}}_3,\boldsymbol{\mathcal{A}}_4,\boldsymbol{\mathcal{B}}_1,\boldsymbol{\mathcal{B}}_2,\boldsymbol{\mathcal{C}}_1,\boldsymbol{\mathcal{C}}_2,\mathcal{D}$ are given as

$$\begin{aligned}\boldsymbol{\mathcal{A}}_1 &= \boldsymbol{I}_{N_1}\otimes\boldsymbol{\alpha}_1\\ &\quad+\left[\boldsymbol{A}_1\boldsymbol{F}^{-1}+\delta_2\boldsymbol{A}_2\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\right]\otimes(\boldsymbol{\beta}_1\boldsymbol{\gamma}_1)\\ \boldsymbol{\mathcal{A}}_2 &= \left[\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\right]\otimes(\boldsymbol{\beta}_1\boldsymbol{\gamma}_2)\\ \boldsymbol{\mathcal{A}}_3 &= \left[\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\right]\otimes(\boldsymbol{\beta}_2\boldsymbol{\gamma}_1)\\ \boldsymbol{\mathcal{A}}_4 &= \boldsymbol{I}_{N_2}\otimes\boldsymbol{\alpha}_2\\ &\quad+\left[\boldsymbol{A}_4\boldsymbol{G}^{-1}+\delta_1\boldsymbol{A}_3\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\right]\otimes(\boldsymbol{\beta}_2\boldsymbol{\gamma}_2)\\ \boldsymbol{\mathcal{B}}_1 &= \left[\boldsymbol{F}^{-1}\boldsymbol{b}_1+\delta_2\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2\right]\otimes\boldsymbol{\beta}_1\\ \boldsymbol{\mathcal{B}}_2 &= \left[\boldsymbol{G}^{-1}\boldsymbol{b}_2+\delta_1\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\boldsymbol{b}_1\right]\otimes\boldsymbol{\beta}_2\\ \boldsymbol{\mathcal{C}}_1 &= \left[\boldsymbol{c}_1\boldsymbol{F}^{-1}+\delta_2\boldsymbol{c}_2\boldsymbol{G}^{-1}\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\right]\otimes\boldsymbol{\gamma}_1\\ \boldsymbol{\mathcal{C}}_2 &= \left[\boldsymbol{c}_2\boldsymbol{G}^{-1}+\delta_1\boldsymbol{c}_1\boldsymbol{F}^{-1}\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\right]\otimes\boldsymbol{\gamma}_2\\ \mathcal{D} &= d+\delta_1\boldsymbol{c}_1\boldsymbol{F}^{-1}\left[\boldsymbol{b}_1+\delta_2\boldsymbol{A}_2(\boldsymbol{I}_{N_2}-\delta_2\boldsymbol{A}_4)^{-1}\boldsymbol{b}_2\right]\\ &\quad+\delta_2\boldsymbol{c}_2\boldsymbol{G}^{-1}\left[\boldsymbol{b}_2+\delta_1\boldsymbol{A}_3(\boldsymbol{I}_{N_1}-\delta_1\boldsymbol{A}_1)^{-1}\boldsymbol{b}_1\right]. \qquad (18)\end{aligned}$$

Equation (18) denotes the coefficient matrices of $H_d(z_1,z_2)$.

Though (18) looks complicated, it is very suitable for the analysis of $H_d(z_1,z_2)$ on the state-space equations because (18) is described as the closed form and consists of simple algebraic operations such as matrix addition, subtraction, multiplication, inverse, and Kronecker product. This is also suitable for implementation in MATLAB because the necessary task is only to write (18) in MATLAB language without any elaborate algorithm.

V. A Numerical Example

This section gives a numerical example to demonstrate our proposed formulation of the frequency transformation. Consider the following state-space system $(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{A}_4,\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{c}_1,\boldsymbol{c}_2,d)$ of a (1, 1)-order 2-D digital filter

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{A}_3 & \boldsymbol{A}_4\end{bmatrix}=\begin{bmatrix}0.8181 & 0.5295\\ -0.0130 & 0.8163\end{bmatrix}, \begin{bmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\end{bmatrix}=\begin{bmatrix}0.0460\\ 0.3050\end{bmatrix},\\ &[\boldsymbol{c}_1 \quad \boldsymbol{c}_2]=[0.3034 \quad 0.0372], d=9.6000\times 10^{-4}. \qquad (19)\end{aligned}$$

From (4), the transfer function of this filter is obtained as

$$H(z_1, z_2) = \frac{0.0010 + 0.0132z_1^{-1} + 0.0106z_2^{-1} + 0.0289z_1^{-1}z_2^{-1}}{1 - 0.8181z_1^{-1} - 0.8163z_2^{-1} + 0.6747z_1^{-1}z_2^{-1}}. \quad (20)$$

For this filter, we apply the following frequency transformation, for example:

$$z_1^{-1} \leftarrow T_1(z_1) = \frac{z_1^{-1} + 0.5}{1 + 0.5z_1^{-1}}, z_2^{-1} \leftarrow T_2(z_2) = \frac{z_2^{-1} + 0.7}{1 + 0.7z_2^{-1}}. \quad (21)$$

Employing this transformation by (5), we have the following transfer function of the transformed system:

$$H_d(z_1, z_2) = \frac{0.0981 + 0.1471z_1^{-1} + 0.1186z_2^{-1} + 0.1712z_1^{-1}z_2^{-1}}{1 - 0.5144z_1^{-1} - 0.2553z_2^{-1} + 0.1716z_1^{-1}z_2^{-1}}. \quad (22)$$

From this transfer function, we obtain the state-space model as

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{bmatrix} = \left[\begin{array}{c|cc} 0.5144 & 1 & 0 \\ \hline -0.0403 & 0.2553 & 0 \\ 0.2322 & 0.1186 & 0 \end{array}\right], \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline 0.2553 \\ 0.1186 \end{array}\right],$$
$$[\mathcal{C}_1 \quad \mathcal{C}_2] = [0.1975 \mid 0.0981 \quad 1], \mathcal{D} = 0.0981. \quad (23)$$

This state-space model has larger dimension than the order of $H_d(z_1, z_2)$, which is quite unacceptable in state-space analysis of the system.

We can avoid this problem by using our proposed formulation of the frequency transformation (18). From (21), $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1, \delta_1)$ and $(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_2, \delta_2)$ in (18) are given by

$$\begin{aligned} (\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1, \delta_1) &= (-0.5, 1, 0.75, 0.5) \\ (\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_2, \delta_2) &= (-0.7, 1, 0.51, 0.7) \end{aligned} \quad (24)$$

and thus, applying our frequency transformation to (19) yields

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{bmatrix} = \begin{bmatrix} 0.5144 & 1.0562 \\ -0.0381 & 0.2553 \end{bmatrix}, \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} = \begin{bmatrix} 0.5192 \\ 0.7038 \end{bmatrix},$$
$$[\mathcal{C}_1 \quad \mathcal{C}_2] = [0.3804 \quad 0.2041], \mathcal{D} = 0.0981. \quad (25)$$

Calculating the transfer function from this state-space model, we obtain

$$H_d(z_1, z_2) = \frac{0.0981 + 0.1471z_1^{-1} + 0.1186z_2^{-1} + 0.1712z_1^{-1}z_2^{-1}}{1 - 0.5144z_1^{-1} - 0.2553z_2^{-1} + 0.1716z_1^{-1}z_2^{-1}} \quad (26)$$

which corresponds with (22). Therefore, it is confirmed that our proposed formulation is valid as the frequency transformation of 2-D digital filters.

We emphasize that the state-space model (25), which is obtained through our proposed formulation, has the same dimension as the order of $H_d(z_1, z_2)$. Hence this model is more suitable for the state-space analysis of $H_d(z_1, z_2)$ than the state-space model represented by (23).

VI. Conclusion

This letter has proposed the frequency transformation in terms of the 2-D state-space equations. The proposed frequency transformation enables us not only to obtain the state-space representation of transformed filters easily, but also to analyze the transformed filters by the state-space equations effectively.

In this letter, we have restricted ourselves to 1-D all-pass transformations, as we stated in Section III. Since the generalized discussion is very important from a theoretical point of view, the state-space formulation of 2-D all-pass transformations is a task of our future work.

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