Quantum Mechanical Probabilities Not Restricted to a Moment of Time

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Quantum Mechanical Probabilities Not Restricted to a Moment of Time

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A theoretical aspect of quantum mechanical probabilities is studied. In particular, it is investigated whether quantum mechanical probabilities can be defined for alternatives which are not restricted to a moment of time. Firstly the meaning and the status of this investigation in quantum mechanics are explained. Secondly a general framework is constructed within which we make the investigation. Thirdly a necessary mathematical tool is reviewed and extended. Lastly the general framework is applied to concrete examples with the help of the mathematical tool. Examples of alternatives are found which are not restricted to a moment of time and for which quantum mechanical probabilities can be defined with clear measurement theoretical meanings if an initial condition of the particle belongs to a specific class. (This is the doctoral thesis of the author.)

key words: quantum mechanical probabilities, interference, sum over paths, decoherence functional, alternatives

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Abstract

Chap. I. Introduction
The standard quantum theory viewed as a probability theory has a special property. That is, probabilities are defined only for alternatives at a single moment of time. Then the following question arises: Is it possible to define quantum mechanical probabilities for alternatives not restricted to a single moment of time? This is the theme of this thesis. In this chapter we explain the aim and the background of this thesis and give an outline of the construction of it.

Chap. II. General Framework
A general framework is constructed which judges whether or not quantum mechanical probabilities can be defined for a given ES (Event Space). The framework provides two conditions: the classifiability condition (C·1) and the no-interference condition (C·2). C·1 requires that the propagator for the particle is decomposable into "components" each of which is associated with each alternative of the ES. C·2 is the consistency condition between the superposition principle for amplitudes and the sum rule for probabilities; it requires vanishing of interference between different components.

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Euclidean lattice method is a mathematical technique which gives a new definition to sum over paths, a definition which may be wider than that of Feynman's path integral in configuration space. In Feynman's path integral, time is skeletonized but space is continuous. By contrast in Euclidean lattice method, we go over to Euclidean time and skeletonize not only time but also space, making Euclidean spacetime lattice. A random walk is defined on the lattice; a discrete sum over paths is introduced which sums up random-walk probabilities over discrete paths of the walk. The sum over paths in configuration space is then defined by Wick-rotating the "diffusion limit" of the discrete sum over paths. This technique was developed by Hartle for a free particle. This chapter reviews and extends it to the case of a nonzero potential. Some formulae are also provided for later use. The Euclidean lattice method makes the general framework developed in the previous chapter applicable to concrete examples of ES.

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Chap. V. Application of General Framework to Concrete Examples (II)
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method. Positive results are obtained. As predicted in the previous chapter, C·1 holds for ESIV and V. In each case it is shown that there exists a specific class of initial amplitudes for which the interference vanishes between different alternatives. Therefore C·2 holds and probabilities can be defined if an initial amplitude belongs to the specific class. Values of probabilities are calculated. It is argued that, owing to the restriction of an initial amplitude, resultant probabilities are interpretable within the familiar measurement theory and they gain clear measurement theoretical meanings which are becoming to the values of the probabilities.

Chap. I. Introduction

§1. Introduction

1.1 *Posing a problem*

Quantum theory describes physical world with probabilities. The theory consists of a mathematical framework and physical interpretation. The mathematical framework provides a formulation which constructs positive quantities from complex valued amplitudes in such a way that they fulfill axioms for probabilities. Physical interpretation reads the positive quantities as physical probabilities and associates them with a set of outcomes of an experiment. We are interested in a probability-theoretical aspect of quantum theory. Quantum theory viewed as a probability theory has a special property. That is, probabilities are defined only for a specific class of sample spaces whose alternatives are all associated with a single moment of time. By a sample space, we mean a set of alternatives. An alternative is a member of the sample space. Physically an alternative is an occurrence or an event (in the sense of probability theory) of an experiment. Let us confine ourselves to nonrelativistic quantum mechanics (NRQM) for a particle and explain more about the special property.

We shall write a sample space as ES. (Read it as exhaustive set of alternatives or event space.) Since all measurements can eventually be reduced to position measurements (spacetime picture),[1] we shall work with alternatives in Newtonian spacetime. Alternatives, for example, in momentum space (with time) are therefore not studied here. In spacetime picture, the following probability for position is the only probability in quantum mechanics:

$$P(\Delta X,T) = \int_{\Delta X} dX |\Psi(X,T)|^2 \,, \tag{I·1·1}$$

which is normalized to

$$\sum_j P(\Delta X_j,T) = 1, \text{ i.e., } \int_{-\infty}^{\infty} dX |\Psi(X,T)|^2 = 1\,. \tag{I·1·2}$$

where $\Psi(X,T)$ is Schrödinger's wave function. Probability (I · 1 · 1) is assigned to the alternative (occurrence) that a particle is found in a certain spatial domain ΔX at a moment of time T. The sample space for (I · 1 · 1) is the set of non-overlapping domains $\{\Delta X_j\}$ on $\mathcal{S}_T$, that is, a hypersurface of constant time T in Newtonian spacetime. We shall call this sample space **EST**. EST lies on $\mathcal{S}_T$ in the sense that all the alternatives of EST are associated with domains ΔX_j on $\mathcal{S}_T$. Since (I · 1 · 1) is, up to now, the only probability in spacetime picture, we can say that a probability-definable ES in quantum mechanics is restricted to EST. However this is not always the case in probability theory. In Brownian motion for instance, there exist probabilities whose sample space is not EST. An example is the probability for the first hitting time. A particle starting from $(X, 0)$ and undergoing a Brownian motion crosses the spatial origin $X = 0$ many times (we consider (1+1)-dimensional case). The first time $T \in \Delta T$ at which the particle hits (crosses or touches) the origin is a random variable. The probability that the first hitting time T lies in ΔT is given by

$$P(\Delta T, X) = \int_{\Delta T} dT \left(\frac{X^2}{2\pi T^3} \right)^{1/2} \exp\left(-\frac{X^2}{2T} \right) , \qquad (\mathrm{I} \cdot 1 \cdot 3)$$

normalized to

$$\int_0^\infty dT\, P(T, X) = 1 . \qquad (\mathrm{I} \cdot 1 \cdot 4)$$

These should be compared with Eqs. (I · 1 · 1) and (I · 1 · 2). The sample space for (I · 1 · 3) is a countable set $\{\Delta T_j\}$ of non-overlapping temporal domains. All these domains lie on a single surface $\mathcal{S}_X$, a surface of constant $X (= 0)$. The sample space therefore lies on $\mathcal{S}_X$. Of course in Brownian motion, there also exist probabilities whose sample space is EST. They are probabilities for position at a moment of time. In this way, in Brownian motion, probabilities whose sample space lies on $\mathcal{S}_X$ coexist with probabilities whose sample space lies on $\mathcal{S}_T$, namely, probability-definable ES is not restricted to EST.

On the basis of the above observation, we pose the following problem which is the theme of this thesis:

Is it possible to define quantum mechanical probabilities for ES other than EST? (I · 1 · 5)

To put the question more concretely,[2)] let us consider a general surface $\mathcal{S}$ and two surfaces $\mathcal{S}_{T_A}$ and $\mathcal{S}_{T_B}$ of constant time which are arranged as in Fig. 1. Suppose that we prepare a particle at a spacetime point $A = (X_A, T_A)$, and consider particle's motion from A to another spacetime point $B = (X_B, T_B)$ $(T_A < T_B)$. A classical particle traveling from A to B intersects $\mathcal{S}$ a certain number of times at certain places. (In the presence of a potential, the particle intersects $\mathcal{S}$ more than once in general.) In Fig. 1, the number of intersection is three and the places of intersection are $\lambda = \lambda_1$, λ_2 and λ_3, where λ is a suitably introduced coordinate on $\mathcal{S}$. The definiteness of the number and the places of intersection is of course due to the uniqueness of the classical path. In quantum mechanics however, the particle is associated with no unique path as illustrated clearly in the sum-over-paths (i.e., the path-integral) formulation . The number and the places of intersection of the particle with $\mathcal{S}$ are then expected not to be unique, since the number and the places of intersection of virtual paths with $\mathcal{S}$ vary from a virtual path to another (see Fig. 2). If the number and the places of intersection are actually 'observed' on a fixed $\mathcal{S}$, a single 'observation' will yield a definite result. For instance one will find that the particle has 'intersected' $\mathcal{S}$ three times at $\lambda = \lambda_1$, λ_2 and λ_3. Another 'observation' will yield another definite result. The possible description of such variable outcomes must be a probabilistic one. In this circumstance, we consider an ES which is the set of all possible numbers and places for the particle to intersect $\mathcal{S}$. An alternative of this ES is an occurrence that the particle intersects $\mathcal{S}$ a certain number of times at certain places. This ES we name **ESI.** Since all the alternatives of ESI are associated with 'measurements on $\mathcal{S}$', This kind of ES we shall call **ES on** $\mathcal{S}$. An ES on $\mathcal{S}$ is an example of ES which is not EST. Of course, when $\mathcal{S}$ is $\mathcal{S}_T$, ES on $\mathcal{S}$ reduces to EST. For ESI, the problem $(\mathrm{I} \cdot 1 \cdot 5)$ asks, "Can quantum mechanical probabilities be defined which predict the number of times and places the particle intersects $\mathcal{S}$?" We can think of other examples of ES on $\mathcal{S}$. The set of all possible places where the particle is first found on $\mathcal{S}$ is such an example, which we shall name ESIII. There are also other kinds of ES. which are not EST and are not ES on $\mathcal{S}$. Consider a spacetime domain Ω as shown in Fig. 3. Then we can think of two alternatives "Yes" and "No". "Yes" is the occurrence that a particle which started at time T_A is 'found' in Ω and 'No' is the complement to 'Yes'. The set $\{\mathrm{Yes}, \mathrm{No}\}$ is an ES which is neither EST nor ES on $\mathcal{S}$, which we shall name ESV. For ESV, we ask "Can quantum mechanical probabilities be defined which predict whether the particle is 'found' in Ω or not?"

1.2 *The aim of this thesis*

In the above paragraph, we used single quotes ' ' for terms which are more or less measurement theoretical. One must be careful about such terms. Measurement theoretical meanings are not self evident for them, because the familiar measurement theory of quantum mechanics deals with only an instantaneous measurement or a sequence of instantaneous measurements. As a matter of fact, problem $(\mathrm{I} \cdot 1 \cdot 5)$ has not been well-posed yet because 'ES other than EST' has not been given a definite measurement theoretical meaning. It seems difficult to pose problem $(\mathrm{I} \cdot 1 \cdot 5)$ in such a way that the meaning of ES is clear from the beginning, because such an ES is new to quantum mechanics. In view of this, this thesis has two aims: First, it aims to present problem $(\mathrm{I} \cdot 1 \cdot 5)$ in such a way that it is well-posed at least mathematically, so that the setting itself of the problem does not suffer from measurement theoretical issues; it also aims to construct a general framework which can answer to problem $(\mathrm{I} \cdot 1 \cdot 5)$ when an ES is given. The second aim is to apply the general framework to particular ES to give an answer to (I · 1 · 5) within the framework. Here we shall briefly see how the problem (I · 1 · 5) becomes well-posed, taking ESV as an example of ES other than EST. Taking spacetime points A and B respectively in the past and in the future of the spacetime domain Ω, we first introduce two amplitudes

$$\Phi(B;\mathrm{Yes};A) \equiv \sum_{B\leftarrow\Omega\leftarrow A} e^{iS}, \qquad \Phi(B;\mathrm{No};A) \equiv \sum_{B\leftarrow\text{outside }\Omega\leftarrow A} e^{iS}, \qquad (\mathrm{I}\cdot 1\cdot 6)$$

where the sums are respectively over paths which pass through Ω and over paths which never pass through it on the way from A to B (see Fig. 3). These amplitudes are purely mathematical objects. We then construct positive quantities $P(\mathrm{Yes})$, $P(\mathrm{No})$ and $P(\mathrm{Yes\ or\ No})$ from these amplitudes by generalizing the rules of calculating constant-time probabilities in the sum-over-paths formulation, rules which correctly work in calculating probability (I · 1 · 1) from an amplitude. Exploring the possibility that the positive quantities thus constructed fulfill axioms for probabilities, we formulate two conditions for amplitudes (I · 1 · 6) under which the positive quantities fulfill the axioms. It then follows that, when the two conditions are satisfied, the positive quantities are, at least mathematically, probabilities. The argument so far has nothing to do with measurements. Measurement theoretical issues arise when the two conditions are satisfied, namely, probabilities are defined. We expect that the resultant probabilities are not only mathematical ones but also associated with physical meanings, since the probabilities are constructed from amplitudes (I · 1 · 6) defined by the behavior of paths with respect to Ω. For example, we expect

$$P(\mathrm{Yes}) = \text{the probability of finding the particle in } \Omega. \qquad (\mathrm{I}\cdot 1\cdot 7)$$

This is to be understood as a trial of giving a definite measurement theoretical meaning to the phrase on the right-hand side, phrase whose meaning is not self-evident in the familiar quantum mechanics in which only an instantaneous measurement is considered. Investigations for other ES are also made in this way. For ESIII for example, we consider amplitudes $\Phi(B;\Delta;A)$, each of which is the sum of e^{iS} over paths which connect A and B and whose first hitting (i.e., crossing or touching) of a given surface $\mathcal{S}$ lies in a domain Δ. Using these amplitudes instead of (I · 1 · 6), we construct positive quantities $P(\Delta)$ according to the generalized rules of constructing a probability from an amplitude in the sum-over-paths formulation. We explore whether these positive quantities fulfill axioms for probabilities. When they are fulfilled, the positive quantities become probabilities and we raise the problem of interpretation: Can the probability $P(\Delta)$ constructed from $\Phi(B;\Delta;A)$ be interpréted as the probability that the first place where the particle is found on $\mathcal{S}$ lies in Δ? These are what we actually investigate in this thesis in answer to (I · 1 · 5). The meaning of an ES which was intuitively introduced as an exhaustive set of alternatives is precisely a distinct set of amplitudes defined by the behavior of paths with respect to a given domain. (I · 1 · 6) is an example of such a set. An ES is specified by two elements: a domain (surface, spacetime domain and so on) and what behavior of paths we pay attention to. Different ES can be considered with respect to the same domain. For example, with respect to one surface $\mathcal{S}$, we consider ESI by paying attention to how may times and at what locations each path intersects $\mathcal{S}$ and consider ESIII by paying attention only to the first location each path hits $\mathcal{S}$. This we simply say that ESI is the set of possible numbers of times and locations the particle intersects $\mathcal{S}$ and ESIII is the set of possible first locations the particle hits $\mathcal{S}$, pretending that a path in the sum-over-paths quantum mechanics is an observable, which is of course not actually so. Thus we re-present problem (I · 1 · 5) as the set of following questions:

Q1: For a given ES, are probability axioms fulfilled by positive quantities constructed from the set of amplitudes according to the generalized rules of constructing probabilities from amplitudes in the sum-over-paths formulation? (I · 1 · 8)

Q2: When the axioms are fulfilled, are the probabilities associated with measurement theoretical meaning becoming to the values of the probabilities? (I · 1 · 9)

When these two questions are positively answered, the way of saying is allowed that ESIII is the set of possible locations where the particle is first found $\mathcal{S}$. Keeping all these in mind, we shall use this way of saying at any stage of investigations of the two questions, just for simplicity. For instance, when the first question Q1 is negatively answered for ESIII, we say that quantum mechanical probabilities cannot be defined for the first place where the particle is found on $\mathcal{S}$.

1.3 *Background*

Our problem was motivated by the work of Hartle in Ref. 3). He argued that the role of time in the sum-over-paths formulation of NRQM is not so central as that in the Hamiltonian formulation is. In the argument he investigated a definability of "wave function on $\mathcal{S}$", which is an extension of Schrödinger's wave function on $\mathcal{S}_T$. The wave function was considered in the sum-over-paths formulation and was essentially of the form of

$$\sum_{\vec{\lambda} \leftarrow A} e^{iS}, \qquad (\mathrm{I} \cdot 1 \cdot 10)$$

where S is the action for a nonrelativistic particle; $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \cdots)$ stands for a suitable set of places on $\mathcal{S}$ and the sum is over paths which connect A and $\vec{\lambda}$ and which lie only in the past of $\mathcal{S}$. The absolute square of this wave function was expected to give probabilities for finding a particle on $\mathcal{S}$ at places $\vec{\lambda}$. (Although Hartle did not mention the probabilistic interpretation of this wave function, the way he tried to construct it suggests this probabilistic interpretation.) What he proved was that the wave function does not exist in quantum mechanics. This conclusion is quite natural if one takes account of the above-mentioned interpretation of the wave function. If such a wave function was to exist, then its classical limit, which should describe classical motion of the particle, would make a definite prediction as to the number and the places the particle is found on $\mathcal{S}$; this definite prediction is made from the information only about the past of $\mathcal{S}$, because in the sum $(\mathrm{I} \cdot 1 \cdot 10)$ paths to be summed over lie only in the past of $\mathcal{S}$. This is clearly impossible. Places where a classical particle is fund on $\mathcal{S}$ cannot be determined from the information only about the past of $\mathcal{S}$, simply because a classical path is not confined in the past of $\mathcal{S}$. In this way, existence of wave function which predicts how one finds a particle on $\mathcal{S}$ from the information only about the past of $\mathcal{S}$ manifestly conflicts with causality, as clearly understood in the classical limit. What we would like to emphasize here is that what he disproved is the existence of a wave function on $\mathcal{S}$ but not the existence (i.e., definability) of probabilities concerning how one finds a particle on $\mathcal{S}$. These considerations motivated us to investigate the existence of such probabilities, which led us to $(\mathrm{I} \cdot 1 \cdot 5)$.

1.4 *Outline*

In this thesis we investigate the problem posed as $(\mathrm{I} \cdot 1 \cdot 5)$ within nonrelativistic quantum mechanics for a particle. We make the problem well-posed, construct a general framework, apply it to concrete examples of ES and obtain interesting results. The outline of this paper is as follows. In Chap. II, making problem $(\mathrm{I} \cdot 1 \cdot 5)$ well-posed, we construct a general framework which judges whether or not quantum mechanical probabilities can be defined for a given ES. The framework is constructed in the sum-over-paths formulation of quantum mechanics. For a general ES, we first provide the rule of assigning a probability amplitude to an individual alternative. Next we propose a particular way of constructing positive quantities from probability amplitudes, quantities which we want to interpret as probabilities. We then formulate two conditions under which the positive quantities fulfill axioms for probabilities. The two conditions shall be called the classifiability condition (C·1) and the no-interference condition (C·2).

In this way, the problem of probability-definability for a general ES is reduced to the problem of whether the two conditions hold or not for the ES. As we shall discuss, C·2 is the consistency condition between the superposition principle for amplitudes and the sum rule for probabilities. In Chap. III we make a mathematical preparation for calculations of sums over paths such as (I · 1 · 6). The Euclidean lattice method is reviewed and extended which gives a rigorous definition to such a sum over paths. This seems wider than Feynman's path integral. This method skeletonizes not only time (Euclidean) but also space. A sum over paths is defined on this Euclidean lattice as a discrete sum of a random-walk probabilities. In Chap. IV and V we apply the general framework to concrete examples of ES with the help of Euclidean lattice method. We investigate following five ES in (1+1)-dimensions as typical examples of ES other than EST:

- ESI : the set of possible numbers of times and possible locations a particle **intersects** a steplike surface in the spacetime,[2)]
- ESII : the set of possible numbers of times and possible locations a particle is **found** on a steplike surface,[2)]
- ESIII : the set of possible locations a particle is **first found** on a steplike surface,[4)]
- ESIV : the set {Yes, No}, where "Yes" is to find a particle in a **temporal domain** ΔT at constant X and "No" is the complement to "Yes",[4)]
- ESV : the set {Yes, No}, where "Yes" is to find a particle in a **spacetime domain** $\Omega \equiv \Delta X \times \Delta T$ and "No" is the complement to "Yes".[5)]

Chap. IV deals with ESI∼ESIII. For ESI and II, C·1 does not hold. For ESIII, C·1 holds but C·2 fails. Thus probabilities cannot be defined for these three ES. We argue that whether C·1 holds or fails is governed by two factors: the non-differentiable property of paths and the "coarseness" of alternatives. Chap. V deal with ESIV and V. For each ES, C·1 holds. It is concluded in each case that C·2 holds and therefore probabilities can be defined if an initial amplitude for the particle belongs to a specific class. Owing to the restriction of an initial amplitude, resultant probabilities are well understood within the familiar measurement theory and hence they gain clear measurement theoretical meanings which are becoming to the values of the probabilities. In Chap. VI, we first summarize this study and then discuss several issues and questions which may arise. We also pose remaining problems. Appendices and references are given in the end.

Chap. II. General Framework

We construct a general framework for the investigation of definability of QP (quantum mechanical probabilities) for ES unrestricted to a moment of time. We proceed as follows:

§1 We rewrite Eq. (I · 1 · 1) into sum-over-paths fashion and observe how QP for EST are constructed from relevant amplitudes in the sum-over-paths formulation.

§2 We generalize the sum-over-paths rules of constructing QP from relevant amplitudes to a general ES. Requiring the consistency of this generalization with mathematical axioms for probabilities and adding other observations as well, we formulate two conditions under which QP can be defined for the ES. Whether QP can be defined or not for a given ES is judged by these two conditions. If both of them hold, then QP can be defined.

§1. Sum-over-paths reconstruction of QP for EST

We take spacetime points $A \equiv (X_A, T_A)$ and $B \equiv (X_B, T_B)$ as shown in Fig. 1. In the sum-over-paths formulation,[1),6)] the quantum mechanical propagator $\Phi(B;A)$ which describes particle's motion from A to B is defined by

$$\begin{aligned}\Phi(B;A) &\equiv \sum_{B\leftarrow A} e^{iS} \\ &\equiv \lim_{\epsilon\to 0}\prod_{j=1}^{N-1}\int_{-\infty}^{\infty} dX_j (2\pi i\epsilon)^{-N/2}\exp\left[i\sum_{k=0}^{N-1}\left\{\frac{(X_{k+1}-X_k)^2}{2\epsilon}+\epsilon V(\frac{X_{k+1}+X_k}{2})\right\}\right],\end{aligned} \tag{II · 1 · 1}$$

where $X_0 \equiv X_A$, $X_N \equiv X_B$ and $\epsilon N = T_B - T_A$; the sum is over all the paths which move forward in time to connect A and B, whose exact definition is given by the last right-hand side. This is the propagator of Schrödinger's wave function $\Psi(X,T)$:

$$\Psi(B) = \int_{-\infty}^{\infty} dX_A \Phi(B;A)\Psi(A), \tag{II · 1 · 2}$$

where $\Psi(A) \equiv \Psi(X_A, T_A)$ and so on. Let $\Psi(A)$ be an initial amplitude. Probability (I · 1 · 1) can be rewritten as follows:

$$P(\Delta X, T) = \int dX_B \left|\int dX_A \Phi(B;\Delta X, T; A)\Psi(A)\right|^2 \tag{II · 1 · 3}$$

with

$$\Phi(B;\Delta X,T;A) \equiv \sum_{B\leftarrow\Delta X\leftarrow A} e^{iS} = \int_{\Delta X} dX \Phi(B;X,T)\Phi(X,T;A)\,, \qquad (\mathrm{II}\cdot 1\cdot 4)$$

where the sum is over (virtual) paths which start from A, move forward in time to cross $\mathcal{S}_T$ in the interval ΔX at time T and then arrive at B; this sum over paths can be factored into two sums over paths and one integration: the sum over paths from A to (X,T) such that $X \in \Delta X$, that from (X,T) to B, and the integration over all possible values X of crossing of $\mathcal{S}_T$. Because of the very definition of Feynman's path integral, the two sums over paths yield propagators $\Phi(X,T;A)$ and $\Phi(B;X,T)$; this explains the last equality in Eq. (II · 1 · 4). Equivalence of expression (II · 1 · 3) to (I · 1 · 1) can be seen by substituting Eq. (II · 1 · 4) into Eq. (II · 1 · 3) and performing the integration over X_B by use of the following well-known property of the propagator

$$\int dX_{B'}\Phi^*(B';X_{B1},T_B)\Phi(B';X_{B2},T_B) = \delta(X_{B1}-X_{B2})\,. \qquad (\mathrm{II}\cdot 1\cdot 5)$$

Probabilities $P(\Delta X_j,T)$ of course satisfy axioms for probabilities:

$$0 \le P(\Delta X,T) \le 1 \qquad (\mathrm{II}\cdot 1\cdot 6)$$

$$P(\Delta X_j \cup \Delta X_k,T) = P(\Delta X_j,T) + P(\Delta X_k,T) \qquad (j \neq k) \qquad (\mathrm{II}\cdot 1\cdot 7)$$

$$\sum_j P(\Delta X_j,T) = 1\,, \qquad (\mathrm{II}\cdot 1\cdot 8)$$

where ΔX_j ($j = 1,2,\cdots$) are non-overlapping spatial domains at time T covering $\mathcal{S}_T$ as j exhausts all integers; for $j \neq k$, $P(\Delta X_j \cup \Delta X_k,T)$ is the probability to find the particle in ΔX_j or in ΔX_k at time T. From the wave-function expression (I · 1 · 1), it is trivial that these axioms are satisfied. Axioms (II · 1 · 6) and (II · 1 · 8) is obvious from the positivity and the normalization of the right-hand side of (I · 1 · 1), which follows from the normalization of the initial amplitude

$$\int_{-\infty}^{\infty} dX_A|\Psi(A)|^2 = 1\,, \qquad (\mathrm{II}\cdot 1\cdot 9)$$

which we always assume. Axiom (II · 1 · 7) is also trivial from the additivity of the integration

$$\int_{\Delta X_j \cup \Delta X_j} dX = \int_{\Delta X_j} dX + \int_{\Delta X_k} dX \quad (j \neq k). \qquad (\mathrm{II}\cdot 1\cdot 10)$$

Next let us see how these axioms are satisfied by the sum over paths expression (II · 1 · 3). The positivity $P > 0$ is trivial from (II · 1 · 3). To see how the sum rule (II · 1 · 7) is satisfied, we note that the sum-over-paths formulation constructs the probability for a union of alternatives as follows:

$$\begin{aligned} &P(\Delta X_j \cup \Delta X_k, T) \\ &= \int dX_B \left| \int dX_A \Big(\Phi(B; \Delta X_j, T; A) + \Phi(B; \Delta X_k, T; A) \Big) \Psi(A) \right|^2 \quad (j \neq k). \end{aligned} \qquad (\mathrm{II}\cdot 1\cdot 11)$$

Note that, according to the superposition principle, the amplitude to pass through ΔX_j and that through ΔX_k are added before absolutely squared. Expanding the right-hand side, we once have

$$P(\Delta X_j \cup \Delta X_k, T) = P(\Delta X_j, T) + P(\Delta X_k, T) + 2\mathrm{Re}D[\Delta X_j; \Delta X_k] \qquad (\mathrm{II}\cdot 1\cdot 12)$$

with

$$\begin{aligned} &D[\Delta X_j; \Delta X_k] \\ &\equiv \int dX_B \iint dX_A dX_{A'} \Phi^*(B; \Delta X_j, T; A) \Phi(B; \Delta X_k, T; A') \Psi^*(A) \Psi(A'). \end{aligned} \qquad (\mathrm{II}\cdot 1\cdot 13)$$

The cross term ReD (2ReD in the strict sense) expresses the quantum mechanical interference between the two alternatives: to find the particle in ΔX_j at time T and to find it in ΔX_k at the time. Of course there should be no interference between the two alternatives; otherwise the sum rule fails. Vanishing of the cross term is proved by using Eq. (II · 1 · 4) and performing the integration over X_B on the right-hand side of (II · 1 · 13); the integral vanishes owing to Eq. (II · 1 · 5) to give

$$D[\Delta X_j; \Delta X_k] = 0 \quad (j \neq k). \qquad (\mathrm{II}\cdot 1\cdot 14)$$

In this way, vanishing of interference (to be called no-interference, for brevity) between different alternatives makes Eq. (II · 1 · 11) consistent with Eq. (II · 1 · 7). In other words, because of no-interference between different alternatives, the superposition principle for amplitudes consists with the sum rule for probabilities. Normalization (II · 1 · 8) is proved as follows:

$$\begin{aligned}\sum_j P(\Delta X_j,T) &= \sum_j \int dX_B \left| \int dX_A \Phi(B;\Delta X_j,T;A)\Psi(A) \right|^2 \\ &= \int dX_B \left| \int dX_A \sum_j \Phi(B;\Delta X_j,T;A)\Psi(A) \right|^2 \\ &= \int dX_B \left| \int dX_A \Phi(B;A)\Psi(A) \right|^2 \\ &= \int dX_A \, |\Psi(A)|^2 \\ &= 1 ,\end{aligned} \tag{II · 1 · 15}$$

where the second equality is guaranteed by the vanishing of interferences, the third equality by

$$\begin{aligned}\sum_j \Phi(B;\Delta X_j,T;A) &= \int dX \Phi(B;X,T)\Phi(X,T;A) \\ &= \Phi(B;A),\end{aligned} \tag{II · 1 · 16}$$

and the fourth equality by Eq. (II · 1 · 5). Since we always assume that the initial amplitude is normalized, we have the last equality in Eq. (II · 1 · 15). The first axiom (II · 1 · 6) is now obvious from (II · 1 · 15) and the positivity of P. By the way Eq. (II · 1 · 16) shows that the amplitudes for passing through ΔX_j on the way from A to B are summed up to recover the total amplitude, namely the propagator connecting A and B. Hence we call the amplitude $\Phi(B;\Delta X_j,T;A)$ the "component" of $\Phi(B;A)$ associated with the alternative that one finds a particle in ΔX_j at time T. Since a component is associated with an alternative, we shall often say "interference between different components" instead of "interference between different alternatives". Equation (II · 1 · 16) is understood in the context of "path classification" as illustrated in Fig. 4. The (virtual) paths which define the propagator $\Phi(B;A)$ by Eq. (II · 1 · 1) are classifiable into distinct classes of paths; the paths belonging to the class specified by ΔX_j are all the paths which pass through ΔX_j at time T on the way from A to B.

From the above observation, we learn two things.

(1) Exclusiveness and no-interference.

It is an empirical fact that to find a particle in ΔX_j and to find it in ΔX_k

$(j \neq k)$ at a moment of time are mutually exclusive. In the sum-over-paths formulation, this fact is *guaranteed* by no-interference between the two components $\Phi(B; \Delta X_j, T; A)$ and $\Phi(B; \Delta X_k, T; A)$, where the interference is given by ReD with D defined by Eq. (II · 1 · 13). In short, exclusiveness is guaranteed by no-interference.

(2) **Normalization.**
Normalization (II · 1 · 8) is derived from the normalization of an initial amplitude. In the sum-over-paths formulation, as seen from Eq. (II · 1 · 15), this derivation is based on the no-interference between different alternatives and on the path-classification relation (II · 1 · 16).

To summarize: Because the propagator $\Phi(B; A)$ is decomposable into components as Eq. (II · 1 · 16) and because there is no interference between different components, QP can be defined for EST with values given by Eq. (II · 1 · 3).

§2. General Framework

2.1 *Construction*

Our problem is to construct a general framework which judges whether or not QP can be defined for a given ES. We concentrate on (I · 1 · 8) in this subsection, so that measurement theoretical issues are not discussed here. Understanding QP for EST as in §1, we get a clue to the construction of the framework. By introducing the notion of decomposition of propagator for a general ES, we can define the interference between different components in the same way as Eq. (II · 1 · 13); if thus defined interference vanishes, then QP can be defined for the ES with values given by expressions similar to Eq. (II · 1 · 3).

We shall formulate this idea. Consider a general ES=$\{\mathcal{O}_j\}$. First we introduce the notion of component of propagator. As discussed in Chap. I, an $\mathcal{O}_j$ represents a possible way of finding a particle in a suitable domain in Newtonian spacetime. Let the suitable domain be bounded by two surfaces $\mathcal{S}_{T_A}$ and $\mathcal{S}_{T_B}$ of constant time T_A and T_B, respectively, such that $-\infty < T_A < T_B < \infty$. (This is always possible because outcomes of a physical experiment are always obtained within a finite time interval.) As a straightforward generalization of Eq. (II · 1 · 4), we define a component of the propagator $\Phi(B; A)$:

$$\Phi(B; \mathcal{O}_j; A) \equiv \sum_{B \leftarrow \mathcal{O}_j \leftarrow A} e^{iS}, \qquad (\text{II} \cdot 2 \cdot 1)$$

where the sum is over paths specified by $\mathcal{O}_j$ on the way from A to B. To be specific, we give some examples. If the ES is ESI, then $\mathcal{O}_j=\Delta(\vec{l}_j)$ and the paths specified by $\mathcal{O}_j$ are all the paths which intersect the surface $\mathcal{S}$ at j-places $\Delta(\vec{l}_j)$ on the way from A to B; the sum of e^{iS} over all such paths defines the component $\Phi(B;\Delta(\vec{l}_j);A)$ which is associated with the alternative $\mathcal{O}_j=\Delta(\vec{l}_j)$. If the ES is ESV, which consists of two alternatives $\mathcal{O}_0\equiv$"Yes" and $\mathcal{O}_1\equiv$"No", the paths specified by $\mathcal{O}_0$ are all the paths which pass through the spacetime domain Ω on the way from A to B; the paths specified by $\mathcal{O}_1$ are all the paths never crossing Ω. The sum of e^{iS} over the paths specified by "Yes" defines $\Phi(B;\Omega;A)$, the component associated with "Yes". The component associated with "No" is defined in a similar way. Intuitively, one might expect for an arbitrary ES that the sum of all the components gives the propagator $\Phi(B;A)$. However, as will be discussed later, this is not always the case. For such an ill ES, we cannot define QP. Hence we pose the following equation as the first condition to be satisfied by the ES:

$$\mathrm{C}\cdot 1 \quad : \quad \Phi(B;A)=\sum_j \Phi(B;\mathcal{O}_j;A)\,. \qquad (\mathrm{II}\cdot 2\cdot 2)$$

Condition C·1 requires that the virtual paths which define the propagator by Eq. (II · 1 · 1) are classifiable into distinct classes of paths, where each class is specified by an $\mathcal{O}_j$. Hence we call C·1 the "classifiability condition". When C·1 holds for a given ES, we construct positive quantities from the components as follows:

$$P(\mathcal{O}_j)\equiv\int dX_B\left|\int dX_A\Phi(B;\mathcal{O}_j;A)\Psi(A)\right|^2, \qquad (\mathrm{II}\cdot 2\cdot 3)$$

which we want to interpret as QP of the occurrence of $\mathcal{O}_j$. The initial amplitude $\Psi(A)$ is the same as that in Eq. (II · 1 · 3); it is Schrödinger's wave function at an initial time T_A. The positive quantity which we want to interpret as QP for the union of $\mathcal{O}_j$ and $\mathcal{O}_k$ $(j\neq k)$ is

$$P(\mathcal{O}_j\cup\mathcal{O}_k)\equiv\int dX_B\left|\int dX_A\Big(\Phi(B;\mathcal{O}_j;A)+\Phi(B;\mathcal{O}_k;A)\Big)\Psi(A)\right|^2, \qquad (\mathrm{II}\cdot 2\cdot 4)$$

in which, according to the superposition principle, two components (amplitudes) are added before absolutely squared. Positive quantities (II · 2 · 3) and (II · 2 · 4) are straightforward generalizations of (II · 1 · 3) and (II · 1 · 11), respectively. Expanding the right-hand side of (II · 2 · 4), we have

$$P(\mathcal{O}_j\cup\mathcal{O}_k)=P(\mathcal{O}_j)+P(\mathcal{O}_k)+2\mathrm{Re}D[\mathcal{O}_j;\mathcal{O}_k] \qquad (\mathrm{II}\cdot 2\cdot 5)$$

with

$$D[\mathcal{O}_j;\mathcal{O}_k] \equiv \int dX_B \iint dX_A dX_{A'} \Phi^*(B;\mathcal{O}_j;A)\Phi(B;\mathcal{O}_k;A')\Psi^*(A)\Psi(A') . \quad (\text{II} \cdot 2 \cdot 6)$$

In order for this to be consistent with the sum rule for probabilities, the cross term must vanish. We thus require the second condition

$$\text{C} \cdot 2 \quad : \quad \text{Re}D[\mathcal{O}_j;\mathcal{O}_k] \propto \delta_{jk} . \quad (\text{II} \cdot 2 \cdot 7)$$

Real part of D may be interpreted to be the interference between the two alternatives $\mathcal{O}_j$ and $\mathcal{O}_k$. The condition C·2 requires that every interference between different alternatives vanishes so that the superposition principle consists with the sum rule. The quantity $D[\mathcal{O}_j;\mathcal{O}_k]$ is essentially what has been called the "decoherence functional" for pairs $(\mathcal{O}_j;\mathcal{O}_k)$ of alternatives by Gell-Mann and Hartle.[8] They define a decoherence functional for a more general initial state, that is, a mixed state. The word decoherence is used by them, while we use the word no-interference, both of which are the same thing.

It is easy to see that, if both C·1 and C·2 hold, then axioms for probabilities are satisfied which are, for a general ES,

$$0 \leq P(\mathcal{O}_j) \leq 1, \quad (\text{II} \cdot 2 \cdot 8)$$

$$P(\mathcal{O}_j \cup \mathcal{O}_k) = P(\mathcal{O}_j) + P(\mathcal{O}_k) \quad \text{for} \quad j \neq k , \quad (\text{II} \cdot 2 \cdot 9)$$

and

$$\sum_j P(\mathcal{O}_j) = 1. \quad (\text{II} \cdot 2 \cdot 10)$$

The proof is completely parallel to that for EST. The sum rule (II · 2 · 9) readily follows from C·2. Normalization (II · 2 · 10) is proved as follows:

$$\begin{aligned} \sum_j P(\mathcal{O}_j) &= \sum_j \int dX_B \left| \int dX_A \Phi(B;\mathcal{O}_j;A)\Psi(A) \right|^2 \\ &= \int dX_B \left| \int dX_A \sum_j \Phi(B;\mathcal{O}_j;A)\Psi(A) \right|^2 \\ &= \int dX_B \left| \int dX_A \Phi(B;A)\Psi(A) \right|^2 \\ &= \int dX_A \, |\Psi(A)|^2 \\ &= 1 , \end{aligned} \quad (\text{II} \cdot 2 \cdot 11)$$

where the second equality is guaranteed by C·2, the third equality by C·1 and the fourth by Eq. (II · 1 · 5). The first axiom (II · 2 · 8) is obvious from the positivity and the normalization of P.

Now we have achieved the following framework:

Step1. Given an ES, we calculate components $\Phi(B;\mathcal{O}_j;A)$ according to (II · 2 · 1).

Step2. We investigate whether or not the components satisfy the classifiability condition C·1, namely Eq. (II · 2 · 2). If they do not, we conclude, at this stage, that QP cannot be defined for the ES.

Step3. When C·1 is satisfied, we evaluate the decoherence functional (II · 2 · 6). If C·2 holds for any pairs $(\mathcal{O}_j;\mathcal{O}_k)$ of alternatives $(j \neq k)$, we conclude that QP can be defined for the ES with values (II · 2 · 3). Otherwise QP cannot be defined.

2.2 *Favorable features of the framework*

Here we stress favorable features of our framework.

(1)**No violation of causality:**

Remember Hartle's wave function on $\mathcal{S}$ discussed in Chap. I. There we argued that construction of QP for an ES on $\mathcal{S}$ from such a wave function manifestly conflicts with causality. The reason was because in the sum-over-paths definition of the wave function e^{iS} was summed over those paths which lie *in the past of* $\mathcal{S}$. By contrast, as seen from Eq. (II · 2 · 1), our amplitudes $\Phi(B;\mathcal{O}_j;A)$ are defined by summing e^{iS} over paths which pass through the domains relevant to the ES. When the ES is an ES on a surface $\mathcal{S}$, paths to be summed over intersect the surface $\mathcal{S}$; paths are therefore *not confined in the past of* $\mathcal{S}$. By constructing (II · 2 · 3) from such amplitudes, we do not violate causality.

(2)**QP's independence of a final surface:**

A necessary condition for QP is that it must not depend on the choice of a final surface $\mathcal{S}_{T_B}$ so long as it lies in the future of the domains relevant to the ES. Our construction of (II · 2 · 3) satisfies this. This is easy to check. Take $\mathcal{S}_{T_{B'}}$, a surface of constant time $T_{B'}$, in the future of $\mathcal{S}_{T_B}$ $(T_B < T_{B'})$ which is assumed to be in the future of the domains. If $\mathcal{S}_{T_{B'}}$ is used instead of $\mathcal{S}_{T_B}$, then $\int dX_B$ and $\Phi(B;\mathcal{O}_j;A)$ in Eq. (II · 2 · 3) are respectively replaced by $\int dX_{B'}$ and

$$\Phi(B';\mathcal{O}_j;A) = \sum_{B' \leftarrow \mathcal{O}_j \leftarrow A} e^{iS} \,. \tag{II · 2 · 12}$$

The sum on the right-hand side can be decomposed into three parts: the sum over all the paths specified by $\mathcal{O}_j$ on the way from A to B, the sum over all the paths from

B to B', and the integration over all possible values of X_B where paths intersect $\mathcal{S}_{T_B}$. Therefore

$$\Phi(B';\mathcal{O}_j;A) = \int dX_B \Phi(B';B)\Phi(B;\mathcal{O}_j;A). \qquad \text{(II}\cdot 2\cdot 13)$$

Replacing $\Phi(B;\mathcal{O}_j;A)$ in Eq. (II · 2 · 3) by the above, we perform the integration over $X_{B'}$ in Eq. (II · 2 · 3) by use of Eq. (II · 1 · 5). The result is identical to the original version of Eq. (II · 2 · 3) and the proof is completed. For the same reason, Eq. (II · 2 · 4) and the decoherence functional (II · 2 · 6) are also independent of $\mathcal{S}_{T_B}$.

(3)**"Automatic" normalization:**

In our framework the normalization of QP is not made in such an artificial way that

$$P_{\text{normalized}}(\mathcal{O}_j) = \frac{P(\mathcal{O}_j)}{\sum_k P(\mathcal{O}_k)}. \qquad \text{(II}\cdot 2\cdot 14)$$

If Eq. (II · 2 · 4) is not employed in constructing the general framework, then (II · 2 · 14) will be the only way of normalizing (II · 2 · 3). However we must not dispense with Eq. (II · 2 · 4) because quantum mechanics cannot dispense with superposition principle. In our framework, as seen from Eq. (II · 2 · 11), probabilities are automatically normalized as a consequence of no-interference (Re$D = 0$), the path-classification relation (II · 2 · 2) and of the normalization of an initial amplitude. This is indeed satisfactory, because the unity of the sum of all the probabilities, namely normalization of probabilities, has to be the direct consequence of the existence of a particle and of the exhaustiveness and exclusiveness of the alternatives. In our framework, existence of a particle is guaranteed since it exists at an initial time, or equivalently since an initial amplitude is normalized; exhaustiveness and exclusiveness are taken into account by Eqs. (II · 2 · 2) and (II · 2 · 7).

2.3 *Do C·1 and C·2 hold?*

For EST, each component $\Phi(B;\Delta X,T;A)$ is directly related to $\Phi(B;A)$ by Eq. (II · 1 · 4), which makes C·1 and C·2 hold. For a general ES, the relationship between each component $\Phi(B;\mathcal{O}_j;A)$ and the propagator is not self-evident. Is there a possibility that both of them hold for a general ES?

In order for C·1 to hold for a given ES=$\{\mathcal{O}_j\}$, paths contributing to the sum (II · 1 · 1) have to be classified with respect to the label $\mathcal{O}_j$. Paths dominantly contributing to (II · 1 · 1) are known to be everywhere non-differentiable with respect to time. Such paths cannot be classified by an arbitrary label. For example, non-differentiable paths intersect a general surface an infinite number of times. Then it is suspected that paths cannot be classified according to how many times they intersect the surface. This is indeed the case as we will experience in Chap. IV (ESI and II). The number of times of intersection is an inadequate label for path-classification. Whether C·1 holds for an ES depends on whether non-differentiable paths can be classified with respect to $\mathcal{O}_j$ of the ES.

One cannot expect C·2 to hold for every ES for which C·1 holds. The decoherence functional (II · 2 · 6) depends on (i) components $\Phi(B;\mathcal{O}_j;A)$ and (ii) an initial amplitude $\Psi(A)$. Whether C·2 holds or not depends on these two elements. The remarkable feature of EST is that D vanishes identically, that is, it vanishes *for an arbitrary initial amplitude* because of the special form (II · 1 · 4) of the components. For a general ES this mechanism of vanishing of interference may not be expected. However it can be possible that C·2 holds for a suitable class of initial amplitudes. In fact in Chap. V, we will experience such cases (ESIV and V). There, by calculating components and substituting them into the decoherence functional, we find that only a specific combination of the initial amplitude contributes to the decoherence functional and that there exists a class of initial amplitudes for which the specific combination vanishes. In this way, dependence of D on the initial amplitude gives a chance for C·2 to hold. Of course, such a luck does not happen for all ES. As also shown in Chap. IV, C·2 for ESIII does not hold for any initial amplitudes.

When C·2 holds for an ES under a specific class of initial amplitudes, our conclusion is as follows: QP can be defined for the ES only when an initial amplitude belongs to the specific class. One may think it strange to restrict initial amplitudes in defining probabilities. However it is not strange at all. In Chap. V, we discuss in concrete examples (ESIV and V) why it is not strange. There we will understand that restriction of initial amplitudes plays an essential role in the interpretability of QP within the familiar measurement theory, that is, the measurement theory for an observation made at a moment of time.

2.4 *Measurement*

From a measurement-theoretical point of view, one may ask (1) the meaning of an ES other than EST and (2) the meaning of QP for a given ES which is judged to be definable by our framework.

(1) As seen from the end of §2.1, the application of our framework begins with giving an ES = $\{\mathcal{O}_j\}$. For example, ESV is specified to be the set of two alternatives "Yes" and "No": "Yes" is to find a particle in a spacetime domain Ω and "No" is not to find it in the domain. At this very first stage, the meaning of "(not) to find a particle in Ω" is not stated. This cannot be helped because the familiar measurement theory of quantum mechanics describes only an instantaneous measurement or a sequence of instantaneous measurements. Once one starts talking about an ES other than EST, one faces the absence of its measurement-theoretical meaning. This is not an obstacle

in carrying out calculations, because an ES is mathematically well-defined. It is defined by the behavior of virtual paths with respect to a suitable domain in spacetime. For example, "(not) to find a particle in Ω" is defined by virtual paths which do (not) cross Ω. This mathematical definition is enough to calculate components (II · 2 · 1) and thus enough to investigate C·1 and C·2. Our framework runs without a measurement theoretical meaning of the ES and judges the definability of QP for the ES.

(2) One may then ask, "What on earth is the measurement theoretical meaning of QP for an ES when the framework concludes the existence of QP?" This is perhaps the kind of question which is not answered by the framework itself. In this thesis, this is not completely answered in general terms. However an important clue to this question is obtained by applying our framework to concrete examples. In Chap. V we conclude that QP can be defined for ESIV and V when an initial amplitude belongs to a specific class. As a remarkable consequence of the restriction of the initial amplitude, values of QP turns out to be closely related to values of constant-time probabilities $|\Psi(X,T)|^2$ in the domain relevant to the ES in question, which is a temporal domain ΔT for ESIV and a spacetime domain Ω for ESV. This fact, when combined with the fact that the measurement theoretical meaning of constant-time probabilities is clear in the familiar measurement theory, enables us to interpret the QP for the ES within the familiar measurement theory. For example, we can give a clear measurement theoretical meaning to "QP to find a particle in Ω" within the familiar measurement theory.

(1)&(2). Meaning of "QP to find a particle in Ω" in turn clarifies the meaning of "to find a particle in Ω" itself. In this way, measurement theoretical meanings of ESIV and V turn out to be clear within the familiar measurement theory. Whether this is deep-rooted or accidental is an interesting problem worthy of further study, which is however beyond the scope of this thesis. Anyway, whether or not QP can be defined for a given ES is judged by our framework making no reference to measurements. In QP-definable cases, as far as we have investigated, physical meanings of QP and that of ES turn out to be clear within the familiar measurement theory. This in turn convinces us that our framework is correctly constructed.

. We have constructed a general framework in this chapter. In the application, our first task is to calculate the sum over paths (II · 2 · 1). The next chapter provides the "Euclidean lattice method" which gives the precise definition of the sum.

Chap. III. Euclidean Lattice Method

§1. Need of new definition

Taking ESI as an example, we explain the need of a new definition of sum over paths. ESI is a set of possible numbers of times and places where a particle intersects $\mathcal{S}$. We shall simplify surface $\mathcal{S}$ to steplike surface $\mathcal{S}_{\mathbf{step}}$ as shown in Fig. 5, as we will do so in Chap. IV. Then in the investigation of C·1 for ESI, we have to calculate a sum over paths of the following type

$$\sum_{B\leftarrow\Delta(\vec{T}_n)\leftarrow A} e^{iS[X(T)]}, \qquad \text{(III}\cdot 1\cdot 1)$$

where the sum is over paths which link A to B and intersect the vertical part of $\mathcal{S}_{\mathbf{step}}$ n times and no more with intersections $\Delta(\vec{T}_n) \equiv \Delta T_1 \times \cdots \times \Delta T_n$ as shown in Fig. 6. Here we should note that the usual sum over paths is of the following form:

$$\sum_{B\leftarrow\Delta\vec{X}_n\leftarrow A} e^{iS[X(T)]}\ , \qquad \text{(III}\cdot 1\cdot 2)$$

where $\Delta\vec{X}_n \equiv \Delta X_1 \times \Delta X_2 \times \cdots \times \Delta X_n$. The sum is over paths which start from A, move forward in time to intersect $\mathcal{S}_{T_1}$ at ΔX_1, $\mathcal{S}_{T_2}$ at $\Delta X_2, \cdots$, and $\mathcal{S}_{T_n}$ at ΔX_n in this order and arrive at B. For an infinite number n of time slicing, sum (III · 1 · 2) is given by the last right-hand side of Eq. (II · 1 · 1) with $\int_{-\infty}^{\infty} dX_j$ replaced by $\int_{\Delta X_j} dX_j$. (Notation: n here is $N-1$ in Eq. (II · 1 · 1).) The set of spatial intervals $\Delta\vec{X}_n$ is called a cylindrical set or quasi-intervals.[7] The case $n = 3$ is shown in Fig. 7. As (III · 1 · 2) shows, in the usual "time slicing" definition of sum over paths, paths which can be summed up are restricted to those paths that are expressible as a cylindrical set. By contrast, paths to be summed up in (III · 1 · 1) are specified by the set of temporal intervals $\Delta(\vec{T}_n)$; this set is not a cylindrical set. Therefore the usual definition cannot calculate (III · 1 · 1). Generally speaking, for an ES other than EST, we must evaluate a sum over paths which are not expressible as a cylindrical set. To get over this difficulty, we need a more flexible definition of sum over paths in which paths that can be summed up are not restricted to those expressible as a cylindrical set. The definition must of course be consistent with the usual definition for a cylindrical set. Such a flexible definition has already been noted and used by Hartle[3] in his investigation of "wave function on $\mathcal{S}$." The definition is based on a random-walk representation of Feynman's path integration. In the next section, we review this definition, partially quoting Hartle's work.

§2. Euclidean Lattice Method

2.1 *Definition*

Consider a spacetime lattice with integer-valued coordinates $[x, t]$ as shown in Fig. 8. We introduce a random-walk on this lattice: A 'particle' moves forward in 'time' t on the lattice. At each point $[x, t]$, the 'particle' can move to $[x+1, t+1]$ with a 'probability' of $1/2$ or to $[x-1, t+1]$ with the same 'probability'. By iterating this step, the 'particle' performs a random-walk, leaving a discrete 'path' (a sequence of points) on the lattice. This random-walk is symmetric in the sense that the 'probability' to move to the right and that to the left take the same value. A priori this random walk has no direct relation to the real particle's motion on the real spacetime (X, T), motion which is described by the propagator Φ. We shall use single quotes ' ' for what are relevant to this random-walk to distinguish them from usages for a real particle. We also use the notation that a point on the lattice and that on the real spacetime are denoted by a lowercase italic and by an uppercase italic letter, respectively, such as $a \equiv [x_a, t_a]$ and $A \equiv (X_A, T_A)$.

Starting from an initial point a, the 'probability' $u[b; a]$ that a 'particle' arrives at a point b ($t_a < t_b$) is calculated by the following discrete sum over 'paths': We first associate each 'path' from a to b with the weight $(1/2)^p$ where p is the number of the steps of the 'path' and is equal to $t_b - t_a$, because 'paths' move forward in 'time'. The 'probability' is then obtained by summing each weight over all possible 'paths'. If the walk is restricted to a certain region by additional boundary conditions, then the sum is restricted accordingly. When there is no such restriction, we have

$$\begin{aligned} u[b; a] &= \sum_{b \leftarrow a} \left(\frac{1}{2}\right)^p \\ &= \frac{1}{2^t} \binom{t}{\frac{x+t}{2}} \end{aligned} \tag{III·2·1}$$

with

$$p = t \equiv t_b - t_a, \; x \equiv x_b - x_a, \tag{III·2·2}$$

where the sum is over all possible 'paths' which move forward in 'time' to connect a and b; the last equality in Eq. (III · 2 · 1) is obtained by counting the number of the 'paths' since p is common to all the paths (see Appendix A). When $x + t$ is odd, the binomial coefficient is defined to be zero in accordance with the fact that no 'path' connects the end points in such a case. This fact we call "odd-even asymmetry".

This discrete sum over 'paths' can be related to the usual sum over paths, namely Feynman's path integral. In particular, the quantum mechanical propagator $\Phi(B;A)$ for a free particle can be derived from the 'probability' $u[b;a]$. Let us set up a Euclidean spacetime (X,τ) corresponding to the real spacetime (X,T) and relate the spacetime lattice $[x,t]$ to the real spacetime via this Euclidean spacetime. We lay the lattice on the Euclidean spacetime with spatial spacing η_1 and temporal spacing η_2 so that

$$x = X/\eta_1, \ t = \tau/\eta_2 . \qquad \text{(III} \cdot 2 \cdot 3)$$

A point $[x,t]$ on the lattice is identified with the point $(\eta_1 x, \eta_2 t)$ on the Euclidean spacetime; conversely (X,τ) is identified with $[X/\eta_1, \tau/\eta_2]$. Let us denote a point on the Euclidean spacetime by a lowercase Greek letter such as $\alpha \equiv (X_\alpha, \tau_\alpha)$. In what follows, a point on the lattice denoted by a lowercase italic is identified with the point on the Euclidean spacetime denoted by the corresponding lowercase Greek. For example, $a = [x_a, t_a]$ is identified with $\alpha = (X_\alpha, T_\alpha)$, where coordinate values are related with each other according to Eq. (III · 2 · 3), so that

$$x_a = X_\alpha/\eta_1, \ t_a = \tau_\alpha/\eta_2 . \qquad \text{(III} \cdot 2 \cdot 4)$$

We write this simply as

$$a = \alpha/\eta . \qquad \text{(III} \cdot 2 \cdot 5)$$

Now let us consider the quantity $u/2\eta_1$. This is a spatial 'probability' density on the Euclidean spacetime, where the factor is not η_1 but $2\eta_1$ to take account of odd-even asymmetry. (We also use single quotes for what are relevant to the Euclidean spacetime.) We introduce the following continuum limit

$$\text{Lim} : \eta_1 \to 0 \text{ keeping } X, \ \tau \text{ and } \eta_2/\eta_1^2 \text{ fixed such that } \eta_2/\eta_1^2 = \text{const} \equiv m . \qquad \text{(III} \cdot 2 \cdot 6)$$

This limit gives (see Appendix A)

$$\text{Lim} \frac{u[b;a]}{2\eta_1} = \Phi_E(\beta;\alpha) \equiv \sqrt{\frac{m}{2\pi\tau}} \exp\left(-\frac{mX^2}{2\tau}\right) \qquad \text{(III} \cdot 2 \cdot 7)$$

with

$$X \equiv X_\beta - X_\alpha, \ \tau \equiv \tau_\beta - \tau_\alpha . \qquad \text{(III} \cdot 2 \cdot 8)$$

In taking the limit, we used Eqs. (III · 2 · 1), (III · 2 · 4), (III · 2 · 6) and Stirling's formula. The quantity Φ_E is the well-known 'probability' density of Brownian motion (on the Euclidean spacetime). (The 'probability' that a 'particle', starting from α and undergoing a Brownian motion with a diffusion constant $1/2m$, arrives in a small interval ΔX around X_β at 'time' τ_β is given by $\Phi_E(\beta;\alpha)\Delta X$. The continuum limit (III · 2 · 6) we call the diffusion limit.) What was shown here is nothing but a constructive method of defining a Brownian motion, starting from a random-walk and using the diffusion limit in which $\eta_2/\eta_1^2 = \text{const}$ (diffusion limit) plays an essential role. Now, by the Wick rotation

$$\tau = iT, (X_\alpha = X_A, X_\beta = X_B), \qquad \text{(III} \cdot 2 \cdot 9)$$

the density Φ_E is converted into the quantum mechanical propagator with the unit $\hbar = 1$:

$$\Phi_E(X_B, iT_B; X_A, iT_A) = \sqrt{\frac{m}{2\pi i T}} \exp\left(i\frac{mX^2}{2T}\right) = \Phi(B;A) \qquad \text{(III} \cdot 2 \cdot 10)$$

with

$$X \equiv X_B - X_A, \ T \equiv T_B - T_A. \qquad \text{(III} \cdot 2 \cdot 11)$$

At this stage m is identified with the mass of the real particle ($m/\hbar$ in the ordinary unit). Having started from a random-walk on the Euclidean spacetime lattice, we have now arrived at the quantum mechanical propagator. In this way, the sum over paths (II · 1 · 1) is the corresponding sum over 'paths' (III · 2 · 1) on the Euclidean lattice combined with the diffusion limit (III · 2 · 6) and the Wick rotation to real time. This is summarized as follows:

$$\sum_{\text{paths}\in W} \exp(iS) = \text{Wick rotation}\left\{\text{Lim} \sum_{\text{'paths'}\in w} \left(\frac{1}{2}\right)^p\right\}. \qquad \text{(III} \cdot 2 \cdot 12)$$

In the case we have just studied, paths and 'paths' are not restricted at all, that is, $W = (T_B, T_A) \times R$ with R being the whole space $X \in (-\infty, \infty)$ and $w = [t_b, t_a] \times r$ with r being the whole space $x \in [-\infty, \infty]$ on the lattice, except that end points are fixed at A and B for paths and at a and b for 'paths'. Equation (III · 2 · 12) also works to derive the path-classification relation (II · 1 · 16) from the corresponding 'path'-classification relation

$$u[b;a] = \sum_{x_c} u[b;c]u[c;a] \quad (t_a < t_c < t_b). \qquad \text{(III} \cdot 2 \cdot 13)$$

(This is obtained by classifying 'paths' from a to b according to the location x_c they cross the 'surface' s_{t_c} of constant 'time' t_c. See Fig. 9.) Let us divide both sides by $2\eta_1$

and take the diffusion limit. Noting Eq. (III · 2 · 7) and that $\sum_{x_c}(2\eta_1)$ becomes $\int dX_\gamma$ in the limit, we have

$$\Phi_E(\beta;\alpha) = \int dX_\gamma\, \Phi_E(\beta;\gamma)\Phi_E(\gamma;\alpha)\,. \qquad (\text{III}\cdot 2\cdot 14)$$

By the Wick rotation, we have

$$\Phi(B;A) = \int dX_C\, \Phi(B;C)\Phi(C;A)\,. \qquad (\text{III}\cdot 2\cdot 15)$$

Integrations over X_γ and X_C range from $-\infty$ to ∞. We should note that the summand and the integrands on the right-hand sides of Eqs. (III · 2 · 13) ~ (III · 2 · 15) obey the scheme (III · 2 · 12) with W and w shrinking to X_C at time T_C and x_c at 'time' t_c, respectively. Here we make the following observation:

(i) The time slicing definition (II · 1 · 1) of a sum over paths results from Eq. (III · 2 · 15).

(ii) Equation (III · 2 · 15) results from Eq. (III · 2 · 12).

From these two, it follows that the sum over paths defined by Feynman's path integral is, at least, *included* in the sum over paths defined by Eq. (III · 2 · 12). Furthermore we note the following fact:

(iii) The right-hand side of (III · 2 · 12) is *well-defined for arbitrary w* because the sum on the side is merely a discrete sum; for instance, we can carry out the discrete sum which corresponds to (III · 1 · 1), as will be shown in the next chapter.

In view of these circumstances, *we define a sum over paths for an arbitrary W by the right-hand-side of Eq.* (III · 2 · 12) *with the rule* (III · 2 · 6). This definition of a sum over paths is named the Euclidean lattice method by Hartle who first proposed this definition. He calculated particular sums over paths by use of this method in his investigation of "wave function on a general surface".

Defining a sum over paths by Eq. (III · 2 · 12) with (III · 2 · 6), we make the right-hand side of Eq. (II · 2 · 1) well-defined. This enables us to apply our framework constructed in Chap. II to concrete examples of ES. In the next subsection, we calculate necessary sums over paths in advance.

2.2 *Application and useful formulae*

We begin with a combinatorial problem. Let us evaluate the following sum over 'paths':

$$f[x,t] \equiv f[0,t;x,0] \equiv \sum_{[0,t] \leftrightarrow [x,0]} \left(\frac{1}{2}\right)^t, \qquad (\mathrm{III}\cdot 2\cdot 16)$$

where the sum is over 'paths' which connect $[x,0]$ and $[0,t]$ and which are restricted to the half space $x \geq 0$ as shown in Fig. 10. (The symbol $\leftrightarrow$ is to stress that 'paths' are restricted to the half space $x \geq 0$.) Note that 'paths' are allowed to touch $x = 0$ on the way. Consider the 'probability' (III · 2 · 1) with $a = [x,0]$ and $b = [0,t]$:

$$u[0,t;x,0] = \sum_{[0,t] \leftarrow [x,0]} \left(\frac{1}{2}\right)^t = \frac{1}{2^t}\binom{t}{\frac{|x|+t}{2}}. \qquad (\mathrm{III}\cdot 2\cdot 17)$$

Here and in what follows, the symbol $|\ \ |$ is to make formulae valid for $x < 0$. The sum (III · 2 · 17) is contributed from the 'paths' invading the region $x < 0$ which is forbidden in (III · 2 · 16). By subtracting this contribution from (III · 2 · 17), we can evaluate the sum (III · 2 · 16). To do this, we introduce a random-walk which connect $[-x-2\mathrm{sgn}x, 0]$ and $[0,t]$. The starting point $[-x-2\mathrm{sgn}x,0]$ is the spatial mirror image of $[x,0]$ with respect to $x = -\mathrm{sgn}x$. The 'probability' for this walk is given by

$$u[0,t;-x-2\mathrm{sgn}x,0] = \sum_{[0,t] \leftarrow [-x-2\mathrm{sgn}x,0]} \left(\frac{1}{2}\right)^t = \frac{1}{2^t}\binom{t}{\frac{|x|+t}{2}+1}. \qquad (\mathrm{III}\cdot 2\cdot 18)$$

As understood from Fig. 11, there is one-to-one correspondence between 'path' to be subtracted from (III · 2 · 17) and a 'path' which contributes to (III · 2 · 18); furthermore they have the same weight $(1/2)^t$. It thus follows that (see Appendix A)

$$f[x,t] = u[0,t;x,0] - u[0,t;-x-2\mathrm{sgn}x,0] = \frac{1}{2^t}\frac{|x|+1}{t+1}\binom{t+1}{\frac{|x|+t}{2}+1}. \qquad (\mathrm{III}\cdot 2\cdot 19)$$

We first apply the Euclidean lattice method to

$$\sum_{(0,T)\leftrightarrow(X,0)} e^{iS}, \qquad (\text{III}\cdot 2\cdot 20)$$

where $X > 0$ for simplicity and the sum is over paths which link $(X, 0)$ to $(0, T)$ and which do not invade the region $X < 0$ until the end time T. The sum over 'paths' corresponding to (III · 2 · 20) is then

$$\begin{aligned} \bar{f}[x,t] &\equiv \bar{f}[0,t;x,0] \\ &\equiv \sum_{[0,t]\leftrightarrow[x,0]} \left(\frac{1}{2}\right)^t, \end{aligned} \qquad (\text{III}\cdot 2\cdot 21)$$

where the sum is over 'paths' which connect $[x, 0]$ and $[0, t]$ never hitting $x = 0$ before 'time' t (see Fig. 12). This is essentially the same as (III · 2 · 16) *except* that 'paths' do not touch $x = 0$ on the way. (The symbol $\leftrightarrow$ here should be read accordingly.) Let us divide the 'paths' into two parts, 'paths' from $[x, 0]$ to $[1, t-1]$ and a 'path' from $[1, t-1]$ to $[0, t]$; the sum (III · 2 · 21) is decomposed accordingly. Since the former 'paths' are of the type which contributes to (III · 2 · 16) with an absorbing barrier placed at $x = 1$ instead of $x = 0$ and the latter is merely one step, we obtain

$$\begin{aligned} \bar{f}[x,t] &= \frac{1}{2} \times f[0,t-1;x-1,0] \\ &= \frac{1}{2^t}\frac{|x|}{t}\binom{t}{\frac{|x|+t}{2}}. \end{aligned} \qquad (\text{III}\cdot 2\cdot 22)$$

We shall give some comments on this, for this is the starting point of all formulae we derive below. In the theory of random walk, it is well known that a 'particle' starting from a point $x \neq 0$ cannot stay only in the half space $x > 0$ or $x < 0$ and thus crosses $x = 0$ without fail. The first 'time' t at which the 'particle' hits $x = 0$ becomes a random variable. (III · 2 · 22) is nothing but the 'probability' for this first hitting time. All the 'paths' whose first hitting of $x = 0$ occurs at 'time' t contribute to the sum (III · 2 · 21) to give (III · 2 · 22). 'Probabilities' (III · 2 · 22) are, as it must be, normalized to

$$\sum_{t=|x|}^{\infty} \bar{f}[x,t] = 1. \qquad (\text{III}\cdot 2\cdot 23)$$

The proof is given in Appendix A. By the way, the difference between "to intersect" and "to hit (i.e., intersect or touch)" is clear on the lattice, as we have just experienced in calculating sums (III · 2 · 16) and (III · 2 · 21). This is one of the advantages which the Euclidean lattice method has. So long as we use Feynman's path integral (in configuration space), it seems difficult to clarify such a difference because of the continuity of paths. Since the difference between ESI and ESII lies in the contribution from paths which touch the surface $\mathcal{S}$, the Euclidean lattice method plays an essential role in defining ESI and II.

Returning to our main concern, we take the diffusion limit of (III · 2 · 22). We write lattice coordinates of (III · 2 · 22) in terms of Euclidean coordinates according to (III · 2 · 3), divide (III · 2 · 22) by $2\eta_2$ and then let $\eta_1 \to 0$ obeying the rule (III · 2 · 6). The result is (see Appendix A)

$$\text{Lim}\frac{\bar{f}[X/\eta_1, \tau/\eta_2]}{2\eta_2} = F(X,\tau) \equiv \left(\frac{mX^2}{2\pi\tau^3}\right)^{1/2} \exp\left(-\frac{mX^2}{2\tau}\right) . \qquad \text{(III} \cdot 2 \cdot 24)$$

By Wick-rotating this result according to (III · 2 · 9), we complete the calculation of (III · 2 · 20) to have

$$\begin{aligned} \sum_{(0,T)\leftrightarrow(X,0)} e^{iS} &= F(X,iT) \\ &= \left[\frac{mX^2}{2\pi(iT)^3}\right]^{1/2} \exp\left(i\frac{mX^2}{2T}\right) . \end{aligned} \qquad \text{(III} \cdot 2 \cdot 25)$$

Next we apply the Euclidean lattice method to

$$\sum_{\substack{B\leftarrow(0,T)\leftrightarrow A \\ T\in\Delta T}} e^{iS} , \qquad \text{(III} \cdot 2 \cdot 26)$$

where the sum is over paths which link A to B and whose first hitting time of $X = 0$ lies in ΔT. For simplicity we put $A = (X_A, 0)$ and $X_A > 0$. The corresponding sum over 'paths' is

$$\sum_{\substack{b\leftarrow[0,t]\leftrightarrow a \\ t\in\Delta t}} \left(\frac{1}{2}\right)^{t_b} , \qquad \text{(III} \cdot 2 \cdot 27)$$

where the sum is over 'paths' which link $a = (x_a, 0)$ $(x_a > 0)$ to b and whose first hitting 'time' of $x = 0$ lies in Δt. Defining $c \equiv (0, t)$ for $t \in \Delta t$, we divide the 'paths' into two parts, 'paths' from a to c and 'paths' from c to b. The former 'paths' are restricted to the half space $x > 0$ and the latter are not restricted at all. The sum (III · 2 · 27) can be decomposed accordingly, giving

$$\sum_{\substack{b \leftarrow [0,t] \leftrightarrow a \\ t \in \Delta t}} \left(\frac{1}{2}\right)^{t_b} = \sum_{t \in \Delta t} \left\{ \sum_{b \leftarrow c} \left(\frac{1}{2}\right)^{t_b - t} \right\} \times \left\{ \sum_{c \leftrightarrow a} \left(\frac{1}{2}\right)^{t} \right\}$$
$$= \sum_{t \in \Delta t} u[b;c] \bar{f}[c;a] . \qquad \text{(III} \cdot 2 \cdot 28)$$

The sum over paths (III · 2 · 26) is obtained from (III · 2 · 28) by taking the diffusion limit and Wick rotating to real time. We divide the last right-hand side of (III · 2 · 28) by $2\eta_1$, rewrite lattice coordinates in terms of Euclidean coordinates and take the limit (III · 2 · 6). Noting $\bar{f}[c;a] = \bar{f}[x_a, t]$ and using Eqs. (III · 2 · 7) and (III · 2 · 24), we have

$$\text{Lim} \frac{1}{2\eta_1} \sum_{t \in \Delta t} u[b;c] \bar{f}[c;a] = \int_{\Delta \tau} d\tau \Phi_E(\beta; 0, \tau) F(X_\alpha, \tau), \qquad \text{(III} \cdot 2 \cdot 29)$$

where Φ_E and F are defined by Eqs. (III · 2 · 7) and (III · 2 · 24), respectively. This is Wick rotated ($\tau = iT, X_\alpha = X_A, X_\beta = X_B$) to give

$$\sum_{\substack{B \leftarrow (0,T) \leftrightarrow (X_A, 0) \\ T \in \Delta T}} e^{iS} = \int_{\Delta T} d(iT) \Phi(B; 0, T) F(X_A, iT). \qquad \text{(III} \cdot 2 \cdot 30)$$

Although we have assumed $X_A > 0$, this is also correct for $X_A < 0$. This is physically because we are dealing with a free particle and is mathematically because the sum over 'paths' from a to c appearing on the first right-hand side of Eq. (III · 2 · 28) remains unchanged if we take $x_a < 0$ and restrict 'paths' in the region $x < 0$. When ΔT is equal to the time difference between the end points, the integral on the right-hand side of Eq. (III · 2 · 30) can be carried out analytically by use of the formula

$$\int_0^{T'} dT (T' - T)^{-1/2} T^{-3/2} \exp\left(-\frac{C_1}{T' - T} - \frac{C_2}{T} \right)$$
$$= \left(\frac{\pi}{C_2 T'} \right)^{-1/2} \exp\left(-\frac{(C_1^{1/2} + C_2^{1/2})^2}{T'} \right), \qquad \text{(III} \cdot 2 \cdot 31)$$

giving

$$\sum_{\substack{B \leftarrow (0,T) \leftrightarrow (X_A, 0) \\ 0 < T < T_B}} e^{iS} = \int_0^{T_B} d(iT) \Phi(B; 0, T) F(X_A, iT) \qquad \text{(III} \cdot 2 \cdot 32)$$

$$= \begin{cases} \Phi(B; X_A, 0) & \text{for } X_B X_A < 0, \qquad \text{(III} \cdot 2 \cdot 33) \\ \Phi(B; -X_A, 0) & \text{for } X_B X_A > 0. \qquad \text{(III} \cdot 2 \cdot 34) \end{cases}$$

The meaning of this formula is clear. When $X_B X_A < 0$, every paths contributing to (III · 2 · 30) hits the spatial origin $X = 0$ at least once. Thus the sum (III · 2 · 30) with $\Delta T = T_B$ is over *all* the paths which connect A and B and therefore gives the propagator connecting the end points; this explains Eq. (III · 2 · 33). When $X_B X_A > 0$, some paths hit the origin but others do not. In this case, note that the sum (III · 2 · 30) remains unchanged by the replacement $X_A \to -X_A$ because of $F(-X_A, iT) = F(X_A, iT)$, which follows from (III · 2 · 25). Thus the paths to be summed over in (III · 2 · 30) can be replaced by the paths which connect $(-X_A, 0)$ and B, which are on the opposite sides of $X = 0$. Since the replaced paths hit the origin at least once, the sum (III · 2 · 30) with $\Delta T = T_B$ gives the propagator which connect $(-X_A, 0)$ and B; this explains Eq. (III · 2 · 34).

Lastly we show that

$$\sum_{B \nleftrightarrow (X_A,0)} e^{iS} = \Phi(B; X_A, 0) - \Phi(B; -X_A, 0) \quad \text{for } X_B X_A > 0\,, \qquad \text{(III} \cdot 2 \cdot 35\text{)}$$

where the sum is over paths which connect $A = (X_A, 0)$ and B and which never hit the origin $X = 0$ on the way. This can, of course, be proved from the corresponding sum over 'paths'. However we can shorten the proof by using Eqs. (III · 2 · 33) and (III · 2 · 34). Since A and B are on the same side of the origin, some paths hit it and others do not. The contribution from paths which hit the origin at least once is given by (III · 2 · 34). Subtracting this from the contribution from all the paths, we have Eq. (III · 2 · 35).

§3. Extension to the case of a nonzero potential

3.1 *Generality*

The role of the Euclidean lattice method is that it gives a precise definition to the sum over paths on the right hand side of (II · 2 · 1), thereby making the general framework applicable to concrete examples of ES, or equivalently, making the question (I · 1 · 8) well-posed. We have reviewed the method and applied it to a free particle. Although this is essentially enough for investigations in later chapters, whether the method is extendible or not to the case of a nonzero potential is a fundamental question. This question must be answered affirmatively to make sure that the question (I · 1 · 8) itself is well-posed even in the presence of a potential, which is one of our basic standpoints. In the following we show that the method is extendible to such a case, clarifying how the scheme (III · 2 · 12) have to be modified when a potential exists. This issue has not been discussed by Hartle.

The Euclidean lattice method summarized as (III · 2 · 12) is made up of two elements:
(i) The diffusion limit which converts a random-walk 'probability' into a 'probability' density of Brownian motion.
(ii) The Wick rotation which converts the 'probability' density into the quantum mechanical amplitude of interest.
The first element (i) is a constructive definition of Brownian motion from random walk. The second element (ii) is understood as the correspondence between the Schrödinger equation for a free particle and the diffusion equation. In other words, for a free particle, the Euclidean lattice method finds its ground in the fact that the free Schrödinger equation Wick rotated to imaginary time describes a continuous stochastic process (Brownian motion) and the continuous process is constructed from a discrete stochastic process (random walk) in the diffusion limit (III · 2 · 6). In the presence of a nonzero potential, the Wick rotated Schrödinger equation does not describe a stochastic process any more as discussed below. In extending the Euclidean lattice method, we wish to keep the point that a sum over paths is constructed from a sum over 'paths' of random walk. Hence we shall proceed as follows: First we relate the Schrödinger equation having a potential term to a Fokker-Planck equation describing a continuous stochastic process. Next we construct the continuous process from a discrete stochastic process, which turns out to be a non-symmetric random walk, that is, a random-walk in which 'probability' to move to the right and that to the left are not the same.

Consider the Schrödinger equation

$$i\frac{\partial\Psi(X,T)}{\partial T} = -\frac{1}{2}\frac{\partial^2\Psi(X,T)}{\partial X^2} + V(X)\Psi(X,T), \qquad \text{(III} \cdot 3 \cdot 1)$$

where we have considered a time-independent potential $V(X)$. Wick rotating this equation according to $\tau = iT$, we have

$$\frac{\partial\Psi_E(X,\tau)}{\partial\tau} = \frac{1}{2}\frac{\partial^2\Psi_E(X,\tau)}{\partial X^2} - V(X)\Psi_E(X,\tau), \qquad \text{(III} \cdot 3 \cdot 2)$$

where $\Psi_E(X,\tau) \equiv \Psi(X,-i\tau)$. This does not describe a stochastic process in the sense that Ψ_E cannot be a 'probability' density because

$$\frac{\partial}{\partial\tau}\int dX\Psi_E(X,\tau) \neq 0 \qquad \text{for } V(X) \neq 0. \qquad \text{(III} \cdot 3 \cdot 3)$$

However Ψ_E is decomposable into a 'probability' density and a function defined by the potential as demonstrated below. We introduce a function $U(X)$ and a constant E and write

$$\Psi_E(X,\tau) = \exp\left(U(X) - E\tau\right)\rho(X,\tau). \qquad \text{(III}\cdot 3\cdot 4)$$

Substituting this into Eq. (III · 3 · 2), we have

$$\begin{aligned}\frac{\partial \rho}{\partial \tau} &= \frac{\rho}{2}\left(\frac{dg}{dX} + g^2 - 2(V - E)\right) \\ &\quad - \frac{\partial}{\partial X}\left(g - \frac{1}{2}\frac{\partial}{\partial X}\right)\rho,\end{aligned} \qquad \text{(III}\cdot 3\cdot 5)$$

where

$$g \equiv g(X) \equiv -\frac{dU(X)}{dX}. \qquad \text{(III}\cdot 3\cdot 6)$$

Let us choose g so that it satisfies

$$\frac{dg}{dX} + g^2 - 2(V - E) = 0. \qquad \text{(III}\cdot 3\cdot 7)$$

Then Eq. (III · 3 · 5) becomes a Fokker-Planck equation

$$\frac{\partial \rho}{\partial \tau} = -\frac{\partial}{\partial X}\left(g - \frac{1}{2}\frac{\partial}{\partial X}\right)\rho. \qquad \text{(III}\cdot 3\cdot 8)$$

Function $\rho(X,\tau)$ satisfies the conservation law

$$\frac{\partial}{\partial \tau}\int \rho dX = 0. \qquad \text{(III}\cdot 3\cdot 9)$$

We can interpret ρ as a 'probability' density of a continuous stochastic process described by (III · 3 · 8). Let us consider the Kernel for the time evolution of ρ, which we shall write $\rho(\beta;\alpha) \equiv \rho(X_\beta, \tau_\beta; X_\alpha, \tau_\alpha)$ $(\tau_\beta > \tau_\alpha)$. This kernel is the solution to the Fokker-Planck equation (III · 3 · 8) satisfying the initial condition

$$\lim_{\tau_\beta \to \tau_\alpha} \rho(\beta;\alpha) = \delta(X_\beta - X_\alpha). \qquad \text{(III}\cdot 3\cdot 10)$$

Define

$$\Phi_E(\beta;\alpha) \equiv \exp\left(-\int_{X_\alpha}^{X_\beta} g(X)dX - E(\tau_\beta - \tau_\alpha)\right)\rho(\beta;\alpha). \qquad \text{(III}\cdot 3\cdot 11)$$

Since $g(X)$ has been chosen to be a solution to (III · 3 · 7), function $\Phi_E(\beta;\alpha)$ thus defined is the solution to Eq. (III · 3 · 2) satisfying the initial condition

$$\lim_{\tau_\beta \to \tau_\alpha} \Phi_E(\beta;\alpha) = \delta(X_\beta - X_\alpha). \qquad \text{(III} \cdot 3 \cdot 12)$$

That is, $\Phi_E(\beta;\alpha)$ is the Euclidean propagator. Wick rotating $\Phi_E(\beta;\alpha)$ back to real time according to $\tau = iT$, we recover the quantum mechanical propagator $\Phi(B;A)$ which is the solution to the Schrödinger equation (III · 3 · 1) satisfying the initial condition

$$\lim_{T_B \to T_A} \Phi(B;A) = \delta(X_B - X_A). \qquad \text{(III} \cdot 3 \cdot 13)$$

Let us summarize the discussion so far. Given a time-independent potential $V(X)$, we choose the function $g(X)$ as a solution to Riccati's differential equation (III · 3 · 7). Then the propagator $\Phi(B;A)$ in the presence of a potential $V(X)$ can be constructed from a probability density $\rho(\beta;\alpha)$ of a continuous stochastic process described by the Fokker-Planck equation (III · 3 · 8).

Comments: Up to now, the constant E is arbitrary so long as it is finite (this is necessary to make the transformation (III · 3 · 4) well defined). Given a constant E, the solution to Eq. (III · 3 · 7) is not unique. Any solution is acceptable at this stage so long as $U(X)$ is finite. The succeeding discussion uniquely determines the value of E and the function $g(X)$.

We turn to find such a discrete stochastic process whose diffusion limit gives the continuous stochastic process described by Eq. (III · 3 · 8). Consider a random walk on the Euclidean lattice $[x,t]$ whose walk to the left and that to the right are associated with site-dependent probabilities $l[x]$ and $r[x]$, respectively, such that

$$l[x] + r[x] = 1,\ \ 0 < l[x],\ \ 0 < r[x]. \qquad \text{(III} \cdot 3 \cdot 14)$$

(The case $l[x] = r[x] = 1/2$ was already studied.) Let $u[b;a]$ be the 'probability' that a 'particle' starting from $a = [x_a, t_a]$ arrives at $[x_b, t_b]$. 'Probability' $u[b;a]$ obeys the difference equation

$$u[b;a] = r[x_b - 1]u[x_b - 1, t_b - 1; a] + l[x_b + 1]u[x_b + 1, t_b - 1; a] \qquad \text{(III} \cdot 3 \cdot 15)$$

and satisfies the initial condition

$$u[a;a] = 1. \qquad \text{(III} \cdot 3 \cdot 16)$$

Solving this difference equation step by step, we obtain the sum-over-'paths' expression for u

$$u[b;a] = \sum_{b \leftarrow a} \mu(\text{'path'}), \tag{III·3·17}$$

where μ('path') is the weight associated with each 'path' connecting a and b; it is a product of l's and r's. For example, consider a 'path' which starts from a and walk to the left $[x_a - 1, t_a + 1]$, next to the right $[x_a, t_a + 2]$, $\cdots$, and finally comes into b from its left $[x_b - 1, t_b - 1]$; the weight associated with this 'path' is given by $\mu = r[x_b - 1] \cdots r[x_a - 1] l[x_a]$. The right-hand side of (III · 3 · 17) is the sum of weights over all the 'paths' connecting the end points. Let us organize this random walk in such a way that it reproduces the continuous stochastic process described by (III · 3 · 8) in the diffusion limit (III · 2 · 6). This is accomplished by identifying the diffusion limit of the difference equation (III · 3 · 15) with the Fokker-Planck equation (III · 3 · 8). Rewriting the difference equation in terms of continuous coordinates X and τ according to (III · 2 · 3), we have

$$\begin{aligned} u[\beta/\eta; \alpha/\eta] =& r[X_\beta/\eta_1 - 1] u[X_\beta/\eta_1 - 1, \tau_\beta/\eta_2 - 1; \alpha/\eta] \\ &+ l[X_\beta/\eta_1 + 1] u[X_\beta/\eta_1 + 1, \tau_\beta/\eta_2 - 1; \alpha/\eta], \end{aligned} \tag{III·3·18}$$

where the notation (III · 2 · 5) is used. Divide both sides by $2\eta_1$ and take the diffusion limit according to the rule (III · 2 · 6). Here we assume that the following limits exist:

$$\rho(\beta;\alpha) \equiv \text{Lim} \frac{1}{2\eta_1} u[\beta/\eta; \alpha/\eta], \tag{III·3·19}$$

$$R(X) \equiv \lim_{\eta_1 \to 0} \frac{r[X/\eta_1] - 1/2}{\eta_1}, \quad L(X) \equiv \lim_{\eta_1 \to 0} \frac{l[X/\eta_1] - 1/2}{\eta_1}. \tag{III·3·20}$$

Quantities on the right-hand sides of (III · 3 · 19) and (III · 3 · 20) are, before limits are taken, spatial 'probability' densities on the lattice; what have been assumed is that these densities become quantities which are fully written in terms of continuous coordinates of the Euclidean spacetime in the limit so that they can be interpreted as 'probability' densities on the Euclidean spacetime. This was the case when there was no potential; we expect that it is also the case even in the presence of a potential. The limit on the right-hand side of (III · 3 · 19) will later be identified with $\rho(\beta;\alpha)$ which has already been introduced, so that we have denoted the limit accordingly. Assuming the existence of these limits, we have

$$\rho(\beta;\alpha) = \mathrm{Lim}\left[\{\eta_1 R(X_\beta-\eta_1)+\frac{1}{2}\}\rho(X_\beta-\eta_1,T_\beta-\eta_2;\alpha) \right. \\ \left. +\{\eta_1 L(X_\beta+\eta_1)+\frac{1}{2}\}\rho(X_\beta+\eta_1,T_\beta-\eta_2;\alpha)\right]. \qquad (\mathrm{III}\cdot 3\cdot 21)$$

Expanding the right-hand side to first order in η_2 and to second order in η_1, we obtain

$$\frac{\partial\rho}{\partial\tau} = -\frac{\partial}{\partial X}\left((R-L)-\frac{1}{2}\frac{\partial}{\partial X}\right)\rho, \qquad (\mathrm{III}\cdot 3\cdot 22)$$

where $\rho=\rho(X,\tau;\alpha)$, $R=R(X)$, $L=L(X)$ and we used Eq. (III · 3 · 14) and $\eta_2/\eta_1^2=1$ (the convention $m=1$ has been used in this section). Identifying Eq. (III · 3 · 22) with Eq. (III · 3 · 8), we have $R-L=g$. From this, with $R+L=0$, we obtain

$$\lim_{\eta_1\to 0}\frac{r[X/\eta_1]-1/2}{\eta_1/2} = g(X), \\ \lim_{\eta_1\to 0}\frac{l[X/\eta_1]-1/2}{\eta_1/2} = -g(X). \qquad (\mathrm{III}\cdot 3\cdot 23)$$

In terms of lattice coordinate x, this reads for an infinitesimal η_1,

$$r[x]=\frac{1}{2}(1+\eta_1 g(\eta_1 x)),\quad l[x]=\frac{1}{2}(1-\eta_1 g(\eta_1 x)). \qquad (\mathrm{III}\cdot 3\cdot 24)$$

In order for this to be consistent with $0\le r[x],\ l[x]\le 1$ for an infinitesimal η_1, function $g(X)$ must be bounded, that is, there exists a positive quantity M such that

$$|g(X)|<M \quad \text{for } {}^\forall X. \qquad (\mathrm{III}\cdot 3\cdot 25)$$

Otherwise (III · 3 · 24) fails no matter how small we choose η_1. In what follows, we argue that the condition (III · 3 · 25) uniquely determines the constant E and the function $g(X)$.

Let us rewrite $g(X)$ into the following form:

$$g(X)=\frac{1}{h(X)}\frac{dh(X)}{dX}, \qquad \text{(III}\cdot3\cdot26)$$

where $h(X)$ is non-vanishing over the entire interval because of (III · 3 · 25), namely,

$$h(X)\neq 0 \qquad \text{for } {}^{\forall}X. \qquad \text{(III}\cdot3\cdot27)$$

Substituting (III · 3 · 26) into Eq. (III · 3 · 7), we obtain

$$-\frac{1}{2}\frac{d^2h(X)}{dX^2}+V(X)h(X)=Eh(X). \qquad \text{(III}\cdot3\cdot28)$$

This is precisely the eigenvalue equation obtained from the time dependent Schrödinger equation (III · 3 · 1). The existence of eigenfunctions $h(X)$ and their properties are familiar issues in quantum mechanics. We assume that this eigen value equation has bound state solutions. This is the case if the minimum value of the potential $V_{\mathbf{min}}$ satisfies the condition $V_{\mathbf{min}} < \mathrm{Min}(V_+, V_-)$, where V_+ and V_- are the respective limits of the potential when X tends towards ∞ and $-\infty$. If the potential does not satisfy this condition, we place infinitely high potential barriers at $X = \pm\infty$ and reconsider that the original potential is the finite-region part of the modified potential, thereby making the condition satisfied. This is not a bad modification since, in a realistic situation, a particle is usually found in a finite region. Considering in this way, we can discuss bound state solutions of Eq. (III · 3 · 28) for an arbitrary potential. Bound state solutions exist for discrete eigen values E; let us arrange these values in the increasing order such that $E_0 < E_1 < \cdots$. Since we are considering 1-dimensional case, the eigen value spectrum is non-degenerate. Let $h_n(X)$ be the nth eigen function having eigenvalue E_n $(n = 0,\ 1,\ \cdots)$. What is important for us is the following fact which we shall call "node theorem".
Node theorem: The nth eigenfunction $h_n(X)$ has n nodes.
This guarantees that there is one and only one solution which satisfies the condition (III · 3 · 27); it is the ground state eigenfunction. The condition (III · 3 · 27) has now uniquely determined the function $h(X)$ and the constant E. In this way, the constant E and the function $g(X)$ which define the transformation (III · 3 · 4) (with (III · 3 · 6)) are uniquely determined by the condition (III · 3 · 25) to be

$$E=E_G,\quad g(X)=\frac{1}{h_G(X)}\frac{dh_G(X)}{dX}=\frac{d\log h_G(X)}{dX}, \qquad \text{(III}\cdot3\cdot29)$$

where $h_G(X)\equiv h_0(X)$ is the ground state eigenfunction of Eq. (III · 3 · 28) and $E_G \equiv E_0$ is the ground state energy.

We have now completed constructing a discrete stochastic process (a random walk) whose diffusion limit gives the continuous stochastic process described by Eq. (III · 3 · 8): Given a potential $V(X)$, solve the eigenvalue equation (III · 3 · 28) to obtain the ground state eigenfunction $h_G(X)$. Define a function $g(X)$ from $h_G(X)$ according to Eq. (III · 3 · 29). The random walk in question is then uniquely constructed on the lattice $[x,t]$ according to the rule (III · 3 · 24) and Eq. (III · 3 · 15).

Here it would be in order to rewrite several equations which include $g(X)$ and $U(X)$ in terms of $h_G(X)$. The basic relationship is, from Eqs. (III · 3 · 6) and (III · 3 · 29),

$$g(X) = \frac{d\log h_G(X)}{dX} = -\frac{dU(X)}{dX}, \qquad \text{(III} \cdot 3 \cdot 30)$$

or equivalently,

$$h_G(X) = \text{const.}\exp\left(-U(X)\right) = \exp\left(\int^X g(X')dX'\right). \qquad \text{(III} \cdot 3 \cdot 31)$$

The 'probability' of random walk in one step is given by

$$r[x] = \frac{1}{2}\left(1 + \frac{d\log h_G(\eta_1 x)}{dx}\right), \quad l[x] = \frac{1}{2}\left(1 - \frac{d\log h_G(\eta_1 x)}{dx}\right). \qquad \text{(III} \cdot 3 \cdot 32)$$

Since the ground state is non-degenerate, $h_G(X)$ is a real function within a phase; $r[x]$ and $l[x]$ are certainly real. Let us define a two-point function $G_E(\beta;\alpha)$:

$$G_E(\beta;\alpha) \equiv \frac{h_G(X_\alpha)e^{E_G\tau_\alpha}}{h_G(X_\beta)e^{E_G\tau_\beta}}. \qquad \text{(III} \cdot 3 \cdot 33)$$

The Euclidean propagator (III · 3 · 11) is then written as

$$\Phi_E(\beta;\alpha) = G_E(\beta;\alpha)\rho(\beta;\alpha). \qquad \text{(III} \cdot 3 \cdot 34)$$

By the Wick rotation which puts $\tau_\beta = iT_B$, $\tau_\alpha = iT_A$, $X_\beta = X_B$ and $X_\alpha = X_A$, we have

$$\Phi(B;A) = G(B;A)\rho(B;A), \qquad \text{(III} \cdot 3 \cdot 35)$$

where

$$G(B;A) \equiv G_E(X_B, iT_B; X_A, iT_A) = \frac{\psi(A)}{\psi(B)} \qquad \text{(III} \cdot 3 \cdot 36)$$

with

$$\psi(X,T) \equiv h_G(X)e^{iE_GT}, \qquad \text{(III} \cdot 3 \cdot 37)$$

which is the Schrödinger's wave function for the ground state. Functions G_E and G

have the following properties:

$$G_E(\beta;\alpha) = G_E(\beta;\gamma)G_E(\gamma;\alpha);$$
$$G(B;A) = G(B;C)G(C;A). \qquad (\mathrm{III}\cdot 3\cdot 38)$$

Summarizing all the discussions so far, we have now acquired a constructive method of defining and calculating the quantum mechanical propagator $\Phi(B;A)$ starting from a sum over 'paths' of a random walk.

(i) Given a potential $V(X)$, we first solve Eq. (III · 3 · 28) for its ground state to obtain $h_G(X)$ and E_G. We then go down to the Euclidean lattice $[x,t]$. To a rightward walk from $[x,t]$ to $[x+1,t+1]$, we assign the weight $r[x]$ defined by Eq. (III · 3 · 32). The weight $l[x]$ is assigned to the leftward walk from $[x,t]$ to $[x-1,t]$. A weight of a 'path', which we denote by μ('path'), is defined to be the product of r's and l's multiplied along the 'path'. Given end points a and b, we sum up μ('path') over all the 'paths' which connect the end points. We denote the result by $u[b;a]$.

(ii) We take the diffusion limit of $u[b;a]$ (divided by $2\eta_1$) according to (III · 2 · 6) and write the result as $\rho(\beta;\alpha)$ as indicated by (III · 3 · 19).

(iii) We Wick rotate $\rho(\beta;\alpha)$.

(iv) We multiply the result by the two point function $G(B;A)$ which is constructed from the ground state wave function as Eq. (III · 3 · 36) with (III · 3 · 37). The result is the propagator $\Phi(B;A)$ which has the sum-over-paths expression (II · 1 · 1).

The above construction is summarized as follows:

$$\sum_{\text{paths}\in W} \exp(iS) = G(B;A) \times \left[\text{Wick rotation}\left\{\text{Lim} \sum_{\text{'paths'}\in w} \mu(\text{'path'})\right\}\right], \qquad (\mathrm{III}\cdot 3\cdot 39)$$

where A and B are the end points of W; W and w are, in the construction of the propagator, the same as those explained after (III · 2 · 12). The procedure of dividing the sum over 'paths' by $2\eta_1$ has not been written out in (III · 3 · 39). This is because we will later apply this scheme to a sum over 'paths' which is divided not by $2\eta_1$ but by $2\eta_2$. Here the dividing procedure is to be understood as being included in the symbol Lim.

It was shown in §2 that, for a free particle, the composition law (III · 2 · 15) can be derived from the composition law (III · 2 · 13) on the lattice according to the scheme

(III · 2 · 12). In the presence of a potential, same thing can be said if the old scheme (III · 2 · 12) is replaced by the new one (III · 3 · 39), which we shall now demonstrate. By classifying the 'paths' contributing to $u[b;a]$ with respect to the crossing of the 'surface' of 'time' t_c, we have the composition law (III · 2 · 13). (Since 'paths' are discrete, this classification is always possible and therefore we can always write down the composition law on the lattice as a well-defined equation.) Dividing both sides of (III · 2 · 13) by $2\eta_1$ and taking the diffusion limit according to (III · 2 · 6), we once have

$$\rho(\beta;\alpha) = \int dX_\gamma \rho(\beta;\gamma)\rho(\gamma;\alpha). \qquad \text{(III} \cdot 3 \cdot 40)$$

According to the scheme (III · 3 · 39), both sides of the above is Wick rotated and then multiplied by the factor $G(B;A)$. Noting the property (III · 3 · 38), we have

$$G(B;A)\rho(B;A) = \int dX_C G(B;C)\rho(B;C)G(C;A)\rho(C;A). \qquad \text{(III} \cdot 3 \cdot 41)$$

Because of (III · 3 · 35), this is the real-time composition law (III · 2 · 15).

Since Feynman's path integral (in configuration space) is obtained from the composition law (III · 2 · 15), we can say that the path integral is included in the scheme (III · 3 · 39). Moreover the sum over 'paths' on the right-hand side of (III · 3 · 39) is merely a discrete sum and is therefore always well-defined. Even a sum over paths for a non-cylindrical set, for which Feynman's path integral is not applicable, can be defined by this scheme. Hence we adopt the scheme (III · 3 · 39) as a flexible definition of a sum over paths. In this way, *the Euclidean lattice method is newly defined to be* (III · 3 · 39) *when there is a time-independent potential.* Therefore, even in the presence of such a potential, the sum over paths on the right-hand side of (II · 2 · 1) is well-defined. This makes sure that the our question (I · 1 · 8) is well-posed even in such cases.

3.2 *Application*

We apply the scheme (III · 3 · 39) to two sums over paths. One is the sum over paths (III · 2 · 25) for the "first hitting amplitude" F and the other is the "half-space sum over paths" (III · 2 · 35) in the presence of a potential $V(X)$.

(I) The first hitting amplitude.
The evaluation of the sum over paths on the left-hand side of (III · 2 · 25) also begins with calculating the corresponding sum over 'paths' (III · 2 · 21) in which the weight $(1/2)^t$ is replaced by μ('path'). We then take the diffusion limit. Assuming the existence of

the limit, we write the result as $\tilde{F}(X,\tau)$.

$$\bar{f}[x,t] \equiv \sum_{[0,t]\leftrightarrow[x,0]} \mu(\text{`path'}),$$
$$\tilde{F}(X,\tau) \equiv \mathrm{Lim}\frac{\bar{f}[X/\eta_1,\tau/\eta_2]}{2\eta_2}. \qquad (\mathrm{III}\cdot 3\cdot 42)$$

The sum for $\bar{f}$ is over 'paths' which connect $[x,0]$ and $[0,t]$ and which are restricted to the right half-space $x>0$ when x in $[x,0]$ is positive or to the left half-space $x<0$ when x in $[x,0]$ is negative; this restriction is expressed by the symbol $\leftrightarrow$. According to the scheme (III · 3 · 39), we Wick rotate $\tilde{F}(X,\tau)$ and then multiply by the factor $G(0,T;X,0)$, thereby having

$$\sum_{(0,T)\leftrightarrow(X,0)} e^{iS} = G(0,T;X,0)\tilde{F}(X,iT). \qquad (\mathrm{III}\cdot 3\cdot 43)$$

For later use we define

$$F(X,\tau) \equiv G_E(0,\tau;X,0)\tilde{F}(X,\tau). \qquad (\mathrm{III}\cdot 3\cdot 44)$$

It follows that

$$\sum_{(0,T)\leftrightarrow(X,0)} e^{iS} = F(X,iT) = G(0,T;X,0)\tilde{F}(X,iT). \qquad (\mathrm{III}\cdot 3\cdot 45)$$

Although we have completely defined the first hitting amplitude $F(X,iT)$, the above definition itself is not suitable for the actual calculation of F. For that purpose, an equation is useful which F satisfies. In fact, (III · 2 · 33) is such an equation. It also holds even in the presence of a potential $V(X)$. We shall first prove this and then solve it formally.

Let us note that Eq. (III · 2 · 28) also holds, provided that weights of the form $(1/2)^t$ are replaced by μ('path'). (This is because Eq. (III · 2 · 28) expresses a manifestly possible 'path' classification.) Consider the special case $\Delta t = t_b - t_a$. The sum on the right-hand side of (III · 2 · 28) then exhausts all the 'paths' from a to b to give $u[b;a]$. Therefore the equation becomes

$$u[b;a] = \sum_t u[b;c]\bar{f}[x_a,t] \quad \text{for } x_a x_b < 0,\ c=[0,t], \qquad (\mathrm{III}\cdot 3\cdot 46)$$

where u and $\bar{f}$ are of course defined by (III · 3 · 17) and by (III · 3 · 42), respectively.

Dividing both sides by $2\eta_1$ and taking the diffusion limit, we have

$$\rho(\beta;\alpha) = \int_0^{T_\beta} d\tau \rho(\beta;0,\tau)\tilde{F}(X_\alpha,\tau), \qquad \text{(III}\cdot 3\cdot 47)$$

where $X_\beta X_\alpha < 0$ and we have used Eqs. (III · 3 · 19) and (III · 3 · 42). Wick rotating and then multiplying by the factor $G(B;A)$, we obtain Eq. (III · 2 · 33), where we have used Eqs. (III · 3 · 35) and (III · 3 · 45).

Let us solve (III · 2 · 33). Since we are considering a time-independent potential, the propagator has the translational invariance in time, so that we shall write $\Phi(B;A)$ as $\Phi(T_B - T_A|X_B, X_A)$, displaying its time dependence explicitly. Equation (III · 2 · 33) then becomes

$$\Phi(T_B|X_B,X_A) = \int_0^{T_B} d(iT)\Phi(T_B - T|X_B,0)F(T|X_A) \qquad \text{for } X_B X_A < 0, \ \text{(III}\cdot 3\cdot 48)$$

where $F(T|X) \equiv F(X,iT)$. This is a Volterra integral equation of the first kind for the unknown function F. According to the theory of integral equation, the solution F is given by

$$iF(T|X_A) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds \frac{\phi(s|X_B,X_A)}{\phi(s|X_B,0)} e^{Ts}, \qquad \text{(III}\cdot 3\cdot 49)$$

where ϕ is the Laplace transform of Φ, namely,

$$\phi(s|X_B,X_A) \equiv \int_0^{\infty} dT e^{-sT}\Phi(T|X_B,X_A). \qquad \text{(III}\cdot 3\cdot 50)$$

On the right-hand side of (III · 3 · 49), the integration contour is an infinite vertical line in the complex s-plane and the constant γ is chosen so that all the singularities of the integrand are on the left-hand side of the contour. Formula (III · 3 · 49) gives the formal expression for the first hitting amplitude F. If we use

$$\Phi(T|X_B,X_A) = < X_B|\exp(-iHT)|X_A >, \qquad \text{(III}\cdot 3\cdot 51)$$

where H is the Hamiltonian, we can perform the time integral in (III · 3 · 50):

$$\begin{aligned}\phi(s|X_B,X_A) &= \int_0^{\infty} dT < X_B|\exp(-(s+iH)T)|X_A > \\ &= < X_B|\frac{1}{s+iH}|X_A > \\ &= \sum_n \frac{\langle X_B|n\rangle\langle n|X_A\rangle}{s+iE_n},\end{aligned} \qquad \text{(III}\cdot 3\cdot 52)$$

where $|n>$ is an eigenstate of the Hamiltonian H with an eigenvalue E_n, namely,

$$H|n>= E_n|n> . \qquad (\mathrm{III}\cdot 3\cdot 53)$$

This shows that the first hitting amplitude F can be defined when the series on the last right-hand side of (III · 3 · 52) converges. Let us assume that the potential is such that makes the series converge. (The series converges for a free particle (see Appendix B). Moreover, if $|\Psi_1>$ and $|\Psi_2>$ are states of unit norm, then $| < \Psi_2|e^{-iHT}|\Psi_1 > | \leq 1$ and the Laplace transform of $< \Psi_2|e^{-iHT}|\Psi_1 >$ certainly exists. Hence our assumption would not be very restrictive.) Solving the eigenvalue equation (III · 3 · 53) and substituting (III · 3 · 52) into (III · 3 · 49), we obtain the formal expression for F in terms of eigenvalues and eigen functions of the Hamiltonian. To explore more about F with concrete examples of $V(X)$ is itself an interesting issue. However we do not need an analytic expression for F in later investigations; what we need there is the only fact that F satisfies the integral equation (III · 3 · 48). So we shall not be engaged in calculating F for concrete V's. For a free particle, the integral equation can be solved explicitly and the result agrees with (III · 2 · 24). This will be shown in Appendix B.

Here, for later use, we shall also extend Eq. (III · 2 · 34) to the case of a nonzero potential. Equation (III · 2 · 34) does not hold in the presence of a general potential. When there is no potential, there is the symmetry $F(-X, iT) = F(X, iT)$ because a path from $(X_A, 0)$ to $(0, T)$ and its mirror-reflected path from $(-X_A, 0)$ to $(0, T)$, where T is the time of first hitting of $X = 0$, are associated with the same amplitude. This symmetry explains why Eq. (III · 2 · 34) holds for the same F that satisfies Eq. (III · 2 · 33), as already discussed in §2.2. A general potential breaks this symmetry and Eq. (III · 2 · 34) fails. However, this symmetry is retained by a symmetric potential $V(-X, T) = V(X, T)$, so that it is reasonable to expect that F satisfies Eq. (III · 2 · 34) in the presence of such a potential. Confining ourselves again to the time-independent case, we shall prove this in the context of the Euclidean lattice method. Let us consider a symmetric and time-independent potential:

$$V(-X) = V(X). \qquad (\mathrm{III}\cdot 3\cdot 54)$$

Consider the quantity on the right-hand side of Eq. (III · 3 · 46). First we shall prove

$$\bar{f}[x_a, t] = \bar{f}[-x_a, t]. \qquad (\mathrm{III}\cdot 3\cdot 55)$$

Recall that $\bar{f}$ is defined in Eq. (III · 3 · 42). Consider a 'path' contributing to the sum defining $\bar{f}[x_a, t]$ and its mirror image with respect to $x = 0$, which contributes to $\bar{f}[-x_a, t]$. Let us pick up a rightward walk linking $[x, t]$ to $[x+1, t+1]$ from the former 'path' and consider its mirror-reflected process from $[-x, t]$ to $[-x-1, t+1]$, which is a leftward walk picked up from the latter 'path'. The weights of the rightward and the leftward walk are respectively $r[x]$ and $l[-x]$, where r and l are defined by Eq. (III · 3 · 32). Note that $h_G(X)$ is even since our potential is even. It then follows from Eq. (III · 3 · 32) that

$$r[x] = l[-x]. \qquad \text{(III} \cdot 3 \cdot 56\text{)}$$

Since this holds for $^\forall x$, the former and the latter path have the same weight and therefore Eq. (III · 3 · 55) holds. Now we write Eq. (III · 3 · 46) with x_a replaced by $-x_a$ and then use Eq. (III · 3 · 55) to have

$$u[b; -x_a, 0] = \sum_t u[b; c]\bar{f}[x_a, t], \quad \text{for } x_a x_b > 0,\ c = [0, t]. \qquad \text{(III} \cdot 3 \cdot 57\text{)}$$

Dividing both sides by $2\eta_1$ and taking the diffusion limit, we have

$$\rho(\beta; -X_\alpha, 0) = \int_0^{T_\beta} d\tau \rho(\beta; 0, \tau)\tilde{F}(X_\alpha, \tau). \qquad \text{(III} \cdot 3 \cdot 58\text{)}$$

Wick rotating and multiplying by the factor $G(B; -X_A, 0)$, we once have

$$\Phi(B; -X_A, 0) = \int_0^{T_B} d(iT)\Phi(B; 0, iT)G(0, T; -X_A, 0)\tilde{F}(X_A, iT), \qquad \text{(III} \cdot 3 \cdot 59\text{)}$$

where we used the property (III · 3 · 38) on the right-hand side. Recall that $h_G(X)$ is even. It then follows from Eqs. (III · 3 · 36) and (III · 3 · 37) that

$$G(0, T; -X_A, 0) = G(0, T; X_A, 0). \qquad \text{(III} \cdot 3 \cdot 60\text{)}$$

Substituting this into Eq. (III · 3 · 59) and using (III · 3 · 45), we obtain Eq. (III · 2 · 34). We have now derived the following formulae in the scheme of the Euclidean lattice method:

$$\sum_{\substack{B \leftarrow (0,T) \leftrightarrow (X_A, 0) \\ 0<T<T_B}} e^{iS}$$

$$= \int_0^{T_B} d(iT)\Phi(B; 0, T)F(X_A, iT) \qquad \text{(III} \cdot 3 \cdot 61\text{)}$$

$$= \begin{cases} \Phi(B; X_A, 0) & \text{for } X_B X_A < 0,\ ^\forall V(X) \qquad \text{(III} \cdot 3 \cdot 62\text{)} \\ \Phi(B; -X_A, 0) & \text{for } X_B X_A > 0,\ V(-X) = V(X). \qquad \text{(III} \cdot 3 \cdot 63\text{)} \end{cases}$$

What has been proved is that F which is defined by Eq. (III · 3 · 45) with (III · 3 · 36) and (III · 3 · 42) satisfies Eqs. (III · 3 · 62) and (III · 3 · 63). This has been proved within the framework of the Euclidean lattice method (III · 3 · 39). By the way, without the knowledge of the Euclidean lattice method, Eqs. (III · 3 · 62) and (III · 3 · 63) could have been obtained from the very meaning of F as the first hitting amplitude. That is, each of these equations would naturally arise when we classify paths with respect to the time at which each path firstly hits $X = 0$, provided that such a classification of paths in the real spacetime makes sense. (Note that the Euclidean lattice method gives a rigorous sense to such a classification.) In this sense we could regard Eqs. (III · 3 · 62) and (III · 3 · 63) as being free from the Euclidean lattice method. Furthermore, once Eqs. (III · 3 · 62) and (III · 3 · 63) have been accepted, we can employ these equations as the definition of F. Therefore, the first hitting amplitude F can be obtained without the knowledge of the Euclidean lattice method. This method of integral equation is extendible to the case of a time-dependent potential $V(X,T)$ to which (III · 3 · 39) is not applicable. Even if such a potential exist, there is no difficulty in considering the path classification with respect to the first hitting time. Thus we can write down

$$\sum_{\substack{B\leftarrow(0,T)\nleftrightarrow(X_A,0)\\ 0<T<T_B}} e^{iS}$$

$$= \int_0^{T_B} d(iT)\Phi(B;0,T)F(X_A,iT) \qquad \text{(III} \cdot 3 \cdot 64)$$

$$= \begin{cases} \Phi(B;X_A,0) & \text{for } X_BX_A < 0,\ {}^{\forall}V(X,T) \qquad \text{(III} \cdot 3 \cdot 65) \\ \Phi(B;-X_A,0) & \text{for } X_BX_A > 0,\ \ V(-X,T) = V(X,T). \qquad \text{(III} \cdot 3 \cdot 66) \end{cases}$$

Whether these equations have solutions or not may depend on the behavior of $V(X,T)$. When they have solutions, we can define the first hitting amplitude F. At the time of this writing, the author does not have a definite idea of extending the Euclidean lattice method to the case of a time-dependent potential $V(X,T)$. (It will be discussed later that elaborating the Euclidean lattice method in the presence of such a potential would not be fruitful.) Whenever F is used in the presence of a time-dependent potential in later investigations, it is to be understood as being defined by integral equations (III · 3 · 65) and (III · 3 · 66); it is of course assumed there that the potential belongs to such a class that guarantees these integral equations to have solutions.

(II)The half-space sum over paths.

Equation (III · 2 · 35) is also extendible to the case of a symmetric potential. The second term $\Phi(B;-X_A,0)$ on the right-hand side is the contribution from the paths which invade the forbidden region $X < 0$ or $X > 0$ on the way from $A = (X_A,0)$ to B which is on the same side of A with respect to $X = 0$. The reason why this way of taking account of the contribution from the forbidden paths goes well is because that a path invading the forbidden region does not change its amplitude e^{iS} when it is mirror-reflected before its first hitting of $X = 0$. This is also the case when the potential has the symmetry $V(-X,T) = V(X,T)$. Thus Eq. (III · 2 · 35) should also hold in the presence of such a potential:

$$\sum_{B \leftrightarrow (X_A,0)} e^{iS} = \Phi(B; X_A, 0) - \Phi(B; -X_A, 0) \quad \text{for } X_B X_A > 0,\ V(-X,T) = V(X,T), \tag{III · 3 · 67}$$

where the sum of e^{iS} is over paths which are restricted to the right half-space $X > 0$ when $X_A, X_B > 0$ or to the left half-space $X < 0$ when $X_A, X_B < 0$; this restriction is expressed by the symbol $\leftrightarrow$. When the potential is time-independent, Eq. (III · 3 · 67) can be proved in the scheme of the Euclidean lattice method as follows. Consider the sum over 'paths',

$$\sum_{b \leftrightarrow [x_a,0]} \mu(\text{'path'}) \quad \text{for } x_b x_a > 0, \tag{III · 3 · 68}$$

where we take x_a, $x_b > 0$ without loss of generality; the sum is over 'paths' which are restricted to the right half-space $x > 0$. This is evaluated as follows:

$$\sum_{b \leftrightarrow [x_a,0]} \mu(\text{'path'}) = u[b;a] - \sum_t u[b;c]\bar{f}[x_a,t], \ (c \equiv [0,t]) \tag{III · 3 · 69}$$

where the first term is contributed from all the 'paths' connecting a and b and the second term represents the contribution from the 'paths' invading the forbidden region $x \le 0$. The summand in the second term is the contribution from the 'paths' whose first hitting of $x = 0$ occurs at 'time' t. If $V(-X) = V(X)$, then Eq. (III · 3 · 57) holds and we substitute it into Eq. (III · 3 · 69) to obtain

$$\sum_{b \leftrightarrow [x_a,0]} \mu(\text{'path'}) = u[b;a] - u[b; -x_a, 0] \quad \text{for } x_a x_b > 0. \tag{III · 3 · 70}$$

It is now straightforward to convert this into Eq. (III · 3 · 67) according to (III · 3 · 39).

§4. Discussion

The Euclidean lattice method has been reviewed and extended as a flexible definition of a sum over paths. Here we shall examine the results obtained by use of the method and discuss whether they can be obtained by use of Feynman's path integral in configuration space.

Let us begin with Eq. (III · 3 · 67). Its physical meaning is clear; the right-hand side is the propagator on a half space $X > 0$ (or $X < 0$). That is, if an amplitude $\Psi_{\mathbf{half}}(A) \equiv \Psi_{\mathbf{half}}(X_A, T_A)$ is given at an initial time $T_A = 0$ so that $\Psi_{\mathbf{half}}(0,0) = 0$ and that it is normalized on a half space to (we consider the half space $(0, \infty)$)

$$\int_0^\infty dX_A |\Psi_{\mathbf{half}}(A)|^2 = 1\,, \qquad \text{(III} \cdot 4 \cdot 1)$$

then the "half-space wave function" defined at a later time T_B by

$$\Psi_{\mathbf{half}}(B) \equiv \int_0^\infty dX_A \Big(\Phi(B; X_A, 0) - \Phi(B; -X_A, 0) \Big) \Psi_{\mathbf{half}}(A) \qquad \text{(III} \cdot 4 \cdot 2)$$

obeys the Schrödinger equation, vanishes at $X_B = 0$ and is normalized to

$$\int_0^\infty dX_B |\Psi_{\mathbf{half}}(B)|^2 = 1\,. \qquad \text{(III} \cdot 4 \cdot 3)$$

Comparing these with Eqs. (II · 1 · 9) and (II · 1 · 2), we understand that the right-hand side of Eq. (III · 3 · 67) is the propagator for a particle confined in a half space. As shown in the previous section, the Euclidean lattice method calculates the half-space propagator by "summing up e^{iS} over the paths which are restricted to the half space". (The precise meaning of the phrase in " " is of course defined by the scheme (III · 3 · 39).) This method already calculated the entire-space propagator Φ when paths ('paths') were not restricted at all. Thus (III · 3 · 67) is the second example in which the Euclidean lattice method reproduces the correct propagator by restricting paths to the region where the propagator is defined. Can similar things be said in Feynman's path integral? It is also the case in Feynman's path integral that the entire-space propagator is calculated when the sum is over all the paths, that is, when the integration ranges are $(-\infty, \infty)$ as the last right-hand side of Eq. (II · 1 · 1). However it is not clear whether the half-space propagator is obtained just by restricting integration ranges to the half space, namely,

$$\lim_{\epsilon\to 0}\prod_{j=1}^{N-1}\int_0^\infty dX_j(2\pi i\epsilon)^{-N/2}\exp\left[i\sum_{k=0}^{N-1}\left\{\frac{(X_{k+1}-X_k)^2}{2\epsilon}+\epsilon V(\frac{X_{k+1}+X_k}{2})\right\}\right]$$
$$\stackrel{?}{=}\Phi(B;A)-\Phi(B;-X_A,T_A), \tag{III·4·4}$$

where $X_0\equiv X_A$, $X_N\equiv X_B$. The usual way of obtaining the half-space propagator in Feynman's path integral is to use the method of images. Given a particle defined on the half-space $(0,\infty)$ with a potential $V(X)$ $(X>0)$, we first extend the potential to the unphysical region $X<0$ so that $V(-X)=V(X)$. Taking end points A and B in the half space, we calculate the left-hand side of (III · 4 · 4) with integration ranges replaced by $\prod_j\int_{-\infty}^{\infty}dX_j$ including the unphysical region $(-\infty,0)$. The result is the propagator $\Phi(B;A)$; this is the solution to the Schrödinger equation defined on the entire space $(-\infty,\infty)$ in the presence of the symmetric potential and it satisfies the initial condition (III · 3 · 13). We next take end points $(-X_A,T_A)$ and B and perform the same path integral. The result is $\Phi(B;-X_A,T_A)$ which is also defined on the entire space and which obeys the same Schrödinger equation and satisfies the initial condition $\lim_{T_B\to T_A}\Phi(B;-X_A,T_A)=\delta(X_B+X_A)$. The half-space propagator is defined to be the solution to the Schrödinger equation which satisfies the initial condition (III · 3 · 13) for X_B, $X_A>0$ and satisfies the boundary condition that it vanishes at $X=0$. From $\Phi(B;A)$ and $\Phi(B;-X_A,T_A)$, such a solution can be constructed by superposing them so that the result satisfies the boundary condition. The result is the right-hand side of (III · 3 · 67) $(T_A=0)$. The half-space propagator is thus constructed in Feynman's path integral. It must be stressed that this construction by the method of images has not calculated the path integral on the left-hand side of (III · 4 · 4). There are several discussions as to whether Eq. (III · 4 · 4) is correct or not.[10] The present author has also explored this issue but has not obtained a definite result. Although one may naively write down the left-hand side of (III · 4 · 4) as giving the half-space propagator, justification of it is not easy or, possibly, it is not correct. In this way, in Feynman's path integral, the correspondence is not necessarily clear between the range of the sum and the range where the resultant propagator is defined. By contrast, the correspondence is very clear in the Euclidean lattice method as shown already. This is a merit of the method.

However the real reason we employ the Euclidean lattice method is the existence of those sums over paths to which Feynman's path integral seems not to be applicable but to which the Euclidean lattice method is applicable. They are sums over paths which are not expressible as a cylindrical set as argued in §1. An example is the sum over paths for the first hitting amplitude F. As already discussed, F could naturally be introduced in quantum mechanics by considering a path classification with respect to the first hitting time; F could then be defined as the solution to the integral equation (III · 3 · 48); this definition is free from the Euclidean lattice method. The important thing is that the Euclidean lattice method correctly reproduces F without difficulty, while Feynman's path integral may not because the paths over which the sum is taken are not expressible as a cylindrical set. Another example of a sum over paths for non-cylindrical set is (III · 1 · 1). Differently from the first hitting amplitude which could be defined without the Euclidean lattice method, it seems difficult to define (III · 1 · 1) unless we use the method. (The author thinks that Feynman's path integral is inapplicable.)

In this way, the Euclidean lattice method is applicable to various kinds of sums over paths, some of which are undefinable in the naive context of Feynman's path integral.

There seems to be no fundamental difficulty in generalizing the discussion in §3.1 to an isolated many-particle system in higher dimensions interacting with each other. At the time of this writing, however, the author has not explored this issue yet. It is a remaining problem. One might think it a more urgent problem to generalize the Euclidean lattice method to the case of a time-dependent potential. However a problem of n particles in such a potential should properly be regarded as a reduced problem of an isolated system of more than n particles. Thus it is included in the problem already mentioned. The generalization to more general cases of a system consisting of particles and fields is, of course, a problem. It deserves to be studied separately from the theme of this thesis and we shall not discuss it here.

Chap. IV. Application of General Framework to Concrete Examples (I)

Probability-undefinable cases

Throughout this chapter we deal with a particle in (1+1)-dimensional Newtonian spacetime. To be specific we work with

ESI : the set of possible numbers of times and possible locations a particle *intersects* a (hyper) surface in the spacetime,

ESII : the set of possible numbers of times and possible locations a particle is *found* on a surface,

ESIII : the set of possible locations a particle is *first found* on a surface.

A suitable potential is allowed for ESIII.

§1. ESI

We prove that C·1 does not hold for ESI, thereby concluding that QP cannot be defined.

Suppose that a particle intersects a surface $\mathcal{S}$. Let us divide $\mathcal{S}$ into a countable set of non-overlapping domains $\{\Delta(l) \mid l = 0, \pm 1, \pm 2, \cdots\}$ such that $\mathcal{S} = \bigcup_l \Delta(l)$ (see Fig. 13). The number and the places for the particle to intersect $\mathcal{S}$ can be specified by a subset of these domains. Thus, an alternative of ESI is specified by $\Delta(\vec{l}_n)$; n is a (odd) number of times a particle intersects $\mathcal{S}$ and

$$\Delta(\vec{l}_n) \equiv \Delta(l_1) \times \Delta(l_2) \times \cdots \times \Delta(l_n) \quad (l_1 < \cdots < l_n) \tag{IV · 1 · 1}$$

denotes n-places of intersection. ESI is expressed as

$$\text{ESI} = \{\Delta(\vec{l}_n)\}. \tag{IV · 1 · 2}$$

We follow the steps which we showed at the end of §2.1 of Chap. II. The component of propagator to be associated with an alternative $\Delta(\vec{l}_n)$ is defined and denoted by

$$\Phi(B; \Delta(\vec{l}_n); A) \equiv \int_{\Delta(\vec{l}_n)} d\vec{\lambda}\, \Phi_n(B; \vec{\lambda}; A) \equiv \sum_{B \leftarrow \Delta(\vec{l}_n) \leftarrow A} e^{iS[X(T)]}, \tag{IV · 1 · 3}$$

where λ is a coordinate on $\mathcal{S}$ and

$$\vec{\lambda} \equiv (\lambda_1, \cdots, \lambda_n) \text{ such that } \lambda_1 < \cdots < \lambda_n, \quad \int_{\Delta(\vec{l}_n)} d\vec{\lambda} \equiv \prod_{j=1}^{n} \int_{\Delta(l_j)} d\lambda_j\,. \tag{IV · 1 · 4}$$

The sum on the right-hand side of (IV · 1 · 3) is over paths which start from A, move forward in time to intersect $\mathcal{S}$ at n domains $\{\Delta(l_j) \mid j = 1, \cdots, n\}$ and arrive at B (cf. Fig. 1). The classifiability condition (II · 2 · 2) takes the following form for ESI:

$$\begin{aligned}\Phi(B;A) &= \sum_{n=1}^{\infty}{}' \sum_{\vec{l}_n} \Phi(B;\Delta(\vec{l}_n);A) \\ &= \sum_{n=1}^{\infty}{}' \int d\vec{\lambda}\, \Phi_n(B;\vec{\lambda};A),\end{aligned} \qquad \text{(IV} \cdot 1 \cdot 5\text{)}$$

where $\sum'_n$ stands for the sum over all positive odd integers.

Here we make a simplification: As a surface $\mathcal{S}$, we consider a steplike surface $\mathcal{S}_{\mathbf{step}}$ as shown in Fig. 14. This simplification makes analytic investigations easy. The essence of the problem is not lost by this simplification, since there is no geometrical relationship between Newtonian space and Newtonian time. As the coordinate λ on $\mathcal{S}_{\mathbf{step}}$ we adopt

$$\lambda = \begin{cases} X_C & (X_C < 0, T = T_C) \\ T & (X = 0, T_C < T < T_D) \\ X_D & (X_D > 0, T = T_D)\,, \end{cases} \qquad \text{(IV} \cdot 1 \cdot 6\text{)}$$

where X_C and X_D are the space coordinates on $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$, respectively. With this coordinate, domains on $\mathcal{S}_{\mathbf{step}}$ are given by

$$\Delta(l) = \begin{cases} \Delta X_C(l) & \text{on } \mathcal{S}_{T_C} \\ \Delta T(l) & \text{on } \mathcal{S}| \\ \Delta X_D(l) & \text{on } \mathcal{S}_{T_D}\,, \end{cases} \qquad \text{(IV} \cdot 1 \cdot 7\text{)}$$

where $\mathcal{S}|$ is the vertical (i.e., $X = 0$) part of $\mathcal{S}_{\mathbf{step}}$. (For instance, l runs from $-\infty$ to -101 on $\mathcal{S}_{T_C}$, from -100 to 100 on $\mathcal{S}|$ and from 101 to ∞ on $\mathcal{S}_{T_D}$. The surfaces $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ are of course restricted to $X_C < 0$ and $X_D > 0$, respectively.)

The question of whether Eq. (IV · 1 · 5) holds or not is the question of whether paths connecting A and B are classifiable or not according to how many times (n) and at what locations ($\vec{\lambda}$) they intersect the steplike surface. Since paths move forward in time to intersect $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ once and only once, we can concentrate on whether such classification is possible or not when paths intersect $\mathcal{S}|$ on the way from a point $C \equiv (X_C, T_C)$ on $\mathcal{S}_{T_C}$ to a point $D \equiv (X_D, T_D)$ on $\mathcal{S}_{T_D}$. The path-classification between $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ falls into the following two types according as C and D are on the same or the opposite side of $X = 0$.

$$\Phi(D;C) = \sum_n{}' \int d\vec{T}\,\tilde{\Phi}_n(D;\vec{T};C) \qquad \text{for } X_C < 0 < X_D\,, \qquad (\text{IV}\cdot 1\cdot 8)$$

$$\Phi(D;C) = \sum_n{}'' \int d\vec{T}\,\tilde{\Phi}_n(D;\vec{T};C) \qquad \text{for } X_C\,, X_D < 0\,, \qquad (\text{IV}\cdot 1\cdot 9)$$

where $\sum''_n$ stands for the sum over all positive even integers (including zero); $\Phi_n(B;\vec{T};A)$ is defined by

$$\int_{\Delta(\vec{T}_n)} d\vec{T}\,\Phi_n(B;\vec{T};A) \equiv \sum_{B\leftarrow\Delta(\vec{T}_n)\leftarrow A} e^{iS[X(T)]}, \qquad (\text{IV}\cdot 1\cdot 10)$$

where the sum is over paths which intersect $\mathcal{S}|$ at $T_1 < \cdots < T_n$ on the way from A to B. If $\{\tilde{\Phi}_n\}$ "exist to satisfy" Eqs. (IV · 1 · 8) and (IV · 1 · 9), then one can construct $\{\Phi_n\}$ which satisfy Eq. (IV · 1 · 5). Conversely, if either of them fails to hold, then it is concluded that C. 1 does not hold for ESI on $\mathcal{S}_{\text{step}}$. Here we should comment on the phrase "exist to satisfy". The amplitudes $\{\tilde{\Phi}_n\}$ exist in the sense that they are defined by sums over paths (IV · 1 · 10). However, the precise definition of the sums is given by the Euclidean lattice method. It is apriori unknown whether the amplitudes thus defined satisfy Eq. (IV · 1 · 8) and (IV · 1 · 9).

In what follows, we prove that Eq. (IV · 1 · 8) does not hold, thereby concluding that QP cannot be defined for ESI. For simplicity we take the symbol ˜ off $\tilde{\Phi}_n$ and replace C and D in it by A and B, respectively. We call Eq. (IV · 1 · 8) with these changes C. 1 for ESI on $\mathcal{S}|$. In the context of the Euclidean lattice method, C. 1 for ESI on $\mathcal{S}|$ is to be derived in the following way:

$$u[b;a] = \sum_{n=1}^{N}{}' \sum_{\vec{t}} u_n[b;\vec{t};a]\,, \qquad (\text{IV}\cdot 1\cdot 11)$$

$$\Big\downarrow \text{Lim}$$

$$\Phi_E(\beta;\alpha) = \sum_{n=1}^{\infty}{}' \int d\vec{\tau}\,\Phi_{E_n}(\beta;\vec{\tau};\alpha)\,, \qquad (\text{IV}\cdot 1\cdot 12)$$

$\downarrow$Wick rotation

$$\Phi(B;A) = \sum_{n=1}^{\infty}{}' \int d\vec{T}\,\Phi_n(B;\vec{T};A)\,, \tag{IV·1·13}$$

where

$$\begin{aligned} &x_b < 0 < x_a,\ X_\beta < 0 < X_\alpha,\ X_B < 0 < X_A \\ &\sum_{\vec{t}} \equiv \sum_{t_a < t_1, \cdots, t_n < t_b},\ \int d\vec{\tau} \equiv \prod_{j=1}^{n} \int_{\tau_\alpha}^{\tau_{j+1}} d\tau_j \quad (\tau_{n+1} \equiv \tau_\beta) \\ &\int d\vec{T} \equiv \prod_{j=1}^{n} \int_{T_A}^{T_{j+1}} dT_j \quad (T_{n+1} \equiv T_B). \end{aligned} \tag{IV·1·14}$$

Let $s|$ be the spatial origin $x = 0$ on the lattice. The quantity $u_n[b;\vec{t};a]$ is the 'probability' that a 'particle' starting from a intersects $s|$ at 'times' $t_1, \cdots, t_n$ in this order and arrives at b (we denote by N the maximum number of intersection). This 'probability' is given by the following sum over 'paths':

$$u_n[b;\vec{t};a] = \sum_{b \leftarrow \vec{t} \leftarrow a} (\frac{1}{2})^{t_b - t_a}\,, \tag{IV·1·15}$$

where the sum is over 'paths' which intersect $s|$ at $t_1 < \cdots < t_n$ on the way from a to b. It is certain that there holds Eq. (IV·1·11) with (IV·1·15). This is because (i) the 'paths' defining $u[b;a]$ ($x_a < 0 < x_b$) are uniquely classifiable according to how many times (n) and at what locations ($\vec{t} = t_1 < \cdots < t_n$) they intersect $s|$ and (ii) every 'path' has the same weight $(1/2)^{t_b - t_a}$. This 'path'-classification is expressed as Fig. 15 in which the diagram on the left-hand side is the abbreviation of the 'paths' defining $u[b;a]$; the zigzag diagram on the right-hand side is the abbreviation of all the 'paths' which intersect $s|$ at $t_1 < \cdots < t_n$ on the way from a to b. Note that 'paths' are allowed to touch $s|$ between one intersection and the next intersection. Observing Eqs. (IV·1·11) $\sim$ (IV·1·13), we note that

(1) The transformation $u \to \Phi_E \to \Phi$ on the left-hand sides has no problem because of Eqs. (III·2·7) and (III·2·10).

(2) The mathematical relationship between Φ_{E_n} and Φ_n is unique on a suitable assumption on the analytic property of Φ_{E_n} (see Appendix C):

$$\Phi_n(X_B, T_B; T_n, \cdots, T_1; X_A, T_A) = i^n \Phi_{E_n}(X_B, iT_B; iT_n, \cdots, iT_1; X_A, iT_A)\,. \tag{IV·1·16}$$

Therefore *we have only to investigate whether the diffusion limit converts the right-hand side of Eq.* (IV·1·11) *into the form of the right-hand side of Eq.* (IV·1·12).

Now let us calculate the sum over 'paths' (IV · 1 · 15). This sum is broken down into partial sums and then recomposed as follows: The sum is over 'paths' which are contained in the zigzag diagram on the right-hand side of Fig. 15. The zigzag diagram can be decomposed into sub-diagrams of the type of Fig. 10 and that of Fig. 16. The sum over 'path' (IV · 1 · 15) is then given by a product of partial sums over 'paths', each of which is over 'paths' that are contained in a sub-diagram. There are two types of partial sum. One is over 'paths' of the type of Fig. 10 and we have already evaluated this type of sum; the sum over 'paths' contained in Fig. 10 is given by Eq. (III · 2 · 19). The other type of partial sum is the special case of the former type; the sum over 'paths' contained in Fig. 16 is given by putting $x = 0$ in Eq. (III · 2 · 19), namely by

$$g[t] \equiv f[0,t] = \frac{1}{2^t}\frac{1}{t+1}\binom{t+1}{\frac{t}{2}}. \qquad \text{(IV}\cdot 1\cdot 17)$$

Therefore we obtain

$$u_n[b;\vec{t};a] = f[x_b, t_b - t_n]g[t_n - t_{n-1}]\cdots g[t_2 - t_1]f[x_a, t_1 - t_a]. \qquad \text{(IV}\cdot 1\cdot 18)$$

In the course of this 'path'-decomposition, we also obtain the precise expressions for the multiple time summation in Eq. (IV · 1 · 11) and for the maximum number of intersection N:

$$\sum_{\vec{t}} \equiv \prod_{j=1}^{n} \sum_{t_j = t_a + |x_a| + 2(j-1)}^{t_{j+1}-2}, \qquad (t_{n+1} \equiv t_b - x_b + 2) \qquad \text{(IV}\cdot 1\cdot 19)$$

where each sum $\sum_{t_j}$ is taken either over even integers or over odd integers because of the odd-even asymmetry, and

$$N = \frac{(t_b - x_b) - (t_a + |x_a|)}{2} + 1. \qquad \text{(IV}\cdot 1\cdot 20)$$

(One can confirm that Eq. (IV · 1 · 11) is satisfied by (III · 2 · 1) and (IV · 1 · 18) ~ (IV · 1 · 20); Eq. (IV · 1 · 11) becomes an identity concerning binomial coefficients. The confirmation is however not easy. We have numerically confirmed it. Analytical confirmation is left as a problem of enumerative combinatorics.)

Now we examine the diffusion limit of Eq. (IV · 1 · 11) substituted with (IV · 1 · 18). Dividing both sides of Eq. (IV · 1 · 11) by $2\eta_1$ and writing lattice coordinates a and b

in terms of the corresponding Euclidean coordinates α and β, we first have, with the notation (III · 2 · 5),

$$\mathrm{Lim}\frac{u[\beta/\eta;\alpha/\eta]}{2\eta_1}=\mathrm{Lim}\sum_{n=1}^{N}{}'\sum_{\vec{t}}\frac{u_n[\beta/\eta;\vec{t};\alpha/\eta]}{2\eta_1}. \qquad \text{(IV} \cdot 1 \cdot 21)$$

The maximum number of intersection $N=(\tau_\beta-\tau_\alpha-m\eta_1(X_\beta+|X_\alpha|))/(2\eta_2)+1$ (cf. Eq. (IV · 1 · 20)) becomes infinite in the limit. The left-hand side of Eq. (IV · 1 · 21) is $\Phi_E(\beta;\alpha)$. A necessary condition for Eq. (IV · 1 · 21) to turn into the form of Eq. (IV · 1 · 12) is that the right-hand side of Eq. (IV · 1 · 21) can be transformed into the following double limit:

$$\lim_{N_0\to\infty}\sum_{n=1}^{N_0}{}'\,\mathrm{Lim}\sum_{\vec{t}}(2\eta_2)^n\frac{u_n}{(2\eta_2)^n 2\eta_1}, \qquad \text{(IV} \cdot 1 \cdot 22)$$

where N_0 is an odd integer independent of η_1. This is because that the sum over n in Eq. (IV · 1 · 13) and hence in Eq. (IV · 1 · 12) is understood to be defined by $\lim_{N_0\to\infty}\sum'^{N_0}_{n=1}$ according to the mathematical definition of a sum of an infinite series. If the transformation is possible, then the value of the double limit (IV · 1 · 22) must coincide with $\Phi_E(\beta;\alpha)$. Let us evaluate the double limit explicitly. We change the variables for the multiple 'time' summation defined by Eq. (IV · 1 · 19) from t_j to $\tau_j(\equiv\eta_2 t_j)$ $(j=1,2,\cdots,n)$. The ranges of the summation in terms of τ_j are

$$\tau_\alpha+m\eta_1|X_\alpha|+2\eta_2(j-1)\le\tau_j\le\tau_{j+1}-2\eta_2, \qquad (j=1,2,\cdots,n) \qquad \text{(IV} \cdot 1 \cdot 23)$$

where $\tau_{n+1}\equiv\tau_\beta-m\eta_1X_\beta+2\eta_2$. In the diffusion limit, these ranges tend to

$$\tau_\alpha<\tau_1<\tau_2<\cdots<\tau_n<\tau_\beta\,, \qquad \text{(IV} \cdot 1 \cdot 24)$$

and accordingly

$$\sum_{\vec{t}}(2\eta_2)^n\to\int d\vec{\tau}\,. \qquad \text{(IV} \cdot 1 \cdot 25)$$

Finally, the double limit (IV · 1 · 22) can be written as

$$\lim_{N_0\to\infty}\sum_{n=1}^{N_0}{}'\int d\vec{\tau}\;\mathrm{Lim}\frac{u_n[\beta/\eta;\tau_n/\eta_2,\cdots,\tau_1/\eta_2;\alpha/\eta]}{(2\eta_2)^n 2\eta_1}\,. \qquad \text{(IV} \cdot 1 \cdot 26)$$

If this is to recover $\Phi_E(\beta;\alpha)$, then we must identify

$$\Phi_{E_n}(\beta;\vec{\tau};\alpha) = \mathrm{Lim}\frac{u_n[\beta/\eta;\tau_n/\eta_2,\cdots,\tau_1/\eta_2;\alpha/\eta]}{(2\eta_2)^n 2\eta_1}\,. \qquad (\mathrm{IV}\cdot 1\cdot 27)$$

Substituting Eq. (IV · 1 · 18) into Eq. (IV · 1 · 26) and using the following formulae,

$$\mathrm{Lim}\frac{f[X/\eta_1,\tau/\eta_2]}{4\eta_2} = F(X,\tau) \equiv |X|\left(\frac{m}{2\pi\tau^3}\right)^{1/2}\exp\left(-\frac{mX^2}{2\tau}\right), \qquad (\mathrm{IV}\cdot 1\cdot 28)$$

$$\mathrm{Lim}\frac{g[\tau/\eta_2]}{4\eta_2\eta_1} = G(\tau) \equiv \left(\frac{m}{2\pi\tau^3}\right)^{1/2}, \qquad (\mathrm{IV}\cdot 1\cdot 29)$$

where F is given by Eq. (III · 2 · 24), we find

$$\begin{aligned}&\mathrm{Lim}\frac{u_n}{(2\eta_2)^n 2\eta_1}\\ &= \lim_{\eta_1\to 0}\eta_1^n 2^{n+1} m F(X_\beta,\tau_\beta-\tau_n)G(\tau_n-\tau_{n-1})\cdots G(\tau_2-\tau_1)F(X_\alpha,\tau_1-\tau_\alpha)\\ &\propto \lim_{\eta_1\to 0}\eta_1^n = 0\,.\end{aligned} \qquad (\mathrm{IV}\cdot 1\cdot 30)$$

It then follows that the double sum (IV · 1 · 26) vanishes and fails to be $\Phi_E(\beta;\alpha)$. Therefore Eq. (IV · 1 · 11) cannot be of the form of Eq. (IV · 1 · 12). This means that Eq. (IV · 1 · 13) fails to hold. Accordingly Eq. (IV · 1 · 5) (C·1 for ESI on $\mathcal{S}_{\mathbf{step}}$) does not hold. We therefore conclude that QP cannot be defined for ESI on $\mathcal{S}_{\mathbf{step}}$ because of the failure of C·1. In this way, our framework judges "at **Step2**"(see the end of §2.1 of Chap. II) that QP cannot be defined. Discussion about this result, together with that for ESII, is given in §2.2.

§2. ESII

2.1 *Analysis*

Here we prove that C·1 does not hold for ESII, thereby concluding that QP cannot be defined again. We also simplify a surface $\mathcal{S}$ to be the steplike surface $\mathcal{S}_{\mathbf{step}}$ shown in Fig. 14. An alternative of ESII is also expressed as $\Delta(\vec{l}_n)$ with (IV · 1 · 1) and (IV · 1 · 7) where n is the number of times a particle can be *found* on $\mathcal{S}_{\mathbf{step}}$ and $\Delta(\vec{l}_n)$ denotes n-places of finding. The difference between "to intersect $\mathcal{S}_{\mathbf{step}}$ (ESI)" and "to be found on $\mathcal{S}_{\mathbf{step}}$ (ESII)" is to be understood as the difference between a component of ESI and that of ESII. For ESII, the component of propagator to be associated with an alternative specified by $\Delta(\vec{l}_n)$ is also defined by the sum over paths (IV · 1 · 3), but the sum is over those paths which *hit* (i.e., intersect or touch) $\mathcal{S}_{\mathbf{step}}$ at n domains $\Delta(\vec{l}_n)$ on the way from A to B. The distinction between "to hit" and "to intersect" is made on the Euclidean lattice.

The formal discussion from Eq. (IV · 1 · 1) to (IV · 1 · 16) for ESI also applies here with minor changes of the notation and the terminology. (In equations corresponding to Eqs. (IV · 1 · 5), (IV · 1 · 9) and so on, the sums over n are not restricted to odd or even integers, because the number of times a path hits $\mathcal{S}_{\mathbf{step}}$ can be even or odd. In what follows, anytime we refer to equations of previous subsection, they are to be interpreted as equations for ESII with these minor changes.)

We can begin with Eq. (IV · 1 · 11) of ESII-version to prove the failure of C·1. This time however, we shall provide a somewhat more elegant proof which does not need an analytic expression of u_n. In particular, starting from an identity concerning 'probabilities' of a random walk , we prove that Φ_{E_n} in Eq. (IV · 1 · 12) vanishes for any finite n. (Notation (IV · 1 · 14) is still valid here.) The identity is the following:

$$u[b;a] = \sum_{t=t_a+|x_a|}^{t_b-x_b} \bar{f}[x_b, t_b - t]u[0,t;a], \qquad (x_a < 0 < x_b) \qquad \text{(IV} \cdot 2 \cdot 1)$$

where $\bar{f}[x,t]$ is defined and given by Eqs. (III · 2 · 21) and (III · 2 · 22), respectively. This is obtained by classifying the 'paths' from a to b according to the last hitting (i.e., intersecting or touching) 'time' of $s|$. This 'path'classification is schematically represented as Fig. 17. We take the diffusion limit of the identity:

$$\mathrm{Lim}\frac{u[\beta/\eta;\alpha/\eta]}{2\eta_1} = \mathrm{Lim} \sum_{t=\tau_\alpha/\eta_2+|X_\alpha|/\eta_1}^{\tau_\beta/\eta_2-X_\beta/\eta_1} \frac{\bar{f}[X_\beta/\eta_1, \tau_\beta/\eta_2 - t]u[0,t;\alpha/\eta]}{2\eta_1}. \qquad \text{(IV} \cdot 2 \cdot 2)$$

Changing the summation variable from t to $\tau \equiv \eta_2 t$, we find

$$\Phi_E(\beta;\alpha) = \int_{\tau_\alpha}^{\tau_\beta} d\tau\, F(X_\beta, \tau_\beta - \tau)\Phi_E(0,\tau;\alpha) \qquad \text{(IV} \cdot 2 \cdot 3)$$

with F given by Eq. (III · 2 · 24). This is also an identity, confirmed by use of formula (III · 2 · 31), and is understood as expressing the 'path'-classification shown in Fig. 18 which is the continuum version of Fig. 17. In the figure, the diagram on the right-hand side is the abbreviation of the 'paths' (of Euclidean Brownian motion) whose last hitting of $\mathcal{S}|$ is at 'time' τ. In identity (IV · 2 · 3), the integrand $F\Phi_E$ multiplied by $d\tau$ (and divided by $\Phi_E(\beta;\alpha)$) is the 'probability' that the 'time' of the last hitting of $\mathcal{S}|$ lies somewhere in $d\tau$. Next we note

$$u_{n+1}[b;t_{n+1},\cdots,t_1;a]=\bar{f}[x_b,t_b-t_{n+1}]u_n[0,t_{n+1};t_n,\cdots,t_1;a]\,, \qquad \text{(IV}\cdot 2\cdot 4)$$

which follows directly from the very meanings of $\bar{f}$ and u_n as explained in Fig. 19.

We are now ready to prove the failure of ESII version of Eq. (IV · 1 · 12). Suppose that it holds for $X_\alpha < 0 < X_\beta$, then a modified equation

$$\Phi_E(0,\tau_\beta;\alpha)=\Phi_{E_0}(0,\tau_\beta;\alpha)+\sum_{n=1}^{\infty}\int d\vec{\tau}\,\Phi_{E_n}(0,\tau_\beta;\vec{\tau};\alpha) \qquad \text{(IV}\cdot 2\cdot 5)$$

should hold when the final point lies on $\mathcal{S}|$. The extra term Φ_{E_0} takes account of the existence of 'paths' on the lattice which start from a, never hit $s|$ before 'time' t_b and reach $[0,t_b]$ on $s|$ (see Fig. 20). This extra term is therefore calculated as follows

$$\Phi_{E_0}(0,\tau;\alpha)=\mathrm{Lim}\frac{1}{2\eta_1}\bar{f}[X_\alpha/\eta_1,(\tau-\tau_\alpha)/\eta_2]\,. \qquad \text{(IV}\cdot 2\cdot 6)$$

However from Eq. (III · 2 · 24) and the relationship $\eta_2/\eta_1^2=$ const, it follows that the right-hand side of Eq. (IV · 2 · 6) vanishes. Hence Eq. (IV · 1 · 12) can be assumed to hold for $X_\alpha < 0 \le X_\beta$. Paying attention to this fact, we substitute Eq. (IV · 1 · 12) into the right-hand side of identity (IV · 2 · 3), replace the integration variable τ by τ_{n+1}, and change the order of the integration over τ_{n+1} and the summation over n. The result is

$$\Phi_E(\beta;\alpha)=\sum_{n=1}^{\infty}\int d\vec{\tau}_{n+1}\,F(X_\beta,\tau_\beta-\tau_{n+1})\Phi_{E_n}(0,\tau_{n+1};\vec{\tau}_n;\alpha)\,, \qquad \text{(IV}\cdot 2\cdot 7)$$

where we have explicitly displayed the subscripts of integration variables; $\vec{\tau}_n\equiv(\tau_1,\cdots,\tau_n)$. If Eq. (IV · 1 · 12) is to be derived from (IV · 1 · 11), then Φ_{E_n} should be given by Eq. (IV · 1 · 27). Now let us see what happens if this is the case. We use first Eqs. (III · 2 · 24) and (IV · 1 · 27), then identity (IV · 2 · 4), and finally Eq. (IV · 1 · 27) again, to rewrite the integrand of (IV · 2 · 7) as

$$\begin{aligned} &F(X_\beta, \tau_\beta - \tau_{n+1})\Phi_{E_n}(0, \tau_{n+1}; \vec{\tau}_n; \alpha) \\ &= \mathrm{Lim}\frac{1}{2\eta_2}\bar{f}[X_\beta/\eta_1, (\tau_\beta - \tau_{n+1})/\eta_2] \\ &\times \frac{1}{2\eta_1(2\eta_2)^n}u_n[0, \tau_{n+1}/\eta_2; \tau_n/\eta_2, \cdots, \tau_1/\eta_2; \alpha/\eta] \\ &= \mathrm{Lim}\frac{1}{2\eta_1(2\eta_2)^{n+1}}u_{n+1}[\beta/\eta; \tau_{n+1}/\eta_2, \cdots, \tau_1/\eta_2; \alpha/\eta] \\ &= \Phi_{E_{n+1}}(\beta; \vec{\tau}_{n+1}; \alpha)\,. \end{aligned} \qquad (\mathrm{IV}\cdot 2\cdot 8)$$

Substituting the above into the right-hand side of Eq. (IV · 2 · 7), we have

$$\begin{aligned} \Phi_E(\beta; \alpha) &= \sum_{n=1}^{\infty}\int d\vec{\tau}_{n+1}\, \Phi_{E_{n+1}}(\beta; \vec{\tau}_{n+1}; \alpha) \\ &= \sum_{n=2}^{\infty}\int d\vec{\tau}_n\, \Phi_{E_n}(\beta; \vec{\tau}_n; \alpha)\,, \end{aligned} \qquad (\mathrm{IV}\cdot 2\cdot 9)$$

which is Eq. (IV · 1 · 12) with the term Φ_{E_1} deleted. This procedure can be iterated an arbitrary number of times. Consequently we obtain

$$\Phi_E = \sum_{n=1}^{\infty}\int d\vec{\tau}_n\, \Phi_{E_n} = \sum_{n=2}^{\infty}\int d\vec{\tau}_n\, \Phi_{E_n} = \sum_{n=3}^{\infty}\int d\vec{\tau}_n\, \Phi_{E_n} = \cdots\,. \qquad (\mathrm{IV}\cdot 2\cdot 10)$$

Since each Φ_{E_n} is positive because of Eq. (IV · 1 · 27), Eq. (IV · 2 · 10) proves

$$\Phi_{E_n}(\beta; \vec{\tau}_n; \alpha) = 0 \qquad \text{for finite } n\,. \qquad (\mathrm{IV}\cdot 2\cdot 11)$$

These Φ_{E_n} cannot satisfy Eq. (IV · 1 · 12) and thus Eq. (IV · 1 · 11) cannot be converted into the form of Eq. (IV · 1 · 12). Therefore we conclude that QP cannot be defined for ESII on $\mathcal{S}_{\mathrm{step}}$ because of the failure of C·1.

2.2 *Discussion about ESI and II*

Physical reason of the failure of C·1 and the *generalization* of our result: We have concluded for both ESI and II on $\mathcal{S}_{\text{step}}$ that QP cannot be defined because C·1 fails. For ESI, lattice version of C·1, namely Eq. (IV · 1 · 11), did hold. However in the diffusion limit the 'probability' (IV · 1 · 27) for any finite number of intersection vanished and C·1 failed. This was also the case for ESII as Eq. (IV · 2 · 11) showed. (Although we did not construct the ESII version of Eq. (IV · 1 · 11), we can construct it by classifying 'paths' from a to b according to how many times and at what locations they hit $s|$. Thus the lattice version of C·1 also holds for ESII, whose diffusion limit however fails.) The physical reason of the failure of C·1 in the limit is as follows: The relationship "η_2/η_1^2 = const" between the temporal and the spatial spacing implies that the velocity of the random walk $\eta_1/\eta_2 \propto 1/\eta_1$ becomes infinite in the diffusion limit. This continuum property of the walk makes a 'path' everywhere *nondifferentiable* with respect to 'time' in the limit. (This property was also quoted by Hartle[3] to give a qualitative justification to his conclusion about "wave function on $\mathcal{S}$.") It is in fact well-known, irrespective of the dimension and the potential (provided that it is a sufficiently good function), that nondifferentiable 'paths' contribute dominantly to the Euclidean sum over paths for a nonrelativistic particle.[7] Since nondifferentiable 'paths' intersect $\mathcal{S}|$ an infinite number of times, the 'probability' density of Euclidean Brownian motion vanishes for a finite number of intersection. The number of 'times' of hitting of $\mathcal{S}|$is also infinite and the 'probability' density vanishes again for a finite number of hitting. This observation applies to cases of (i) a more general surface $\mathcal{S}$ than $\mathcal{S}|$ (or $\mathcal{S}_{\text{step}}$), (ii) a nonzero potential and (iii) higher dimensional Newtonian spacetime. Hence we believe that our results obtained analytically are also correct in such general cases. Since physical reason of the failure of C·1 is now clear in this way, we shall not discuss ESI and II in such general cases.

§3. ESIII

We again use $\mathcal{S}_{\mathbf{step}}$ shown in Fig. 14. For ESIII on $\mathcal{S}_{\mathbf{step}}$, we prove that (i) C·1 holds but that (ii) C·2 does not hold, thereby concluding that QP cannot be defined for ESIII on $\mathcal{S}_{\mathbf{step}}$. A potential is allowed which is spatially symmetric with respect to $X = 0$ between T_C and T_D and which is arbitrary at other times, namely,

$$\begin{aligned} &V(-X,T) = V(X,T) \quad \text{for} \quad T_C < T < T_D, \\ &V(X,T) = \forall \quad \text{for} \quad T_A < T < T_C \text{ or } T_D < T < T_B. \end{aligned} \tag{IV $\cdot$ 3 $\cdot$ 1}$$

ESIII on $\mathcal{S}_{\mathbf{step}}$ is expressed as

$$\mathrm{ESIII} = \{\Delta(l)\}, \tag{IV $\cdot$ 3 $\cdot$ 2}$$

where the alternative specified by $\Delta(l)$ is the occurrence that a particle is first found on $\mathcal{S}_{\mathbf{step}}$ at $\Delta(l)$, whose concrete expression is given by (IV · 1 · 7). The component of propagator which is associated with the alternative specified by $\Delta = \Delta(l)$ is defined and denoted by

$$\Phi(B;\Delta;A) \equiv \int_{\Delta} d\lambda\, \Phi(B;\lambda;A) \equiv \sum_{B\leftarrow\Delta\leftarrow A} \mathrm{e}^{iS[X(T)]}, \tag{IV $\cdot$ 3 $\cdot$ 3}$$

where the sum is over paths which connect A and B and whose *first hitting* of $\mathcal{S}_{\mathbf{step}}$ occurs in Δ. The classifiability condition C·1 takes the following form for ESIII:

$$\begin{aligned} \Phi(B;A) &= \int d\lambda\, \Phi(B;\lambda;A) \\ &\equiv \int_{-\infty}^{0} dX_C \Phi(B;X_C;A) \\ &+ \int_{T_C}^{T_D} dT \Phi(B;T;A) \\ &+ \int_{0}^{\infty} dX_D \Phi(B;X_D;A)\,, \end{aligned} \tag{IV $\cdot$ 3 $\cdot$ 4}$$

where the coordinate λ on $\mathcal{S}_{\mathbf{step}}$ is given by (IV · 1 · 6). In the following, we classify paths from A to B according to the first place they hit $\mathcal{S}_{\mathbf{step}}$ and obtain components $\Phi(B;\lambda;A)$ satisfying the above condition.

We begin with $\Phi(B; X_C; A)$ which is defined by

$$\int_{\Delta X} dX_C \Phi(B; X_C; A) \equiv \sum_{B \leftarrow \Delta X \leftarrow A} e^{iS}, \qquad \text{(IV}\cdot 3\cdot 5)$$

where the sum is over paths from A to B whose first hitting of $\mathcal{S}_{\text{step}}$ occurs in ΔX on $\mathcal{S}_{T_C}$ (see Fig. 21). This is easily calculated. The above sum is decomposed into two partial sums and one integration. That is, a sum over all the paths from A to $C = (X_C, T_C)$ $(X_C < 0)$ which results in $\Phi(C; A)$, a sum over all the paths from C to B which results in $\Phi(B; C)$ and an integration over $-\infty < X_C < 0$. Therefore we have

$$\Phi(B; X_C; A) = \Phi(B; C)\Phi(C; A). \qquad \text{(IV}\cdot 3\cdot 6)$$

Let us turn to find $\Phi(B; T; A)$ which is defined by

$$\int_{\Delta T} dT \Phi(B; T; A) \equiv \sum_{B \leftarrow \Delta T \leftarrow A} e^{iS}, \qquad \text{(IV}\cdot 3\cdot 7)$$

where the sum is over paths whose first hitting of $\mathcal{S}_{\text{step}}$ occurs in ΔT on $\mathcal{S}|$. This sum is also broken down into partial sums and integrations as follows (see Fig. 22):

$$\int_{-\infty}^{\infty} dX_D \int_0^{\infty} dX_C \Phi(B; D) \Bigg(\sum_{\substack{D \leftarrow (0,T) \leftrightarrow C \\ T \in \Delta T}} e^{iS} \Bigg) \Phi(C; A), \qquad \text{(IV}\cdot 3\cdot 8)$$

where the detailed notation in §2.2 of Chap. III is employed for the sum over paths in the brackets; two propagators originate from partial sums over paths from A to C $(X_C > 0)$ and from D to B, in which paths are not restricted at all. The Euclidean lattice method carries out the sum over paths in the brackets. Without loss of generality, we put

$$C = (X_C, 0). \qquad \text{(IV}\cdot 3\cdot 9)$$

The following formula holds:

$$\sum_{\substack{D \leftarrow (0,T) \leftrightarrow C \\ T \in \Delta T}} e^{iS} = \int_{\Delta T} d(iT) \Phi(D; 0, T) F(X_C, iT), \qquad \text{(IV}\cdot 3\cdot 10)$$

where the sum is over all the paths which link C to D and whose first hitting of $X = 0$ occurs in the time interval ΔT. This is obtained from Eq. (III $\cdot$ 2 $\cdot$ 28) (with the weight replaced by μ('path') in general) taken diffusion limit, Wick rotated and then multiplied by the factor $G(D;C)$. From Eqs. (IV · 3 · 7), (IV · 3 · 8) and (IV · 3 · 10), it follows that

$$\begin{aligned}\Phi(B;T;A) &= i\int_{-\infty}^{\infty} dX_D \int_0^{\infty} dX_C\, \Phi(B;D)\Phi(D;0,T)F(X_C,iT)\Phi(C;A) \\ &= i\Phi(B;0,T)\int_0^{\infty} dX_C F(X_C,iT)\Phi(C;A)\,.\end{aligned} \qquad \text{(IV · 3 · 11)}$$

Let us turn to find the last component $\Phi(B;X_D;A)$ $(X_D > 0)$ whose definition is given by Eq. (IV · 3 · 5) with X_C replaced by X_D. The sum is again broken down into partial sums and an integration.

$$\Phi(B;X_D;A) = \Phi(B;D)\int_0^{\infty} dX_C\Big(\sum_{D\leftrightarrow C} e^{iS}\Big)\Phi(C;A)\,, \qquad \text{(IV · 3 · 12)}$$

where the sum in the brackets is over paths which never hit $\mathcal{S}|$ on the way from C to D such that $X_C, X_D > 0$. Noting (IV · 3 · 9), we employ formula (III · 3 · 67) to have

$$\Phi(B;X_D;A) = \Phi(B;D)\int_0^{\infty} dX_C\Big(\Phi(D;C) - \Phi(D;\bar{C})\Big)\Phi(C;A), \qquad \text{(IV · 3 · 13)}$$

where $\bar{C} \equiv (-X_C, 0)$. This completes decomposition (IV · 3 · 4). One can confirm that the components given by Eqs. (IV · 3 · 6), (IV · 3 · 11) and (IV · 3 · 13) satisfy Eq. (IV · 3 · 4) by performing all the integrations occurring on the right-hand side of Eq. (IV · 3 · 4). (To perform the time integral in Eq. (IV · 3 · 4) with (IV · 3 · 11) substituted, we use formulae (III · 3 · 65) and (III · 3 · 66).) Therefore the classifiability condition C·1 holds for ESIII.

Next we examine C·2. Decoherence functional for ESIII is defined by

$$D[\lambda;\lambda'] \equiv \int dX_B \iint dX_A dX_{A'} \Phi^*(B;\lambda;A)\Phi(B;\lambda';A')\Psi^*(A)\Psi(A')\,. \qquad \text{(IV · 3 · 14)}$$

The no-interference condition is then

$$\text{Re}D[\lambda;\lambda'] \propto \delta(\lambda - \lambda')\,, \qquad \text{(IV · 3 · 15)}$$

where λ and λ' are given by Eq. (IV · 1 · 6). There are six cases depending on the choice of λ and λ'. We define

$$\varphi(D) \equiv \int_0^\infty dX_C(\Phi(D;C) - \Phi(D;\bar{C}))\Psi(C)\,, \tag{IV$\cdot$3$\cdot$16}$$

$$\chi(T) \equiv i \int_0^\infty dX_C\, F(X_C, iT)\Psi(C)\,, \tag{IV$\cdot$3$\cdot$17}$$

where $\Psi(C)$ is of course given by $\Psi(C) = \int dX_A \Phi(C;A)\Psi(A)$. We substitute components (IV · 3 · 6), (IV · 3 · 11) and (IV · 3 · 13) into the right-hand side of Eq. (IV · 3 · 14) and carry out the integrations over X_B by use of property (II · 1 · 5). The results are as follows:

$$D[X_C; X_{C'}] = \delta(X_C - X_{C'})|\Psi(C)|^2\,, \tag{IV$\cdot$3$\cdot$18}$$

$$D[X_D; X_{D'}] = \delta(X_D - X_{D'})|\varphi(D)|^2\,, \tag{IV$\cdot$3$\cdot$19}$$

$$D[X_C; X_D] = \Phi^*(D;C)\Psi^*(C)\varphi(D)\,, \tag{IV$\cdot$3$\cdot$20}$$

$$D[T;T'] = (2\pi i(T-T'))^{-1/2}\chi^*(T)\chi(T')\,, \tag{IV$\cdot$3$\cdot$21}$$

$$D[X_D; T] = \Phi(D;0,T)\varphi^*(D)\chi(T)\,, \tag{IV$\cdot$3$\cdot$22}$$

$$D[X_C; T] = \Phi^*(0,T;C)\Psi^*(C)\chi(T)\,. \tag{IV$\cdot$3$\cdot$23}$$

If the real part of the right-hand side of Eq. (IV · 3 · 21) is proportional to $\delta(T-T')$ and if the real parts of the right-hand sides of Eqs. (IV · 3 · 20), (IV · 3 · 22) and (IV · 3 · 23) vanish, then C·2 holds. However none of these occur. For example, the real part of the right-hand side of Eq. (IV · 3 · 21) never vanishes for $T \neq T'$. Therefore C·2 does not hold; our framework judges at "**Step3**" (see the end of §2.1 of Chap. II) that QP cannot be defined.

Discussion:
(1) There is no possibility that a special choice of an initial amplitude $\Psi(A)$ makes C·2 hold. For instance, no matter how we vary $\Psi(C)$, the real part of the right-hand side of Eq. (IV · 3 · 21) does not vanish for $T \neq T'$.
(2) Equation (IV · 3 · 18) tells that there is no interference between "to first find a particle at X_C at time T_C" and "to first find it at $X_{C'}$ at time T_C" ($X_C \neq X_{C'}$). Equation (IV · 3 · 19) also tells this kind of things. This is physically quite natural because (i) the number of times a particle is found on $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ is once and only

once and (ii) there is no interference between "to find a particle at one place at time T" and "to find it another place at the same moment of time" since these two alternatives are mutually exclusive.
(3) Because of the symmetry of the potential between $T_C(=0)$ and T_D, we could use formula (III·3·66). So long as a potential obeys (IV · 3 · 1), concrete expression for the potential is not needed in the above calculations. Expression for the function F is not necessary either.

To summarize: The paths contributing to $\Phi(B;A)$ can be classified with respect to the first place they hit $\mathcal{S}_{\mathbf{step}}$, so that $\Phi(B;A)$ can be decomposed into the components whose expressions are given by (IV · 3 · 6), (IV · 3 · 11) and (IV · 3 · 13); thus C·1 holds for ESIII. However interferences between different components do not vanish except for trivial ones; thus C·2 does not hold. We therefore conclude that QP cannot be defined for ESIII on $\mathcal{S}_{\mathbf{step}}$.

§4. Lessons from probability-undefinable cases

Here we discuss what we can learn about the definability of QP from the investigations so far (ESI∼III). Not only for ESI and II but also for ESIII, QP cannot be defined. However the reason why QP cannot be defined for ESIII is different from those for ESI and II. As discussed in §2.2, paths which dominantly contribute to the sum over paths which defines a propagator are everywhere nondifferentiable with respect to time. Since a nondifferentiable path intersects or hits a surface which is not $\mathcal{S}_T$ an infinite number of times, an amplitude for any finite number of crossing or hitting of the surface vanishes and C·1 fails for ESI and II. Alternatives of ESI and II are too fine to be used as labels for path-classification. Alternatives of ESIII do not refer to the number of times of hitting or intersecting the surface. They only refer to the first hitting. In this sense, alternatives of ESIII are coarser than those of ESI and II. This we shall simply say that *ESIII is a coarse-grained set of ESI(or II)*. For this coarse-grained set, C·1 holds. This means that we can classify nondifferentiable paths by using first hitting place of a surface as a label for path-classification. It is intuitively understandable that one can talk about the first place a nondifferentiable path hits the surface, although one cannot talk about the number of times the path intersects or hits it. Since this observation applies to a more general surface than $\mathcal{S}_{\mathbf{step}}$, we believe that C·1 also holds for such a more general surface. We also believe that C·1 does not hold for an ES which is finer than ESIII and holds for an ES coarser than ESIII; ESIII will be the finest ES for which C·1 hold, since no information finer than the place of first hitting will be specifiable as to how a nondifferentiable path intersects or touches a general $\mathcal{S}$. Therefore so long as C·1 is concerned, we can predict, without calculations, whether it holds or fails for a given ES. If the ES is coarser than ESIII, C·1 will hold and otherwise it will fail. Since ESIV and V are coarser than ESIII, it is predicted in advance that C·1 will hold for the two ES, which is indeed the case as we will see later. This prediction however tells nothing about analytic expressions for components of propagator. Without them, we cannot investigate C·2. For ESIII, C·2 does not hold. Although C·2 is the key to the successful definition of QP, it seems difficult to find a qualitative way of judging whether C·2 holds or fails when C·1 holds. Calculating the decoherence functional seems to be the only way to judge it. In this way, whether C·1 holds or not might be judged qualitatively, however a quantitative investigation is necessary to discuss C·2.

Chap. V. Application of General Framework to Concrete Examples (II)

Probability-definable cases

In this chapter also we deal with a particle in (1+1)-dimensional Newtonian spacetime. A potential with a suitable symmetry is allowed. We work with

ESIV : the set {Yes, No}, where "Yes" is to find a particle in a *temporal domain* ΔT at a constant X and "No" is the complement to "Yes",

ESV : the set {Yes, No}, where "Yes" is to find a particle in a *spacetime domain* $\Omega \equiv \Delta X \times \Delta T$ and "No" is the complement to "Yes".

§1. ESIV

1.1 *Analysis*

We consider the following temporal domain:

$$\Delta T \equiv [0, T_D] \text{ at } X = 0. \qquad (\mathrm{V}\cdot 1\cdot 1)$$

This is $\mathcal{S}|$, namely, the temporal part of $\mathcal{S}_{\mathbf{step}}$. A potential is allowed which belongs to class (IV · 3 · 1). ESIV is expressed as

$$\text{ESIV} = \{\text{Yes}, \text{No}\}, \qquad (\mathrm{V}\cdot 1\cdot 2)$$

where "Yes" is to find a particle in ΔT at $X = 0$ and "No" is not to find it in the domain. The component of propagator to be associated with "Yes" is defined and denoted by

$$\Phi(B; \text{Yes}; A) \equiv \sum_{B \leftarrow \textbf{hitting } \mathcal{S}| \leftarrow A} e^{iS}, \qquad (\mathrm{V}\cdot 1\cdot 3)$$

where the sum is over paths which hit $\mathcal{S}|$ at least once on the way from A to B (see Fig. 23). Similarly the component for "No" is

$$\Phi(B; \text{No}; A) \equiv \sum_{B \leftarrow \textbf{not hitting } \mathcal{S}| \leftarrow A} e^{iS}, \qquad (\mathrm{V}\cdot 1\cdot 4)$$

where the sum is over paths which never hit $\mathcal{S}|$ (see Fig. 24). The classifiability condition

C·1 takes the following form for ESIV:

$$\Phi(B;A) = \Phi(B;\text{Yes};A) + \Phi(B;\text{No};A)\,, \qquad (\text{V}\cdot 1\cdot 5)$$

Let us calculate the two components. The sum over paths on the right-hand side of Eq. (V · 1 · 3) is equivalent to the sum over paths on the right-hand side of Eq. (IV · 3 · 7) with $\Delta T = [0, T_D]$. We have already studied Eq. (IV · 3 · 7) for $\Delta T \in [0, T_D]$ and then obtained Eq. (IV · 3 · 11). Substituting Eq. (IV · 3 · 11) into the left-hand side of Eq. (IV · 3 · 7) and putting $\Delta T = [0, T_D]$, we have

$$\Phi(B;\text{Yes};A) = \int_0^{T_D} d(iT) \int_{-\infty}^{\infty} dX_C \Phi(B;0,T) F(X_C, iT) \Phi(C;A)\,. \qquad (\text{V}\cdot 1\cdot 6)$$

This is understood as classifying all the paths which link A to B and hit $\mathcal{S}|$ with respect to two labels: one is the position X_C where paths intersect $\mathcal{S}_{T_C}$ and the other is the first hitting time T of $\mathcal{S}|$. The paths from $C = (X_C, 0)$ to $(0, T)$ are restricted to the half space $X > 0$ or $X < 0$ and the sum over such paths gives the function F according to formula (III · 3 · 45) with (III · 3 · 42) and (III · 3 · 36). Since paths from A to C and those from $(0, T)$ to B are not restricted at all, familiar propagators appears accordingly in Eq. (V · 1 · 6). Writing

$$\Phi(B;0,T) = \int_{-\infty}^{\infty} dX_D \Phi(B;D) \Phi(D;0,T)\,, \qquad (\text{V}\cdot 1\cdot 7)$$

and using formulae (III · 3 · 64) ~ (III · 3 · 66), we can carry out the time integral in Eq. (V · 1 · 6). The result is

$$\begin{aligned} &\Phi(B;\text{Yes};A) \\ &= \left(\int_0^{\infty} dX_D \int_{-\infty}^{0} dX_C + \int_{-\infty}^{0} dX_D \int_0^{\infty} dX_C\right) \Phi(B;D)\Phi(D;C)\Phi(C;A) \\ &+ \left(\int_0^{\infty} dX_D \int_0^{\infty} dX_C + \int_{-\infty}^{0} dX_D \int_{-\infty}^{0} dX_C\right) \Phi(B;D)\Phi(D;\bar{C})\Phi(C;A)\,, \end{aligned} \qquad (\text{V}\cdot 1\cdot 8)$$

where

$$\bar{C} \equiv (-X_C, 0) \text{ for } C = (X_C, 0). \qquad (\text{V}\cdot 1\cdot 9)$$

Although expression (V · 1 · 6) is correct for an arbitrary potential, expression (V · 1 · 8) is valid only when the potential belongs to class (IV · 3 · 1) (this is the case we are dealing with) because we used formula (III · 3 · 66) in carrying out the time integral in (V · 1 · 6).

Next we calculate the other component (V · 1 · 4). There are two ways for a path from A to B to avoid hitting $\mathcal{S}|$. That is, (i) to cross $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ respectively at C and D such that $X_C, X_D < 0$ and never hit $\mathcal{S}|$ (see Fig. 24), (ii) to cross $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ respectively at C and D such that $X_C, X_D > 0$ and never hit $\mathcal{S}|$. In each case, the partial paths from A to C and those from D to B are not restricted at all and they are summed over to give $\Phi(C;A)$ and $\Phi(D;B)$, respectively. The partial paths from C to D are restricted to a half space and the sum over such paths is given by Eq. (III · 3 · 67). From these, we have

$$\Phi(B;\mathrm{No};A) = \left(\int_{-\infty}^{0} dX_D \int_{-\infty}^{0} dX_C + \int_{0}^{\infty} dX_D \int_{0}^{\infty} dX_C\right) \times \Phi(B;D)(\Phi(D;C) - \Phi(D;\bar{C}))\Phi(C;A)\,. \qquad (\mathrm{V}\cdot 1\cdot 10)$$

One can confirm that components (V · 1 · 8) and (V · 1 · 10) satisfy the classifiability condition (V · 1 · 5). Therefore C·1 holds for ESIV.

Next we investigate C·2. The decoherence functional for the pair of "Yes" and "No" is given by

$$D[\mathrm{Yes};\mathrm{No}] = \int dX_B \iint dX_A dX_{A'} \Phi^*(B;\mathrm{Yes};A)\Phi(B;\mathrm{No};A')\Psi^*(A)\Psi(A')\,. \quad (\mathrm{V}\cdot 1\cdot 11)$$

The no-interference condition is

$$\mathrm{Re}D[\mathrm{Yes};\mathrm{No}] = 0. \qquad (\mathrm{V}\cdot 1\cdot 12)$$

Substituting expressions (V · 1 · 8) and (V · 1 · 10) into the right-hand side of Eq. (V · 1 · 11) and carrying out the integration over X_B by use of formula (II · 1 · 5), we have

$$D[\mathrm{Yes};\mathrm{No}] = \int_0^\infty dX_D \int_0^\infty dX_C \int_0^\infty dX_{C'} \times \Phi^*(D;\bar{C})(\Phi(D;C') - \Phi(D;\bar{C}'))\big(\Psi^*(\bar{C}) + \Psi^*(C)\big)\big(\Psi(\bar{C}') + \Psi(C')\big). \qquad (\mathrm{V}\cdot 1\cdot 13)$$

(We discuss ESV in the next section which includes ESIV as a special case. Calculations of the decoherence functional for ESV is given in Appendix E. Since a special case of the decoherence functional for ESV gives (V · 1 · 13), we shall not exhibit the acutual derivation of (V · 1 · 13) here.) The real part of (V · 1 · 13) is not identically zero. Therefore, in general, C·2 does not hold and QP cannot be defined for ESIV.

However we note that *only the symmetric combination of $\Psi(C)$ (Schrödinger's wave function at time $T_C(=0)$) contributes to the decoherence functional.* Therefore, if $\Psi(C)$ is spatially antisymmetric, namely

$$\Psi(\bar{C}) + \Psi(C) = 0, \qquad (\mathrm{V}\cdot 1\cdot 14)$$

then the interference ReD vanishes. In fact, (V · 1 · 14) is also the necessary condition for ReD[Yes; No] $= 0$ at least for a free particle. We shall prove this here. Since an arbitrary function can be written as the sum of a symmetric function and an antisymmetric function, all we have to do is to prove that a symmetric $\Psi(C)$ never makes ReD vanish. When $\Psi(\bar{C}) = \Psi(C)$, it turns out that

$$\mathrm{Re}D[\mathrm{Yes};\mathrm{No}] = -4\int_0^{\infty} dX_D \left| \int_{-\infty}^{0} dX_C \Phi(D;C)\Psi(C) \right|^2 . \qquad (\mathrm{V}\cdot 1\cdot 15)$$

Therefore C·2 takes the following form:

$$\int_{-\infty}^{0} dX_C \Phi(D;C)\Psi(C) = 0 \qquad \text{for } X_D \geq 0 . \qquad (\mathrm{V}\cdot 1\cdot 16)$$

Since this is required to hold for $^{\forall}X_D \geq 0$, we have $\Psi(C) = 0$. (Details are found in Appendix C.) It now follows that *C·2 holds and thus QP can be defined for ESIV if (and, for a free particle, only if) $\Psi(C)$ is an antisymmetric function.* Let us calculate values of the probabilities when $\Psi(C)$ is such. We substitute component (V · 1 · 8) into formula (II · 2 · 3), carry out the integration over X_B and rearrange the result with the help of Eq. (V · 1 · 14). Finally we obtain

$$P(\mathrm{Yes}) = 0 . \qquad (\mathrm{V}\cdot 1\cdot 17)$$

Since our framework guarantees the total probability to be unity, it follows that

$$P(\mathrm{No}) = 1 , \qquad (\mathrm{V}\cdot 1\cdot 18)$$

which is of course confirmed by substituting (V · 1 · 10) into (II · 2 · 3). Therefore "the particle is never found in the temporal domain". We put an interpretation on this result in the next subsection. Before that we claim the following: The condition (V · 1 · 14) can be regarded as a condition on the initial amplitude $\Psi(A)$ because there is one to one correspondence between $\Psi(C)$ and $\Psi(A)$. Let us consider a class of $\Psi(C)$ which is normalizable and satisfies (V · 1 · 14). We write $\Psi(C)$ belonging to this class as $\Psi_{\mathbf{class}}(C)$. Then our result can be restated as follows: If an initial amplitude belongs to a specific class

$$\Psi_{\mathbf{class}}(A) \equiv \int dX_C \Phi^*(C;A)\Psi_{\mathbf{class}}(C), \qquad (\mathrm{V}\cdot 1\cdot 19)$$

then QP can be defined for ESIV with values (V · 1 · 17) and (V · 1 · 18).

1.2 *Interpretation*

We discuss the meaning of the proposition that "the particle is never found in ΔT ". This proposition sounds like as if it is speaking about something like a "measurement" which is distributed in time but not distributed in space. When one talks about a measurement in the usual sense, the measurement is distributed in space but not in time. For example, a particle is found in a finite spatial interval at a moment of time. One must first of all recognize this point when one is given a proposition of the above kind. Therefore the measurement theoretical meaning of the proposition is not apriori self evident. However this never means that it is impossible to give a measurement theoretical meaning to the proposition within the usual measurement theory. In fact we can give such a meaning to the proposition that is becoming to values (V · 1 · 17) and (V · 1 · 18) of the probabilities. This turns out to be possible because QP are defined only in the restricted situation (V · 1 · 14), which we shall now explain.

We have been dealing with a potential of the type of (IV · 3 · 1). Because of this, Schrödinger's wave function remains to be antisymmetric between $T_C(=0)$ and T_D, provided that it is so at time T_C. Therefore (V · 1 · 14) makes the wave function vanish on $\mathcal{S}|$ in the presence of the potential (IV · 3 · 1):

$$\Psi(0,T) = 0 \qquad \text{for } T \in \Delta T \qquad (\mathrm{V}\cdot 1\cdot 20)$$

Since $P(\text{Yes})$ and (No) are defined only in the situation (V · 1 · 20), interpretation problem of the probabilities is also posed only in the situation. In fact the situation (V · 1 · 20) is by itself very becoming to (V · 1 · 17) and (V · 1 · 18) or to the proposition that "the particle is never found in the temporal domain". Hence we shall *specify* the meaning of the proposition as

"The particle is never found in ΔT at $X = 0$."
≡ Schrödinger's wave function vanishes in ΔT at $X = 0$. (V · 1 · 21)

Although the right-hand side is not yet stated in such terms that are directly related to an instantaneous measurement, this interpretation is much better than the complete lack of the meaning of the left-hand side. *Although we started our investigation without having any concrete idea about the physical meaning of probabilities for ESIV, we have arrived at interpretation* (V · 1 · 21) *which seems more or less becoming to the values of the probabilities.* Of course this concerns only one example of ES. It is uncertain whether the above kind of interpretation of QP for an ES is always possible when our framework judges that the QP are definable for the ES. In addition to this, as already mentioned, interpretation (V · 1 · 21) still leaves ambiguities because the right-hand side is not spoken in the language of measurement. In view of these circumstances, we investigate one more example of ES, that is, ESV which includes ESIV as a special case.

§2. ESV

2.1 *Preparation*

We begin by generalizing sum-over-paths formulae obtained by the Euclidean lattice method. Up to now, we often considered the time at which paths hit the "wall" $X = 0$ for the first time. In the present case we consider "walls" which are positioned not at $X = 0$ but at $X = X_W \neq 0$. Accordingly we define

$$F(W;A) \equiv \sum_{W \leftrightarrow A} e^{iS}, \quad A \equiv (X_A, T_A),\ W \equiv (X_W, T),\ (T_A < T), \qquad (\mathrm{V} \cdot 2 \cdot 1)$$

where the sum is over all the paths which link A to W and which do not invade the region $X < X_W$ when $X_W < X_A$ or the region $X > X_W$ when $X_W > X_A$ until the end time T. This sum over paths is defined by and calculated from the corresponding sum over 'paths' combined with the diffusion limit and the Wick rotation. Function $F(X, iT)$ defined by the sum over paths (III · 3 · 45) is the special case of $F(W;A)$:

$$F(X, iT) = F(0, T; X, 0). \qquad (\mathrm{V} \cdot 2 \cdot 2)$$

From Eq. (III · 2 · 24), for a free particle, it follows that

$$F_{\mathrm{free}}(W;A) = \left[\frac{(X_W - X_A)^2}{2\pi i^3 (T - T_A)^3}\right]^{1/2} \exp\left(i\frac{(X_W - X_A)^2}{2(T - T_A)}\right). \qquad (\mathrm{V} \cdot 2 \cdot 3)$$

Formula (IV · 3 · 10) is generalized to

$$\sum_{\substack{B \leftarrow W \leftrightarrow A \\ T \in \Delta T}} e^{iS} = \int_{\Delta T} d(iT)\Phi(B;W)F(W;A), \qquad (\mathrm{V} \cdot 2 \cdot 4)$$

where the sum is over all the paths which link A to B and whose first hitting of $X = X_W$ occurs in the time interval ΔT (see Fig. 25).

We introduce the following notation for the spatial mirror image of X with respect to X_W:

$$\bar{X}[X_W] \equiv -X + 2X_W, \qquad \bar{A}[X_W] \equiv (\bar{X}_A[X_W], T_A). \qquad (\mathrm{V} \cdot 2 \cdot 5)$$

In the rest of this section we write these simply as $\bar{X}$ and $\bar{A}$. For a general potential, the analytical expression for F is not available. However if the potential V is symmetric with respect to the wall, i.e.,

$$V(\bar{X},T) = V(X,T)\,, \qquad (\text{V}\cdot 2\cdot 6)$$

then F has the property

$$F(W;\bar{A}) = F(W;A)\,. \qquad (\text{V}\cdot 2\cdot 7)$$

With these notations, formulae (III · 3 · 64) ∼ (III · 3 · 66) are generalized to

$$\int_{T_A}^{T_B} d(iT)\,\Phi(B;W)F(W;A)$$
$$= \begin{cases} \Phi(B;A) & \text{for } (X_B - X_W)(X_A - X_W) < 0,\ {}^{\forall}V(X,T) \qquad (\text{V}\cdot 2\cdot 8) \\ \Phi(B;\bar{A}) & \text{for } (X_B - X_W)(X_A - X_W) > 0,\ V(\bar{X},T) = V(X,T). \qquad (\text{V}\cdot 2\cdot 9) \end{cases}$$

Similarly formula (III · 3 · 67) is generalized to

$$\sum_{W \nleftrightarrow A} e^{iS} = \Phi(B;A) - \Phi(B;\bar{A})$$
$$\text{for } (X_B - X_W)(X_A - X_W) > 0,\ V(\bar{X},T) = V(X,T), \qquad (\text{V}\cdot 2\cdot 10)$$

where the sum is over all the paths which never hit the wall of $X = X_W$ on the way from A to B (see Fig. 26).

2.2 *Analysis*

Let a spacetime domain Ω be bounded by $\mathcal{S}_{T_A}$ and $\mathcal{S}_{T_B}$ as shown in Fig. 27:

$$\Omega \equiv \Delta X \times \Delta T$$
$$\Delta X = [-a,a],\ \Delta T = [T_C,T_D],\ \text{where } T_A < T_C < T_D < T_B. \qquad (\text{V}\cdot 2\cdot 11)$$

To make our discussion as general as possible, we take account of a potential $V(X,T)$. A restriction is made, however, on the form of V for $T_C < T < T_D$: (i)To make full use of the formulae in §2.1, we assume the potential to be symmetric with respect to the "left wall" and to the "right wall" of Ω, that is, the temporal intervals $T_C < T < T_D$

at $X = -a$ and $X = a$, respectively.

$$V(\bar{X}[\pm 2a], T) = V(X,T) \qquad \text{for} \qquad T_C < T < T_D . \qquad (\text{V} \cdot 2 \cdot 12)$$

(ii)For another reason which will be understood as we proceed, the potential is assumed to be symmetric with respect to $X = 0$ as well.

$$V(-X,T) = V(X,T) \qquad \text{for} \qquad T_C < T < T_D . \qquad (\text{V} \cdot 2 \cdot 13)$$

From Eqs. (V · 2 · 12) and (V · 2 · 13), the general form of V we deal with is

$$V(X,T) = \begin{cases} \sum_{n=0}^{\infty} V_n(T) \cos \frac{\pi n}{a} X & \text{for } T_C < T < T_D \\ \forall & \text{otherwise.} \end{cases} \qquad (\text{V} \cdot 2 \cdot 14)$$

ESV is expressed as

$$\text{ESV} = \{\text{Yes}, \text{No}\}, \qquad (\text{V} \cdot 2 \cdot 15)$$

where "Yes" is to find a particle in Ω and "No" is not to find it in the domain. The components of propagator to be associated with "Yes" and "No" are respectively

$$\Phi(B; \text{Yes}; A) \equiv \sum_{B \leftarrow \Omega \leftarrow A} e^{iS} , \qquad (\text{V} \cdot 2 \cdot 16)$$

where the sum is over paths which pass through Ω on the way from A to B (see Figs. 28 and 29), and

$$\Phi(B; \text{No}; A) \equiv \sum_{B \leftarrow \text{outside } \Omega \leftarrow A} e^{iS} , \qquad (\text{V} \cdot 2 \cdot 17)$$

where the sum is over paths which do not pass through Ω (see Fig. 30). The classifiability condition C·1 takes the same form as that for ESIV:

$$\Phi(B; A) = \Phi(B; \text{Yes}; A) + \Phi(B; \text{No}; A). \qquad (\text{V} \cdot 2 \cdot 18)$$

First we calculate the component for "Yes". We shall call the spatial interval $-a < X < a$ at $T = T_C$ the "bottom" of Ω. There are three ways for a path from A to B to pass through Ω: (i) to intersect the bottom of Ω (see Fig. 28), (ii) to cross S_{T_C} at C such that $X_C < -a$, hit the left wall and then arrive at B (see Fig. 29), (iii) to cross $\mathcal{S}_{T_C}$ at C such that $X_C > a$, hit the right wall and then arrive at B, where $\mathcal{S}_{T_C}$ is the surface of $T = T_C$. Accordingly the sum over paths on the right-hand side of Eq. (V · 2 · 16) can be decomposed into three sums over paths. Summing over all the paths of type (i) simply yields $\Phi(B;C)\Phi(C;A)$ integrated over $-a < X_C < a$. Every path of type (ii) can be decomposed into a partial path from A to C ($X_C < -a$) and a partial path from C to B whose first hitting time of the left wall lies in $[T_C, T_D]$. The sum over all such partial paths from C to B is given by formula (V · 2 · 4) with $\Delta T = T_D - T_C$. The result must be multiplied by $\Phi(C;A)$, which comes from summing over all the partial paths from A to C, and integrated by X_C over $[-\infty, -a]$. The sum over all the paths of type (iii) is calculated in a similar way. Consequently we obtain

$$\begin{aligned}\Phi(B;\text{Yes};A) = &\int_{-a}^{a} dX_C\,\Phi(B;C)\Phi(C;A)\\ &+\int_{-\infty}^{-a} dX_C \int_{T_C}^{T_D} d(iT)\,\Phi(B;W_-)F(W_-;C)\Phi(C;A)\\ &+\int_{a}^{\infty} dX_C \int_{T_C}^{T_D} d(iT)\,\Phi(B;W_+)F(W_+;C)\Phi(C;A)\,,\end{aligned} \qquad (\text{V}\cdot 2\cdot 19)$$

where

$$W_\pm \equiv (\pm a, T)\,. \qquad (\text{V}\cdot 2\cdot 20)$$

(This is valid for an arbitrary V.) On the assumption of (V · 2 · 12), we rewrite Eq. (V · 2 · 19) as follows. Substituting

$$\Phi(B;W_\pm) = \int_{-\infty}^{\infty} dX_D\,\Phi(B;D)\Phi(D;W_\pm) \qquad (\text{V}\cdot 2\cdot 21)$$

into the right-hand side of Eq. (V · 2 · 19) and using Eqs. (V · 2 · 8) and (V · 2 · 9), we can perform the time integrals in Eq. (V · 2 · 19). With the notation

$$\bar{C}_\pm \equiv \bar{C}[\pm a] = (-X_C \pm 2a, T_C)\,, \qquad (\text{V}\cdot 2\cdot 22)$$

we obtain

$$\begin{aligned}
&\Phi(B;\mathrm{Yes};A)\\
&=\int_{-\infty}^{\infty}dX_D\int_{-a}^{a}dX_C\,\Phi(B;D)\Phi(D;C)\Phi(C;A)\\
&\quad+\int_{-\infty}^{-a}dX_D\int_{-\infty}^{-a}dX_C\,\Phi(B;D)\Phi(D;\bar{C}_-)\Phi(C;A)\\
&\quad+\int_{-a}^{\infty}dX_D\int_{-\infty}^{-a}dX_C\,\Phi(B;D)\Phi(D;C)\Phi(C;A)\\
&\quad+\int_{-\infty}^{a}dX_D\int_{a}^{\infty}dX_C\,\Phi(B;D)\Phi(D;C)\Phi(C;A)\\
&\quad+\int_{a}^{\infty}dX_D\int_{a}^{\infty}dX_C\,\Phi(B;D)\Phi(D;\bar{C}_+)\Phi(C;A)\,.
\end{aligned}\tag{V·2·23}$$

Let us turn to (V · 2 · 17). There are two ways for a path from A to B to avoid passing through Ω: (i′) to cross $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ at C and D, respectively, such that $X_C, X_D < -a$, never hitting the left wall on the way (see Fig. 30), (ii′) to cross $\mathcal{S}_{T_C}$ and $\mathcal{S}_{T_D}$ at C and D, respectively, such that $X_C, X_D > a$, never hitting the right wall, where $\mathcal{S}_{T_D}$ is the surface of $T = T_D$. Accordingly the sum over paths on the right-hand side of (V · 2 · 17) is a sum of two sums over paths: the sum over all the paths of type (i′) and that of (ii′). In each of these two, the sum over partial paths from A to C gives $\Phi(C;A)$ and that from D to B gives $\Phi(B;D)$. The sum over partial paths from C to D never hitting the left or the right wall is, on the assumption of (V · 2 · 12), calculated from formula (V · 2 · 10). The result is

$$\begin{aligned}
&\Phi(B;\mathrm{No};A)\\
&=\int_{-\infty}^{-a}dX_D\int_{-\infty}^{-a}dX_C\,\Phi(B;D)\left(\Phi(D;C)-\Phi(D;\bar{C}_-)\right)\Phi(C;A)\\
&\quad+\int_{a}^{\infty}dX_D\int_{a}^{\infty}dX_C\,\Phi(B;D)\left(\Phi(D;C)-\Phi(D;\bar{C}_+)\right)\Phi(C;A)\,.
\end{aligned}\tag{V·2·24}$$

One can confirm that the classifiability condition (V · 2 · 18) is satisfied by the components (V · 2 · 23) and (V · 2 · 24). Therefore C·1 holds for ESV.

Decoherence functional for ESV is given by

$$D[\mathrm{Yes};\mathrm{No}]\equiv\int dX_B\iint dX_A dX_{A'}\Phi^*(B;\mathrm{Yes};A)\Phi(B;\mathrm{No};A')\Psi^*(A)\Psi(A')\,. \tag{V·2·25}$$

The no-interference condition C·2 is

$$\mathrm{Re}D[\mathrm{Yes};\mathrm{No}]=0. \tag{V·2·26}$$

We substitute Eqs. (V · 2 · 23) and (V · 2 · 24) into the right-hand side of Eq. (V · 2 · 25). The integration over X_B can be done by use of formula (II · 1 · 5). Furthermore we use

$$\Phi(\bar{D};C) = \Phi(D;\bar{C}) \quad (\bar{D} \equiv \bar{D}[0] = (-X_D, T_D)\,,\ \bar{C} \equiv \bar{C}[0]) \qquad (V \cdot 2 \cdot 27)$$

on the assumption of (V · 2 · 13). Noting $\int dX_A \Phi(C;A)\Psi(A) = \Psi(C)$, we finally have

$$\begin{aligned} D[\text{Yes};\text{No}] = &\int_a^\infty dX_D \int_0^\infty dX_C \int_0^\infty dX_{C'} \\ &\times \Phi^*(D;-X_C+a,T_C)\,(\Phi(D;X_{C'}+a,T_C) - \Phi(D;-X_{C'}+a,T_C)) \\ &\times \Big\{(\Psi(X_C-a)+\Psi(-X_C-a))^*\,\Psi(-X_{C'}-a) \\ &\quad + (\Psi(X_C+a)+\Psi(-X_C+a))^*\,\Psi(X_{C'}+a)\Big\}\,, \end{aligned} \qquad (V \cdot 2 \cdot 28)$$

where all the Ψ's in { } refer to time T_C, that is, $\Psi(X_C - a) \equiv \Psi(X_C - a, T_C)$, etc; detailed calculations are given in Appendix E. The appearance of the special combination of Ψ's in { } is a consequence of property (V · 2 · 27). Although the decoherence functional $D[\text{Yes};\text{No}]$ is not identically zero, it vanishes if $\Psi(C)$ satisfies the following two conditions.

$$\Psi(X-a) + \Psi(-X-a) = 0 \qquad (V \cdot 2 \cdot 29)$$

and

$$\Psi(X+a) + \Psi(-X+a) = 0\,, \qquad (V \cdot 2 \cdot 30)$$

where we write X_C simply as X. Let Ψ_+ and Ψ_- be respectively the even and odd parity part of Ψ:

$$\Psi(X) = \Psi_+(X) + \Psi_-(X), \quad \Psi_\pm(-X) = \pm\Psi_\pm(X)\,. \qquad (V \cdot 2 \cdot 31)$$

Conditions (V · 2 · 29) and (V · 2 · 30) are equivalent to

$$\Psi_\pm(X+2a) = \mp\Psi_\pm(X)\,. \qquad (V \cdot 2 \cdot 32)$$

In fact, this is also the necessary condition for { } on the right-hand side of (V · 2 · 28) to vanish for arbitrary $X_{C'}$. This is easily proved by writing Ψ in the brackets in terms of $\Psi_\pm$. The general form of Ψ which satisfies (V · 2 · 32) is given by

$$\Psi(X) = \sum_{n=1}^{\infty} (A_n \sin k_n X + B_n \cos p_n X) \qquad (\mathrm{V} \cdot 2 \cdot 33)$$

with

$$k_n \equiv \frac{\pi n}{a}, \quad p_n \equiv \frac{\pi(2n-1)}{2a}. \qquad (\mathrm{V} \cdot 2 \cdot 34)$$

Our framework judges that C·2 holds and QP can be defined for ESV when $\Psi(C)$ belongs to a specific class (V · 2 · 33), or equivalently, when an initial amplitude belongs to a specific class (V · 1 · 19), where $\Psi_{\mathbf{class}}(C)$ here is $\Psi(C)$ which belongs to class (V · 2 · 33).

Let us calculate values of the probabilities. From formula (II · 2 · 3),

$$P(\mathrm{Yes}) = \int dX_B \left| \int dX_A \Phi(B; \mathrm{Yes}; A)\Psi(A) \right|^2. \qquad (\mathrm{V} \cdot 2 \cdot 35)$$

We substitute expression (V · 2 · 23) into the right-hand side. The integration over X_B is carried out by use of Eq. (II · 1 · 5). We then use Eqs. (V · 2 · 29) and (V · 2 · 30). Finally we obtain

$$P(\mathrm{Yes}) = \int_{-a}^{a} dX_D |\Psi(D)|^2 = \int_{-a}^{a} dX |\Psi(X,T)|^2 \qquad (T_C < T < T_D). \qquad (\mathrm{V} \cdot 2 \cdot 36)$$

The last equality follows from the conservation law:

$$\frac{\partial}{\partial T} \int_{-a}^{a} dX |\Psi(X,T)|^2 = 0 \qquad (T_C < T < T_D). \qquad (\mathrm{V} \cdot 2 \cdot 37)$$

We shall prove this here. $\Psi_\pm(X,T)$ obey the Schrödinger equation. Since the potential is symmetric and has the fundamental period $2a$ for $T_C < T < T_D$, the $2a$-antiperiodicity of Ψ_+ and the $2a$-periodicity of Ψ_- are conserved by the time evolution between T_C and T_D. Thus, throughout the time interval, the form (V · 2 · 33) remains unchanged with time dependent coefficients $A_n(T)$ and $B_n(T)$. Therefore

$$\Psi(\pm a, T) = 0 \qquad (T_C < T < T_D). \qquad (\mathrm{V} \cdot 2 \cdot 38)$$

Thus the current of probability density vanishes at $X = \pm a$ for $T_C < T < T_D$ and the conservation law (V · 2 · 37) holds. From our general framework and Eq. (V · 2 · 36), it follows that

$$P(\text{No}) = \int_{-\infty}^{\infty} dX |\Psi(X,T)|^2 - \int_{-a}^{a} dX |\Psi(X,T)|^2 \qquad (T_C < T < T_D), \qquad (\text{V}\cdot 2\cdot 39)$$

which can of course be confirmed by substituting component (V · 2 · 24) into formula (II · 2 · 3). Quantities (V · 2 · 36) and (V · 2 · 39) are "spacetime probabilities" in the sense that these probabilities are associated with finite spacetime domain Ω. The conservation law (V · 2 · 37) allows us to rewrite the spacetime probabilities into suggestive forms:

$$P(\text{Yes}) = \int_{\Omega} dV |\tilde{\Psi}(X,T)|^2 , \qquad (\text{V}\cdot 2\cdot 40)$$

where

$$\tilde{\Psi}(X,T) \equiv \frac{1}{\sqrt{T_D - T_C}} \Psi(X,T), \quad \int_{\Omega} dV \equiv \int_{T_C}^{T_D} dT \int_{-a}^{a} dX \, . \qquad (\text{V}\cdot 2\cdot 41)$$

$P(\text{No})$ can also be written as an integral over a spacetime volume.

As one may have already noticed, there is a difficulty in the above discussion. Since (V · 2 · 33) is spatially periodic, it is not normalizable on $(-\infty, \infty)$; the first term on the right-hand side of (V · 2 · 39) diverges. In this sense, probabilities (V · 2 · 36) and (V · 2 · 39) are formal ones. One way to get over this difficulty is to adopt a box normalization for Ψ instead of (II · 1 · 9) and reinvestigate C·1 and C·2. This is a possible but cumbersome plan because the propagator in the box is complicated; it is a sum of an infinite number of Φ even for a free particle. We shall avoid such a task. Instead we shall discuss what we can learn from the formal probabilities in the next subsection.

Lastly we look at the special case $a = 0$ or $T_D = T_C$. (i)When $a = 0$, conditions (V · 2 · 29) and (V · 2 · 30) are one and the same condition requiring that the initial amplitude be antisymmetric with respect to $X = 0$. (Expression (V · 2 · 33) cannot be used when $a = 0$.) Given such an initial amplitude, probabilities can be defined for $\{\text{Yes}, \text{No}\}$ of the $a = 0$ case. Since an antisymmetric and normalizable initial amplitude is available, normalization problem does not occur. From expressions (V · 2 · 36) and (V · 2 · 39), values of the probabilities are 0 for "Yes" and 1 for "No". These are precisely what we obtained for ESIV in the previous subsection. (ii)When $T_D = T_C$, domain Ω becomes the spatial domain $[-a, a]$; thus familiar probabilities must be defined without any restriction on the initial amplitude. We can easily confirm this. When $T_D = T_C$, the decoherence functional (V · 2 · 28) identically vanishes because of

$$\lim_{T_D \to T_C} \Phi(D;C) = \delta(X_D - X_C). \qquad (\mathrm{V} \cdot 2 \cdot 42)$$

Therefore the pair of "Yes" and "No" decoheres for an arbitrary initial amplitude and probabilities can always be defined with values (V · 2 · 36) and (V · 2 · 39).

To summarize: The no-interference condition holds when an initial amplitude belongs to a specific class specified by (V · 2 · 33). In such situations, probabilities can be defined for spacetime alternatives {Yes, No} with values (V · 2 · 36) and (V · 2 · 39). This result is correct in the presence of any potential of the form (V · 2 · 14). See Fig. 31.

2.3 *Interpretation and discussion*

Here we shall not refer to the normalization problem. The result for ESV is similar to that for ESIV. That is, QP can be defined only in a restricted situation. Accordingly, interpretation problem is also posed only in the situation. Let us concentrate on expression (V · 2 · 36). It says that "the particle is found in Ω with probability $\int_{-a}^{a} dX|\Psi(X,T)|^2$". This proposition sounds like as if it is speaking about a "measurement" distributed both in space and time, whose meaning is not self evident. As discussed in §1.2, this never means that it is impossible to give a measurement theoretical meaning to the proposition within the usual measurement theory. It is interpreted as follows.

In the restricted situation under which QP can be defined, Schrödinger's wave function belongs to a specific class (V · 2 · 33) at time T_C and thus there holds conservation law (V · 2 · 37). It says that the spatial probability $\int_{-a}^{a} dX|\Psi(X,T)|^2$ is conserved *only for the time interval of* Ω. To be precise, the probability to find the particle at time T on the spatial slice $[-a, a]$ of Ω is independent of T only when $T \in [T_C, T_D]$, provided that no measurement is made before T. (This is indeed a favorable situation to introduce QP which is associated with Ω itself.) Hence the conservation law naturally specifies the meaning of $P(\mathrm{Yes})$ as

$$\begin{aligned}&\text{“Probability to find a particle in } \Omega \text{ is, for example, } \tfrac{1}{3}\text{”} \\ &\equiv \text{Probability to find a particle on the spatial slice of } \Omega \text{ is } \textit{always}\ \tfrac{1}{3}.\end{aligned} \qquad (\mathrm{V} \cdot 2 \cdot 43)$$

The right-hand side is stated in such terms that are directly related to an instantaneous measurement and therefore has no ambiguity in its meaning. Specification (V · 2 · 43) in turn clarifies what kind of measurement "to find a particle in Ω" is associated with. It is not associated with a continuous measurement distributed both in space and time but with a familiar instantaneous measurement which is distributed only in space. In this way, in the situation under which QP can be defined, the measurement theoretical meaning of ESV turns out to be clear. Except for the normalization problem, everything comes to gain a clear measurement theoretical meaning when $\Psi(C)$ belongs to specific class (V · 2 · 33). (In view of this, ESIV gains a status as an example of ES in which the normalization problem does not occur and measurement theoretical meaning is clear.)

The important point here is that the specific class has not been provided by hand but has been selected by the mathematical condition of vanishing interference, namely C·2, making no reference to measurements. Our framework has selected the favorable situation systematically. This convinces us that the framework has been successfully constructed.

The above observation tempts us to say that the QP can be defined for ESV *because* the spatial probability on a spatial slice of Ω is independent of the time of slicing. However we cannot tell whether such a reasoning is correct or not at this stage. The above discussion does *not* insist that the conservation of the spatial probability is the criterion for the definability of the spacetime probabilities. The criterion is the no-interference. The relation between the no-interference and conserved spatial probability is an interesting issue, which we want to study in the future.

Chap. VI. Summary, Discussion and Remaining problems

We have discussed whether QP (quantum mechanical probabilities) can be defined for a set of alternatives which are not restricted to a moment of time. We have investigated this problem within nonrelativistic quantum mechanics for a particle. By taking spacetime picture, we have concentrated on alternatives in Newtonian spacetime. We constructed a general framework in Chap. II and applied it in Chap. IV and V to concrete examples of ES with the help of the Euclidean lattice method reviewed and extended in Chap. III. The general framework provides two conditions: the classifiability condition (C·1) and the no-interference condition (C·2). QP-definability for a given ES is judged by examining whether these conditions hold or not. We obtained both negative and positive results. A lesson from negative results is that the success of C·1 depends on whether the ES serves as a good label in classifying virtual paths which are everywhere nondifferentiable with respect to time. If the ES is coarser than ESIII then C·1 will hold; otherwise it will fail. The positive-result cases (ESIV and V) are summarized as follows: Given an ES (ESIV or ESV) whose measurement theoretical meaning is left unspecified, our framework judges QP-definability for the ES by examining C·1 and C·2 making no reference to measurements and then concludes that QP is definable if an initial amplitude belongs to a specific class. Owing to the restriction of the initial amplitude to the specific class, the resultant probabilities are given clear measurement theoretical meanings within familiar measurement theory. This is somewhat surprising because measurement theoretical issues were completely vague (at least for the author) at the beginning of this study. In this way, everything looks like going well in the positive-result cases, except for the normalization problem for ESV.

Looking through the whole story summarized above, we shall discuss several issues to re-understand the meaning and the status of this study in quantum mechanics.

(1) About what we have actually studied: In the context of sum over paths, there is no conceptual difference between $\Phi(B;\Delta X,T;A)$ defined by (II · 1 · 4) and $\Phi(B;\mathcal{O}_j;A)$ defined by (II · 2 · 1). They are both amplitudes in that e^{iS} are summed up over suitable paths. From the former amplitude, positive quantity (II · 1 · 3) is constructed which fulfills axioms for probabilities. Since this positive quantity agrees with (I · 1 · 1), it is indeed a physical probability in quantum mechanics associated with a clear measurement theoretical meaning. Then two questions naturally arise: (i) Is the positive quantity constructed from the latter amplitude in the same way as (II · 1 · 3) also a probability? (ii) If it is a probability, then is it associated with a physical meaning? These are precisely what we have posed as problems and investigated by examples. The first question was answered in §2 of Chap. II; we formulated two conditions C·1 and C·2 for the positive quantity to be a probability. The second question was answered positively in Chap. V.

(2) About the Euclidean lattice method: We used the Euclidean lattice method to define and calculate the sum over paths (II · 2 · 1). At the stage of this writing, we cannot make a definite statement as to whether this method is wider than Feynman's path integral. However the author feels that it is wider. For example, the naive use of Feynman's path integral may not give Eq. (III · 2 · 35); it is also not sure whether Eq. (III · 2 · 25) is obtained in the context of Feynman's path integral. If the Euclidean lattice method turns out to be indeed wider than Feynman's path integral, one may then think that the results obtained in Chap. IV and V by use of the method are possibly outside of quantum mechanics. This is however not the case.

For ESI and II, although the failure of C·1 was proved by use of the Euclidean lattice method, the failure can be qualitatively understood without using the method as discussed in §2.2 of Chap. IV. Therefore the result of QP's undefinability for ESI and II is understandable without the Euclidean lattice method. For ESIII~V, the use of the Euclidean lattice method was only through the "first hitting amplitude" F defined by Eq. (III · 2 · 25) or by Eq. (III · 3 · 45) in general. Here we make the following observations:
(i) There was no need to know an analytic expression for F in investigating ESIII~V; the only property of F we used is that F satisfies the integral equations (III · 2 · 33) and (III · 2 · 34) ((III · 3 · 65) and (III · 3 · 66) in general).
(ii) These integral equations themselves can be written down without the knowledge of the Euclidean lattice method, as discussed in §3.2 of Chap. III. They directly follow from the very meaning of F as the first hitting amplitude.
(iii) It is considered that whether these integral equations have solutions or not depends on the behavior of the potential. We confine ourselves to such a class of potentials that guarantee the existence of the solutions. Then the first hitting amplitude F can be defined without the help of the Euclidean lattice method.
Therefore all the calculations made for ESIII~V can be carried out without the Euclidean lattice method.

Putting all these together, we can say that the results that QP is undefinable for ESI~III but definable for ESIV and V are not something peculiar which were born from the Euclidean lattice method but are understandable within the usual quantum mechanics.

(3) About measurement theory: To tell the truth, the author began this study as a purely logical problem in quantum mechanics without the confidence of finding probability-definable cases. Measurement theoretical issues were hence out of his scope. However, probability-definable cases have been found and they have naturally posed the problem of interpretation of the probabilities. It is not directly the aim of this thesis to provide a measurement theory for a general ES. Our main concern is to apply the rules of defining probabilities in the sum-over-paths quantum mechanics to an ES other than EST, where the precise meaning of an ES is a set of amplitudes $\{\Phi(B; \mathcal{O}_j; A)\}$. Measurement theoretical discussion enters into our investigation when probabilities are definable; when they are definable, it is expected that each of them is interpretable as the probability of the occurrence of $\mathcal{O}_j$. However the meaning of "the occurrence of $\mathcal{O}_j$" is not self-evident except for EST. Hence what we can do is to examine whether such a proposition as "the probability of the occurrence of $\mathcal{O}_j$ is p" can be given a definite meaning within the familiar measurement theory which deals with an instantaneous measurement. In probability-definable cases studied in Chap. V, definite meanings can be given to such propositions, meanings which are becoming to the values of the probabilities. Whether this is always true or not in probability-definable cases is an interesting problem. However, at the time of this writing, the author cannot say definite things about this problem. Although a general discussion is needed about probability-definable cases, the author does not have a definite idea about the discussion.

(4) About the physical meaning of interference: When the interference $\mathrm{Re}D[\mathcal{O}_j; \mathcal{O}_k]$ defined by Eq. (II · 2 · 6) does not vanish for $j \neq k$, the sum rule (II · 2 · 10) fails and probabilities cannot be defined. In this thesis, we have considered that the interference is a mathematical object which serves as the measure of inconsistency between the superposition principle for amplitudes and the sum rule for probabilities. It is however an interesting question whether the interference has some physical meaning. Again, the author cannot say definite things about this at the time of this writing. Instead here we shall discuss the interference in a concrete example. Decoherence functional $D[\mathrm{Yes}; \mathrm{No}]$ for ESV is written as

$$D[\mathrm{Yes}; \mathrm{No}] = \int dX_B D[B|\mathrm{Yes}; \mathrm{No}] \qquad (\mathrm{VI} \cdot 1 \cdot 1)$$

with

$$D[B|\mathrm{Yes}; \mathrm{No}] \equiv \Psi^*_{\mathrm{Yes}}(B)\Psi_{\mathrm{No}}(B) \qquad (\mathrm{VI} \cdot 1 \cdot 2)$$

and

$$\begin{aligned} \Psi_{\mathrm{Yes}}(B) &\equiv \int dX_A \Phi(B; \mathrm{Yes}; A)\Psi(A) \\ \Psi_{\mathrm{No}}(B) &\equiv \int dX_A \Phi(B; \mathrm{No}; A)\Psi(A). \end{aligned} \qquad (\mathrm{VI} \cdot 1 \cdot 3)$$

From Eq. (V · 2 · 18), it follows that

$$\Psi(B) = \Psi_{\mathrm{Yes}}(B) + \Psi_{\mathrm{No}}(B). \qquad (\mathrm{VI} \cdot 1 \cdot 4)$$

Furthermore, because of (II · 2 · 13), functions $\Psi_{\mathrm{Yes}}(B)$ and $\Psi_{\mathrm{No}}(B)$ obey the Schrödinger equation. Thus we can say that Schrödinger's wave function $\Psi(B)$ consists of two "partial waves", each of which also obeys the Schrödinger equation. The quantity Re$D[B|\text{Yes}; \text{No}]$ is the interference between the partial waves as understood from the observation that the quantity is the cross term which appears when the right-hand side of Eq. (VI · 1 · 4) is absolutely squared. Note that the word "interference" is used in the usual sense here. If physical meanings of the partial waves turn out to be clear, then Re$D[B|\text{Yes}; \text{No}]$ also acquires definite physical meaning, which in turn clarifies the meaning of Re$D[\text{Yes}; \text{No}]$. It must be noted that the question of the physical meanings of the partial waves can be posed separately from the question of definability of probabilities which we have pursued in this thesis. The physical meaning of a partial wave is a remaining problem.

(5) About the role of time: In a broad sense, our study belongs to the study of the role of time in quantum mechanics. By exploring the definability of probabilities for a general ES, we have all the more come to recognize a special property of EST. A special property of EST reflects a special role of time in quantum mechanics. Time play a special role in that a constant-time surface provides alternatives for which QP can be defined for an *arbitrary* (normalized) initial amplitude. By contrast, for ESIV and V, the initial amplitude cannot be arbitrary in order for QP to be definable. It is then conjectured that an ES for which probabilities can be defined for an arbitrary initial amplitude is limited to EST. Whether this is correct or not is a remaining problem. If the conjecture is correct, then time is understood as follows: A set of alternatives for which probabilities can be defined for an arbitrary initial amplitude forms a hypersurface; time is then defined as a parameter which parametrize a sequence of such surfaces. By the way, Newtonian time is already special in classical mechanics. One may therefore take it for granted that time has a special property in quantum mechanics. However classical mechanics is an approximation of quantum mechanics. A fundamental issue like "a special role of time" should be understood at a fundamental level. Our study made in this thesis is the first step toward the understanding the role of time from the viewpoint of probability-definability in quantum theory.

The following are the remaining problems:

(1) Application of our framework to other examples of ES, especially to ESV with non-rectangular spacetime domain with a potential, is worthy to be investigated. This will tell us something about the relation between conserved spatial probability and no-interference.

(2) In ESIV and V, for a specific class of initial amplitudes, decoherence functional itself vanishes before we take its real part. Whether this is characteristic to the two ES or not is an interesting problem.

(3) From the investigations so far, it is conjectured that an ES for which C·2 holds

for an arbitrary initial amplitude is limited to EST. This conjecture is worthy to be investigated. If the conjecture is correct, then the status of time in NRQM may be understood as follows: An ES for which QP is defined for an arbitrary (normalized) initial amplitude defines a surface of constant time.

(4) We have extended the Euclidean lattice method to the case of a nonzero time-independent potential for one particle in one dimension. Extension of the method to more general cases is an interesting issue. If such an extension is successfully made, the Euclidean lattice method will acquire a status as one formulation of quantum mechanics alternative to, or possibly wider or more flexible than, other formulations such as Hamiltonian, Path integral and stochastic quantization.

(5) To construct a relativistic version of our framework. In fact Hartle[8),9)] has constructed independently a general framework which is essentially the same as ours. His ultimate aim is to apply his framework to quantum cosmology. The present author is also interested in such a field, especially in the probability interpretation problem in quantum cosmology. Constructing a relativistic version is one of his long-term objectives.

Historical matter, author's contribution and comments:
The author started his investigation from studying Ref. 3). After the efforts of improving the normalization problem of some probabilities which Hartle discussed there, the author pursued the automatic normalization (see §2.2 of Chap. II) for these probabilities and came to recognize the need of vanishing of interferences between different alternatives, which led him to the no-interference condition. And finally he arrived at the general framework introduced in Chap. II. Historically, Griffith[12)] may be the first one who formulated the idea of defining probabilities for non-interfering alternatives. His interest was the logical interpretation of quantum mechanics; he considered a set of alternatives each of which is a sequence of events each of which is defined at a single moment of time. From our point of view, such alternatives are alternatives of EST but not the kind dealt with in this thesis. Gell-Mann and Hartle also formulated the idea of defining probabilities for non-interfering alternatives[8)] on the background of quantum cosmology. Their framework and our framework are essentially the same. The difference is that the decoherence functional was written down from the beginning in their framework, while the present author reached it by pursuing the automatic normalization. Anyway, Ref. 3) may be seen as the common starting point for them and for the present author, and constructions of respective frameworks were made independently. Applications of such a framework to concrete examples of ES other than EST were first made by the present author.

Sorkin has discussed the sum-over-paths rules of defining probabilities from amplitudes.[11)]

However he did not take account of the no-interference condition and hence he was faced with peculiar things such as causality violation in defining probabilities for alternatives which do not decohere. This should be compared with our results. We have defined probabilities only in the situation in which alternatives decohere. The resultant probabilities do not suffer from such peculiar things. His work and that of author's are mutually independent.

In Chap. III we converted the Schödinger equation into a Fokker-Planck equation by use of the transformation (III · 3 · 4). The transformation itself is a familiar one in the theory of Fokker-Planck equation.[13] The author's contribution lies in the construction of the random walk whose diffusion limit gives the continuous stochastic process described by the Fokker-Planck equation obtained from the Schödinger equation; he has found a random-walk representation of quantum mechanics as (III · 3 · 39). (At the time of this writing, he does not know of a literature dealing with such a representation.)

Just before completing this thesis, the author learned that the integral equation method of calculating the first hitting amplitude (cf. Eqs. (III · 3 · 48) $\sim$ (III · 3 · 52)) is, if the amplitude is converted into 'probability' by the Wick rotation, a familiar issue in the theory of stochastic process.[14] This in turn convinces us that the extension of the Euclidean lattice method has been successfully made; the method gives the same result as the integral equation method.

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Appendix A

A. 1 Counting lattice paths

'Probability' for an unrestricted random walk is given by formula (III · 2 · 1). It is obtained by counting the number of 'paths' contributing to the sum on the first right-hand side of (III · 2 · 1). Without loss of generality, let us consider the case $a = [0,0]$ and $b = [x,t]$ such that $x > 0$, $t > 0$; furthermore we assume that $x + t =$ even so that there is at least one 'path' which connects a and b. Since a 'path' moves forward in 'time', all the 'paths' connecting the end points consist of t steps and are thus associated with the same weight $(1/2)^t$. As understood from Fig. 32, a 'path' of t steps consists of $\frac{t-x}{2}$ steps of leftward walks and $\frac{t+x}{2}$ steps of rightward walks. Thus the number of 'paths' which connect the end points is the number of combination of choosing $\frac{t+x}{2}$ things out of t things, namely, $\binom{t}{\frac{t+x}{2}}$. The sum of the weight $(\frac{1}{2})^t$ over this number of 'paths' is just the product of the number of 'paths' and the weight. The result is (III · 2 · 1).

The sum over 'paths' (III · 2 · 16), which defines $f[x,t]$, is evaluated by the method of images. The result is written in terms of unrestricted random walks as the first right-hand side of Eq. (III · 2 · 19). Here we calculate the first right-hand side to derive the last right-hand side. Assuming $x > 0$ for simplicity and using (III · 2 · 1) and (III · 2 · 18), we have

$$\begin{aligned} f[x,t] &= \frac{1}{2^t}\left[\binom{t}{\frac{x+t}{2}} - \binom{t}{\frac{x+t}{2}+1}\right] \\ &= \frac{t!}{2^t}\left(\frac{1}{a!b!} - \frac{1}{(a+1)!(b-1)!}\right) \quad \left(a \equiv \frac{t+x}{2},\ b \equiv \frac{t-x}{2}\right) \\ &= \frac{t!}{2^t}\left[\frac{1}{(a+1)!b!}(a+1-b)\right] \\ &= \frac{1}{2^t}\frac{x+1}{t+1}\binom{t+1}{\frac{x+t}{2}+1}. \end{aligned} \tag{A · 1}$$

A. 2 Identities

(1) In the binomial formula

$$(1+z)^n = \sum_{j=0}^{n}\binom{n}{j} z^k, \tag{A · 2}$$

let us put $z = 1$, $n = t$ and $j = \frac{x+t}{2}$, changing the summation variable from j to x, to have

$$2^t = \sum_{x=-t}^{t} \frac{t!}{(\frac{t+x}{2})!(\frac{t-x}{2})!}, \tag{A · 3}$$

where the sum is taken over even integers when t is even or over odd integers when t is odd. Dividing both sides by 2^t, we have

$$\frac{1}{2^t} \sum_{x=-t}^{t} \binom{t}{\frac{x+t}{2}} = 1. \tag{A · 4}$$

This identity proves the normalization of the 'probability' (III · 2 · 1) for an unrestricted random walk.

(2) Because of the very meaning of $\bar{f}$ which is defined by Eq. (III · 2 · 21), it is evident that there holds the normalization (III · 2 · 23) for the restricted random walk. However, compared with the unrestricted random walk discussed just above, an analytical proof of the normalization is not so easy. For definiteness, we put $t = 2j$ and $x = 2n$ such that $j \geq n > 0$ in (III · 2 · 23). The identity to be proved is then

$$I(n) \equiv \sum_{j=n}^{\infty} \frac{1}{2^{2j}} \frac{n}{j} \binom{2j}{j+n} = 1 \qquad \text{for } ^{\forall} n \, (= 1,\ 2,\ \cdots). \tag{A · 5}$$

We use the method of generating function to prove the above. As a flexible expression for a binomial coefficient, we use

$$\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}. \tag{A · 6}$$

For $m < n$, it follows that $\binom{m}{n} = 0$. We introduce the following function of x and y, which we call the generating function of $I(n)$:

$$I(n|x, y) \equiv 2n \frac{\partial^{n-1}}{\partial x^{n-1}} \int_0^y dy_{n+1} \int_0^{y_{n+1}} dy_n \cdots \int_0^{y_2} dy_1 (1 - 4xy_1)^{-\frac{3}{2}}. \tag{A · 7}$$

By expanding the integrand in power series of x and y_1, carrying out all the integrals over $y_1 \cdots y_{n+1}$ and then differentiating $(n-1)$ times with respect to x, one can prove that

$$I(n) = I(n|\frac{1}{2}, \frac{1}{2}). \tag{A · 8}$$

We proceed as follows:
(i) We first perform y_k integrations in (A · 7) directly (not by expanding in power series), carry out x differentiations and then substitute $x = y = \frac{1}{2}$. The result is a finite

series of binomial coefficients.
(ii) Next we prove that the value of the finite series is unity. This proof is also made by introducing suitable generating functions.

By integrating $(1-4xy_1)^{-\frac{3}{2}}$ over $y_1 \cdots y_n$, we obtain

$$I(n|x,y) = 2n \frac{\partial^{n-1}}{\partial x^{n-1}} \left[\sum_{k=0}^{n} \frac{2k+1}{(-2x)^{k+1}(2k+1)!!} \frac{y^{n-k}}{(n-k)!} \right] \qquad (A \cdot 9)$$
$$+ \text{ terms which vanish when } x = y = \frac{1}{2}.$$

Dropping the vanishing terms, we carry out x differentiations and substitute $x = y = \frac{1}{2}$. The result is a finite series:

$$I(n) = I(n|\frac{1}{2}, \frac{1}{2}) = (-1)^n \left[1 + \sum_{k=1}^{n} (-4)^k \frac{n}{n+k} \binom{n+k}{2k} \right]. \qquad (A \cdot 10)$$

(At this stage one can numerically confirm that the right-hand side is certainly unity for concrete values of n. Such a numerical confirmation is impossible for the infinite series (III · 2 · 23).) Using the identity

$$\frac{n}{n+k}\binom{n+k}{2k} = \binom{n+k}{2k} - \frac{1}{2}\binom{n+k-1}{2k-1}, \qquad (A \cdot 11)$$

and $\binom{n}{0} = 1$, we have

$$I(n) = (-1)^n \left[\sum_{k=0}^{n} (-4)^k \binom{n+k}{2k} - \frac{1}{2} \sum_{k=1}^{n} (-4)^k \binom{n+k-1}{2k-1} \right]. \qquad (A \cdot 12)$$

To evaluate the right-hand side, we introduce the following functions of x:

$$A(x) \equiv \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-4)^k \binom{n+k}{2k} x^n \qquad (A \cdot 13)$$

$$B(x) \equiv \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-4)^k \binom{n+k-1}{2k-1} x^n. \qquad (A \cdot 14)$$

The two series on the right-hand side of (A · 12) are generated by these functions in the following manner:

$$\sum_{k=0}^{n}(-4)^k\binom{n+k}{2k} = \frac{1}{n!}\frac{d^n}{dx^n}A(x)\bigg|_{x=0}. \tag{A · 15}$$

$$\sum_{k=1}^{n}(-4)^k\binom{n+k-1}{2k-1} = \frac{1}{n!}\frac{d^n}{dx^n}B(x)\bigg|_{x=0}. \tag{A · 16}$$

To obtain a closed form of generating functions $A(x)$ and $B(x)$, we note that the expression (A · 6) allows us to extend the upper bounds of the k summations in (A · 13) and (A · 14) from n to ∞. After this extension, we exchange the order of n summation and k summation and use

$$\sum_{n=0}^{\infty}\binom{n+k}{2k}x^n = \frac{x^k}{(1-x)^{2k+1}} \tag{A · 17}$$

$$\sum_{n=1}^{\infty}\binom{n+k-1}{2k-1}x^n = \frac{x^k}{(1-x)^{2k}}, \tag{A · 18}$$

both of which can be proved by expanding the denominators on the right-hand sides in power series of x. Consequently we have

$$\begin{aligned} A(x) &= \sum_{k=0}^{\infty}(-4)^k\frac{x^k}{(1-x)^{2k+1}} \\ &= \frac{1}{1-x}\sum_{k=0}^{\infty}\left[\frac{-4x}{(1-x)^2}\right]^k \\ &= \frac{1-x}{(1+x)^2} \end{aligned} \tag{A · 19}$$

$$\begin{aligned} B(x) &= \sum_{k=1}^{\infty}(-4)^k\frac{x^k}{(1-x)^{2k}} \\ &= \sum_{k=1}^{\infty}\left[\frac{-4x}{(1-x)^2}\right]^k \\ &= \frac{-4x}{(1+x)^2}. \end{aligned} \tag{A · 20}$$

From Eqs. (A · 12), (A · 15), (A · 16), (A · 19) and (A · 20), it follows that

$$\begin{aligned} I(n) &= (-1)^n \frac{1}{n!}\frac{d^n}{dx^n}\left[A(x) - \frac{1}{2}B(x)\right]\Bigg|_{x=0} \\ &= (-1)^n \frac{1}{n!}\frac{d^n}{dx^n}\frac{1}{x+1}\bigg|_{x=0} \\ &= (-1)^n \frac{1}{n!}\frac{d^n}{dx^n}\frac{1}{x}\bigg|_{x=1} \\ &= 1. \end{aligned} \tag{A · 21}$$

This completes the proof of the identity (A · 5).

(3) Equation (III · 2 · 13) is an identity concerning binomial coefficients. As understood from Fig. 32, the range of the sum in (III · 2 · 13) is $x_b-(t_b-t_c) \leq x_c \leq x_b+(t_b-t_c)$. To be specific, let us take $a = [0,0]$, $b = [l,m]$ and $c = [j,n]$ such that $0 < n < m$ and both $l+m$ and $j+n$ are even. Equation (III · 2 · 13) then reads

$$\binom{m}{\frac{l+m}{2}} = \sum_{j=l-(m-n)}^{l+m-n} \binom{m-n}{\frac{l-j+m-n}{2}}\binom{n}{\frac{j+n}{2}}. \tag{A · 22}$$

We change the summation variable from j to $i = \frac{l-j+m-n}{2}$ (note that the numerator is even); we put $\frac{l+m}{2} = q$ and $m - n = p$. Equation (A · 22) then becomes

$$\binom{n+p}{q} = \sum_{i=0}^{p} \binom{p}{i}\binom{n}{q-i}. \tag{A · 23}$$

This is a well-known identity for binomial coefficients and is proved by expanding both sides of $(1+x)^{n+p} = (1+x)^p(1+x)^n$ in power series of x and comparing coefficients on both sides for the same power of x. Explicitly, the expansion in power series is

$$\sum_{q=0}^{n+p}\binom{n+p}{q}x^{n+p-q} = \sum_{r=0}^{p}\binom{p}{r}x^{p-r}\sum_{s=0}^{n}\binom{n}{s}x^{n-s}. \tag{A · 24}$$

Comparing the coefficients of x^{n+p-q} on both sides, we obtain (A · 23).

(4) Equation (III · 3 · 46) applied to a free particle is also an identity concerning binomial coefficients. Use expressions (III · 2 · 1) and (III · 2 · 22) and put for definiteness

$a = [l, 0]$, $b = [-m, n]$ and $c = [0, j]$ such that l, m, $n > 0$. Equation (III · 3 · 46) then reads

$$\binom{n}{\frac{l+m+n}{2}} = \sum_{j=l}^{n-m} \binom{n-j}{\frac{m+n-j}{2}} \frac{l}{j} \binom{j}{\frac{l+j}{2}}. \tag{A · 25}$$

At the time of this writing, the author has not found an analytical proof of this identity. Instead he has numerically confirmed it for concrete values of l, m and n. To give an analytical proof is left as an elementary problem in enumerative combinatorics.

(5) Equation (IV · 1 · 11) with (IV · 1 · 18) is also an identity concerning binomial coefficients whose analytical proof the author has not found. For definiteness, let us put $a = [k, 0]$ and $b = [-l, m]$ such that k, l, $m > 0$ and write $t_n = j_n$. The identity then takes the following form:

$$\binom{m}{\frac{k+l}{2}} = \sum_{n=1}^{\frac{m-l-k}{2}+1}{}' \sum_{j_n=k+2(n-1)}^{m-l} \sum_{j_{n-1}=k+2(n-2)}^{j_n-2} \cdots \sum_{j_1=k}^{j_2-2} \frac{l+1}{m-j_n+1} \binom{m-j_n+1}{\frac{l+m-j_n}{2}+1}$$
$$\times \frac{1}{j_n-j_{n-1}+1} \binom{j_n-j_{n-1}+1}{\frac{j_n-j_{n-1}}{2}} \times \cdots \times \frac{1}{j_2-j_1} \binom{j_2-j_1+1}{\frac{j_2-j_1}{2}} \frac{k+1}{j_1+1} \binom{j_1+1}{\frac{k+j_1}{2}+1}. \tag{A · 26}$$

The primed sum over n is restricted to odd integers. The author has numerically confirmed this identity for concrete values of k, l and m. An analytical proof is also left as a problem in enumerative combinatorics. Although analytical proofs for (A · 25) and (A · 26) are lacking, combinatoric proofs have been given. The latter have been obtained from manifestly correct 'path' classifications.

A. 3 DIFFUSION LIMIT

We exhibit the actual calculations of diffusion limits. We begin with demonstrating Eq. (III · 2 · 7). In the formula

$$u[b; a] = \frac{1}{2^t} \binom{t}{\frac{x+t}{2}}, \quad (t = t_b - t_a,\ x = x_b - x_a) \tag{A · 27}$$

t, $\frac{x+t}{2}$ and $\frac{x-t}{2}$ go to infinity in the diffusion limit (III · 2 · 6). We shall rewrite the right-hand side by use of Stirling's formula in advance which is

$$z! \simeq \sqrt{2\pi z} z^z e^{-z} \qquad \text{for } z \gg 1. \tag{A · 28}$$

We have

$$
\begin{aligned}
&\frac{1}{2^t}\binom{t}{\frac{x+t}{2}} \\
&\simeq \frac{1}{2^t}\sqrt{2\pi t}t^t e^{-t}\left[\sqrt{\pi(t-x)}\left(\frac{t-x}{2}\right)^{\frac{t-x}{2}} e^{-\frac{t-x}{2}}\sqrt{\pi(t+x)}\left(\frac{t+x}{2}\right)^{\frac{t+x}{2}} e^{-\frac{t+x}{2}}\right]^{-1} \\
&= \sqrt{\frac{2t}{\pi(t^2-x^2)}}\frac{t^t}{(t-x)^{\frac{t-x}{2}}(t+x)^{\frac{t+x}{2}}} \\
&= \sqrt{\frac{2}{\pi t(1-\delta^2)}}(1-\delta^2)^{-\frac{t}{2}}\left(\frac{1-\delta}{1+\delta}\right)^{\frac{x}{2}} \qquad (\delta \equiv \frac{x}{t}).
\end{aligned}
\tag{A·29}
$$

Here we use (III · 2 · 3) to have

$$
\frac{1}{2\eta_1}u[b;a] = \sqrt{\frac{1}{2\pi\tau}\frac{\eta_2}{\eta_1^2(1-\Delta^2)}}(1-\Delta^2)^{-\frac{\tau}{2\eta_2}}\left(\frac{1-\Delta}{1+\Delta}\right)^{\frac{X}{2\eta_1}}, \quad \Delta \equiv \frac{\eta_2 X}{\eta_1 \tau}. \tag{A·30}
$$

In the diffusion limit, we keep the ratio $\eta_2/\eta_1^2 = m$ fixed and let $\eta_1 \to 0$. Since $\Delta = \eta_1 mX/\tau \to 0$ in the limit, the first factor approaches $\sqrt{\frac{m}{2\pi\tau}}$; the second and the third factor are respectively evaluated as follows:

$$
(1-\Delta^2)^{-\frac{mX^2}{\Delta^2 2\tau}} \to e^{\frac{mX^2}{2\tau}}, \quad (1\mp\Delta)^{\pm\frac{mX^2}{\Delta 2\tau}} \to e^{-\frac{mX^2}{2\tau}}. \tag{A·31}
$$

Consequently we obtain

$$
\mathrm{Lim}\frac{1}{2\eta_1}u[b;a] = \sqrt{\frac{m}{2\pi\tau}}e^{\frac{mX^2}{2\tau}}e^{-\frac{mX^2}{\tau}} = \sqrt{\frac{m}{2\pi\tau}}e^{-\frac{mX^2}{2\tau}}. \tag{A·32}
$$

Next we calculate the diffusion limit on the left hand-side of (III · 2 · 24). This is now easy because (we assume $X > 0$ for simplicity)

$$
\bar{f}[x,t] = \frac{x}{t}u[x,t;0,0], \tag{A·33}
$$

as seen from (III · 2 · 1) and (III · 2 · 22). The limit is evaluated as follows:

$$
\begin{aligned}
\mathrm{Lim}\frac{\bar{f}[X/\eta_1,\tau/\eta_2]}{2\eta_2} &= \mathrm{Lim}\frac{1}{2\eta_2}\frac{X/\eta_1}{\tau/\eta_2}u[X/\eta_1,\tau/\eta_2;0,0] \\
&= \frac{X}{\tau}\mathrm{Lim}\frac{u[X/\eta_1,\tau/\eta_2;0,0]}{2\eta_1} \\
&= \frac{X}{\tau}\sqrt{\frac{m}{2\pi\tau}}\exp\left(-\frac{mX^2}{2\tau}\right),
\end{aligned}
\tag{A·34}
$$

giving (III · 2 · 24). By the way, the above quantity, namely $F(X,\tau)$, is normalized to

$$\int_0^\infty d\tau F(X,\tau) = 1. \tag{A · 35}$$

(This can be proved by carrying out the time integral explicitly.) This is the continuum counterpart of (III · 2 · 23). Actually (A · 35) can be obtained as the diffusion limit of (III · 2 · 23); in the limit $\sum_t 2\eta_2$ goes over to $\int d\tau$ and $\bar{f}$ divided by $2\eta_2$ tends to F to give (A · 35).

Appendix B

Confining ourselves to a free particle, we solve the Volterra integral equation (III · 3 · 48). Noting $\Phi(T|X_B,X_A) \equiv \Phi(X_B,T;X_A,0)$, we substitute expression (III · 2 · 10) for the free propagator into the right-hand side of Eq. (III · 3 · 50) to have (we use the convention $m = 1$ here)

$$\begin{aligned}\phi(s|X_B,X_A) &= \int_0^\infty dT e^{-sT}\sqrt{\frac{1}{2\pi iT}}e^{i\frac{X^2}{2T}} \quad (X \equiv X_B - X_A)\\ &= \sqrt{\frac{2}{\pi i}}\int_0^\infty d\lambda \exp\left\{-\left(s\lambda^2 - \frac{iX^2}{2\lambda^2}\right)\right\} \\ &= \sqrt{\frac{1}{2is}}\exp\left(-\sqrt{2sX^2/i}\right),\end{aligned} \tag{B · 1}$$

where $\sqrt{i} \equiv \exp(i\pi/4)$. Substituting this into Eq. (III · 3 · 49) and taking account of $X_BX_A < 0$, we have

$$F(T|X) = -\frac{1}{2\pi}\int_{\epsilon-i\infty}^{\epsilon+i\infty} ds \exp(-f(s)), \tag{B · 2}$$

where

$$f(s) \equiv \sqrt{2sX^2/i} - Ts, \quad \epsilon > 0,\ T > 0. \tag{B · 3}$$

The constant γ in Eq. (III · 3 · 49) has been chosen to be an infinitesimal positive quantity ϵ so that the integration contour sees the singularity $s = 0$ on its left-hand side; the contour is shown in Fig. 33(a). Introduction of the cut which runs from $s = 0$ to $-i\infty$ restricts θ, the phase of s, to the range

$$-\frac{\pi}{2} < \theta < \frac{3}{2}\pi. \tag{B·4}$$

This guarantees the integrand to be single valued. Let us explore the behavior of the integrand on the complex s-plane. Using the polar representation $s = re^{i\theta}$, we have

$$f(s) = \sqrt{2X^2 r}\left(\cos\frac{1}{2}(\theta - \frac{\pi}{2}) + i\sin\frac{1}{2}(\theta - \frac{\pi}{2})\right) - T(\cos\theta + i\sin\theta). \tag{B·5}$$

Because of the restriction (B · 4),

$$\text{Re } f(s) > 0 \ \text{ on the left half-plane } \frac{1}{2}\pi < \theta < \frac{3}{2}\pi. \tag{B·6}$$

This enables us to deform our contour as shown in Fig. 33(b). The resultant contour is $C_R + C_L$. We note that r ranges from ∞ to 0 and $\theta = -\frac{\pi}{2}$ on C_R, while, on C_L, r ranges from 0 to ∞ and $\theta = \frac{3}{2}\pi$. Therefore

$$\begin{aligned} F(T|X) &= -\frac{1}{2\pi}\left[\int_\infty^0 d(re^{-\frac{\pi}{2}i})\exp\left(-f(re^{-\frac{\pi}{2}i})\right) + \int_0^\infty d(re^{\frac{3\pi}{2}i})\exp\left(-f(re^{\frac{3\pi}{2}i})\right)\right] \\ &= -\frac{i}{2\pi}\int_0^\infty dr\left(e^{i\sqrt{Ar}} - e^{-i\sqrt{Ar}}\right)e^{-iTr} \quad (A \equiv 2X^2) \\ &= -\frac{i}{\pi T}\int_0^\infty dq\, q\left(e^{iBq} - e^{-iBq}\right)e^{-iq^2} \quad \left(q^2 = Tr, \quad B \equiv \sqrt{\frac{A}{T}}\right) \\ &= -\frac{i}{\pi T}\int_{-\infty}^\infty dq\, qe^{i(Bq-q^2)} \\ &= -\frac{1}{\pi T}\frac{\partial}{\partial B}\int_{-\infty}^\infty dqe^{i(Bq-q^2)} \\ &= \left[\frac{X^2}{2\pi(iT)^3}\right]^{1/2}\exp\left(i\frac{X^2}{2T}\right). \end{aligned} \tag{B·7}$$

The result correctly reproduces expression (III · 2 · 25) calculated in the scheme of the Euclidean lattice method (III · 2 · 12). (Note $F(T|X) \equiv F(X, iT)$.)

By the way, the Laplace transform (B · 1) can be alternatively calculated by use of the formula (III · 3 · 52). Since we are dealing with a free particle, the formula gives

$$\begin{aligned}\phi(s|X_B,X_A) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dK \frac{e^{iKX}}{s+iK^2/2}, \quad (X \equiv X_B - X_A)\\ &= \frac{-i}{2\pi K_0}\int_{-\infty}^{\infty} dK \left(\frac{e^{iKX}}{K-K_0} - \frac{e^{iKX}}{K+K_0}\right), \quad (K_0 \equiv \sqrt{2is})\end{aligned} \tag{B · 8}$$

where K is the wave vector in the units $\hbar/m = 1$; Re$s > 0$ because γ in Eq. (III · 3 · 49) is positive. It is enough to calculate (B · 8) for positive s. The result can be extended over the right half plane by the analytic continuation. For definiteness, let us consider the case $X > 0$. Then the integral contour can be closed by an infinite semicircle in the upper half plane which contribute nothing to the integral (see Fig. 34). The second term on the last right-hand side of (B · 8) is analytic on the upper half plane and thus contribute nothing to the integral. The contribution from the first term is the residue at $K = K_0$ multiplied by $2\pi i$. Consequently,

$$\begin{aligned}\phi(s|X_A,X_B) &= \frac{-i}{2\pi K_0} 2\pi i e^{iK_0X}\\ &= \sqrt{\frac{1}{2is}}\exp\left(i\sqrt{2is}X\right) \qquad \text{for } X = X_B - X_A > 0.\end{aligned} \tag{B · 9}$$

Since the phase of the exponential term of the above and that on the last right-hand side of Eq. (B · 1) are equally $\frac{3}{4}\pi$, Eq. (B · 9) coincides with Eq. (B · 1). The case $X < 0$ also results in Eq. (B · 9) with X replaced by $-X$. The results are unified into the form of the last right-hand of Eq. (B · 1).

Appendix C

We prove Eq. (IV · 1 · 16), revealing the condition under which it holds. We begin by rewriting Eq. (IV · 1 · 12) as

$$\Phi_E(\beta;\alpha) = \sum_{n=1}^{\infty}{}' \frac{1}{n!}\prod_{j=1}^{n}\int_{\tau_\alpha}^{\tau_\beta} d\tau_j \sum_{\text{perm}} \Phi_{E_n}(\beta;\tau_n,\cdots,\tau_1;\alpha), \tag{C · 1}$$

where $\sum_{\text{perm}}$ is the sum over all permutations of the τ_j in Φ_{E_n}. Performing the Wick rotation such that $\tau_\alpha = iT_A$ and $\tau_\beta = iT_B$ and putting $X_\alpha = X_A$ and $X_\beta = X_B$, we have

$$\Phi(B;A) = \sum_{n=1}^{\infty}{}' \frac{1}{n!} \prod_{j=1}^{n} \int_{iT_A}^{iT_B} d\tau_j \sum_{\text{perm}} \Phi_{E_n}(B;\tau_n,\cdots,\tau_1;A)\,. \tag{C·2}$$

The contour of each $\int_{iT_A}^{iT_B} d\tau_j$ in the complex τ_j plane is such that its end points are fixed on the imaginary τ_j axis at iT_A and iT_B. Apart from this restriction, it is arbitrary on the assumption that $\sum_{\text{perm}} \Phi_{E_n}$ is analytic with respect to $\tau_1,\cdots,\tau_n$ for all odd n. We can then write $\tau_j = iT_j$, where T_j are real variables. In terms of T_j, Eq. (C · 2) becomes

$$\begin{aligned}\Phi(B;A) &= \sum_{n=1}^{\infty}{}' \frac{i^n}{n!} \prod_{j=1}^{n} \int_{T_A}^{T_B} dT_j \sum_{\text{perm}} \Phi_{E_n}(B;iT_n,\cdots,iT_1;A) \\ &= \sum_{n=1}^{\infty}{}' i^n \int d\vec{T}\, \Phi_{E_n}(B;iT_n,\cdots,iT_1;A)\,,\end{aligned} \tag{C·3}$$

where the last multiple time integral is time ordered such that $T_A < T_1 < \cdots < T_n < T_B$. Identifying this with Eq. (IV · 1 · 13), we have Eq. (IV · 1 · 16).

Appendix D

We prove that a symmetric function $\Psi(C)$ which satisfies Eq. (V · 1 · 16) is identically zero. We assume that there is no potential. We then substitute expression (III · 2 · 10) into Eq. (V · 1 · 16) to have

$$\int_{-\infty}^{0} dX e^{-iKX} f(X) = 0 \qquad \text{for } {}^\forall K \geq 0, \tag{D·1}$$

where we have put $\frac{X_D}{T_D - T_C} = K$ and $\exp(i\frac{X_C^2}{2(T_D-T_C)})\Psi(C) = f(X_C)$ and have written X_C simply as X. We can differentiate both sides of (D · 1) with respect to $K(>0)$. After n times differentiation, we have

$$\int_{-\infty}^{0} dX e^{-iKX} X^n f(X) = 0 \qquad \text{for } {}^\forall K > 0. \tag{D·2}$$

Let the set of coefficients $\{C_n | n = 0,1,2,\cdots\}$ be defined by

$$C_n = \left[\frac{1}{n!}\frac{d^n}{dX^n}\left(f(X)e^{-iKX}\right)\right]^*_{X=0}, \tag{D·3}$$

so that

$$\sum_{n=0}^{\infty} C_n X^n = f^*(X) e^{iKX}. \qquad (\mathrm{D}\cdot 4)$$

Let us multiply both sides of (D · 2) by C_n and sum up over all n. The result is

$$\int_{-\infty}^{0} dX\, |f(X)|^2 = 0. \qquad (\mathrm{D}\cdot 5)$$

This proves $f(X) = 0$ and hence $\Psi(C) = 0$ for $X < 0$. Recall that $\Psi(C)$ in Eq. (V · 1 · 16) is symmetric. Thus we conclude

$$\Psi(C) = 0 \qquad \text{for } {}^{\forall}X_C. \qquad (\mathrm{D}\cdot 6)$$

Appendix E

Here we calculate the decoherence functional for ESV explicitly to derive Eq. (V · 2 · 28). In this appendix, we use the abbreviation $C_\pm$ for $\bar{C}_\pm$ defined by Eq. (V · 2 · 22). First we calculate $\int dX_A \Phi(B; \mathrm{Yes}; A)\Psi(A)$. From Eq. (V · 2 · 23), we have

$$\begin{aligned}
&\int dX_A \Phi(B; \mathrm{Yes}; A) \\
&= \int_{-\infty}^{-a} dX_D \int_{-a}^{a} dX_C \Phi(B; D)\Phi(D; C)\Psi(C) && \leftarrow 1 \\
&+ \int_{-a}^{a} dX_D \int_{-a}^{a} dX_C \Phi(B; D)\Phi(D; C)\Psi(C) && \leftarrow 2 \\
&+ \int_{a}^{\infty} dX_D \int_{-a}^{a} dX_C \Phi(B; D)\Phi(D; C)\Psi(C) && \leftarrow 3 \\
&+ \int_{-\infty}^{-a} dX_D \int_{-\infty}^{-a} dX_C \Phi(B; D)\Phi(D; C_-)\Psi(C) && \leftarrow 4 \\
&+ \int_{-a}^{\infty} dX_D \int_{-\infty}^{-a} dX_C \Phi(B; D)\Phi(D; C)\Psi(C) && \leftarrow 5 \\
&+ \int_{-\infty}^{a} dX_D \int_{a}^{\infty} dX_C \Phi(B; D)\Phi(D; C)\Psi(C) && \leftarrow 6 \\
&+ \int_{a}^{\infty} dX_D \int_{a}^{\infty} dX_C \Phi(B; D)\Phi(D; C_+)\Psi(C), && \leftarrow 7
\end{aligned}$$

$$\cdots(\mathrm{E}\cdot 1)$$

where $\Psi(C) \equiv \int dX_A \Phi(C;A)\Psi(A)$. Let us introduce the following abbreviation:

$$\begin{aligned}\Phi_{\pm}(D;C) &\equiv \Phi(D;C) - \Phi(D;C_{\pm}) \\ &= \Phi(D;C) - \Phi(D;-X_C \pm 2a, T_C).\end{aligned} \qquad (\mathrm{E}\cdot 2)$$

Equation (V · 2 · 24) then gives

$$\begin{aligned}&\int dX_A \Phi(B;\mathrm{No};A)\Psi(A) \\ &= \int_{-\infty}^{-a} dX_D \int_{-\infty}^{-a} dX_C \Phi(B;D)\Phi_-(D;C)\Psi(C) && \leftarrow 8 \\ &+ \int_{a}^{\infty} dX_D \int_{a}^{\infty} dX_C \Phi(B;D)\Phi_+(D;C)\Psi(C). && \leftarrow 9\end{aligned} \qquad \cdots(\mathrm{E}\cdot 3)$$

We substitute Eqs. (E·1) and (E·3) into Eq. (V · 2 · 25) and carry out the X_B integral by use of

$$\int dX_B \Phi^*(B;D)\Phi(B;D') = \delta(X_D - X_{D'}) \quad (T_{D'} = T_D). \qquad (\mathrm{E}\cdot 4)$$

The following combinations survive after the integration: 1×8, 4×8, 6×8, 3×9, 5×9 and 7×9, giving

$$\begin{aligned}&D[\mathrm{Yes};\mathrm{No}] \\ &= \int_{-\infty}^{-a} dX_D \left[\left(\int_{-a}^{a} dX_C \Phi(D;C)\Psi(C)\right)^* \int_{-\infty}^{-a} dX_C \Phi_-(D;C)\Psi(C)\right] && \leftarrow 1\times 8 \\ &+ \int_{a}^{\infty} dX_D \left[\left(\int_{-a}^{a} dX_C \Phi(D;C)\Psi(C)\right)^* \int_{a}^{\infty} dX_C \Phi_+(D;C)\Psi(C)\right] && \leftarrow 3\times 9 \\ &+ \int_{-\infty}^{-a} dX_D \left[\left(\int_{-\infty}^{-a} dX_C \Phi(D;C_-)\Psi(C)\right)^* \int_{-\infty}^{-a} dX_C \Phi_-(D;C)\Psi(C)\right] && \leftarrow 4\times 8 \\ &+ \int_{a}^{\infty} dX_D \left[\left(\int_{a}^{\infty} dX_C \Phi(D;C_+)\Psi(C)\right)^* \int_{a}^{\infty} dX_C \Phi_+(D;C)\Psi(C)\right] && \leftarrow 7\times 9 \\ &+ \int_{a}^{\infty} dX_D \left[\left(\int_{-\infty}^{-a} dX_C \Phi(D;C)\Psi(C)\right)^* \int_{a}^{\infty} dX_C \Phi_+(D;C)\Psi(C)\right] && \leftarrow 5\times 9 \\ &+ \int_{-\infty}^{-a} dX_D \left[\left(\int_{a}^{\infty} dX_C \Phi(D;C)\Psi(C)\right)^* \int_{-\infty}^{-a} dX_C \Phi_-(D;C)\Psi(C)\right]. && \leftarrow 6\times 8\end{aligned} \qquad \cdots(\mathrm{E}\cdot 5)$$

To arrange the right-hand side, we note a property of $\Phi_{\pm}(D;C)$ in the presence of a symmetric potential (our potential (V · 2 · 14) is symmetric). With the notation

$$\bar{D} \equiv (-X_D, T_D), \quad \bar{C} \equiv (-X_C, T_C), \tag{E · 6}$$

we see

$$\begin{aligned}\Phi_{\pm}(\bar{D};\bar{C}) &= \Phi(\bar{D};\bar{C}) - \Phi(\bar{D}; X_C \pm 2a, T_C) \\ &= \Phi(D;C) - \Phi(D; -X_C \mp 2a, T_C) \\ &= \Phi_{\mp}(D;C),\end{aligned} \tag{E · 7}$$

where the second right-hand side follows from (V · 2 · 27) which is correct for a symmetric potential. Let us concentrate on the first term (1 × 8) on the right-hand side of Eq. (E·5). Change the integration variables (two X_C's and X_D) according to $X_C \to -X_C$ and $X_D \to -X_D$ and then use the property (E · 7). The result is the second term (3 × 9) with two $\Psi(C)$ replaced by $\Psi(\bar{C})$. Same thing occurs for pairs {4 × 8, 7 × 9} and {5 × 9, 6 × 8}. Accordingly we introduce the following abbreviation:

$$\begin{aligned}\Psi(C;C') &\equiv \Psi^*(C)\Psi(C') + \Psi^*(\bar{C})\Psi(\bar{C}') \\ &= \Psi^*(X_C)\Psi(X_{C'}) + \Psi^*(-X_C)\Psi(-X_{C'}).\end{aligned} \tag{E · 8}$$

Equation (E·5) is arranged as

$$\begin{aligned}&D[\text{Yes};\text{No}] \\ &= \int_a^\infty dX_D \int_{-a}^a dX_C \int_a^\infty dX_{C'} \Phi^*(D;C)\Phi_+(D;C')\Psi(C;C') \\ &+ \int_a^\infty dX_D \int_a^\infty dX_C \int_a^\infty dX_{C'} \Phi^*(D;C_+)\Phi_+(D;C')\Psi(C;C') \\ &+ \int_a^\infty dX_D \int_{-\infty}^{-a} dX_C \int_a^\infty dX_{C'} \Phi^*(D;C)\Phi_+(D;C')\Psi(C;C') \\ &= \int_a^\infty dX_D \int_{-\infty}^a dX_C \int_a^\infty dX_{C'} \Phi^*(D;C)\Phi_+(D;C')\Psi(C;C') \\ &+ \int_a^\infty dX_D \int_a^\infty dX_C \int_a^\infty dX_{C'} \Phi^*(D;C_+)\Phi_+(D;C')\Psi(C;C').\end{aligned} \tag{E · 9}$$

By the change of integration variables $X_C - a \to X_C$ and $X_{C'} - a \to X_{C'}$, the constant a disappears from the upper and the lower bounds of X_C and $X_{C'}$ integrals. We succeedingly change the integration variable X_C by $X_C \to -X_C$ in the first term on the last right-hand side of Eq. (E · 9). Then the two integrals are unified on the last right-hand side. Explicitly,

$$
\begin{aligned}
D[\text{Yes};\text{No}] = \int_a^\infty dX_D \int_0^\infty dX_C \int_0^\infty dX_{C'} \\
&\times \Phi^*(D; -X_C + a, T_C)\left(\Phi(D; X_{C'} + a, T_C) - \Phi(D; -X_{C'} + a, T_C)\right) \\
&\times \Big\{ \left(\Psi(X_C - a) + \Psi(-X_C - a)\right)^* \Psi(-X_{C'} - a) \\
&\quad + \left(\Psi(X_C + a) + \Psi(-X_C + a)\right)^* \Psi(X_{C'} + a) \Big\}.
\end{aligned}
\tag{E $\cdot$ 10}
$$

This is Eq. (V · 2 · 28). The special case $a = 0$ gives the decoherence functional for ESIV, namely, Eq. (V · 1 · 13).

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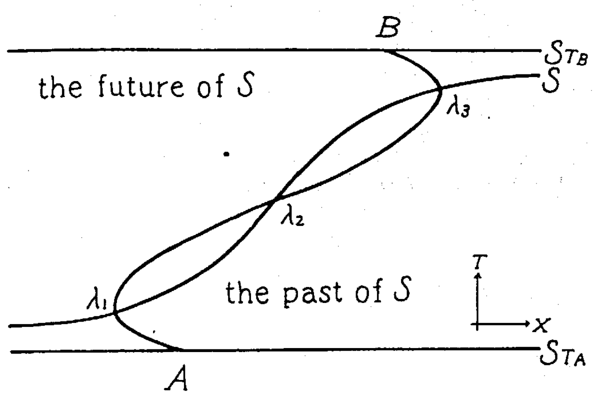

Fig. 1

Intersection of a classical path with a surface S in (1 + 1)-dimensional Newtonian spacetime. A classical path is shown which connects two points A and B lying in the opposite sides of S and which intersects S three times at λ_1, λ_2 and λ_3.

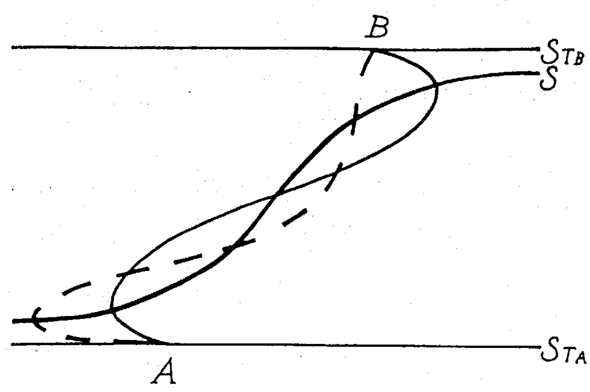

Fig. 2

Two examples of virtual paths which intersect S three times on the way from A to B. Quantum mechanically many virtual paths contribute to the motion from A to B; they intersect S odd number of times, and the number and the places of intersection vary from a virtual path to another.

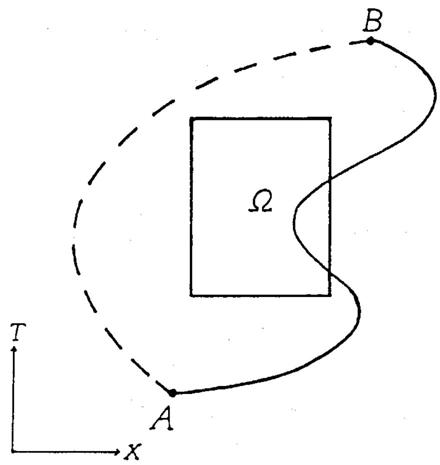

Fig. 3

A spacetime domain Ω. The solid curve is an example of paths which contribute to "Yes"and the broken one to "No".

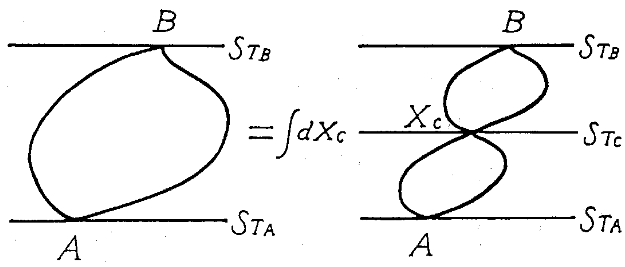

Fig. 4

The path-classification which leads to the composition law (II · 1 · 16). The diagrams on the left- and the right-hand side are the abbreviations of the paths defining $\Phi(B; A)$ and $\Phi(B; C)\Phi(C; A)$, respectively.

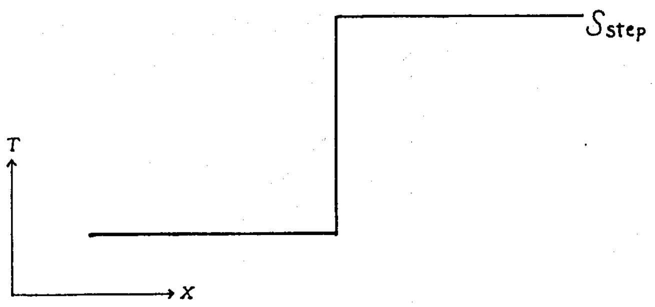

Fig. 5

A steplike surface $\mathcal{S}_{\mathbf{step}}$.

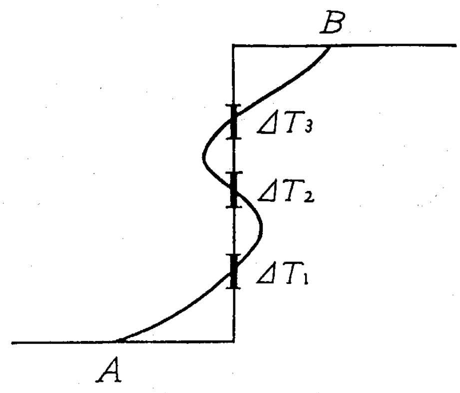

Fig. 6

A path which contributes to sum (III · 1 · 1) with $n = 3$. The paths defining the sum are specified by the set $\Delta\vec{T}_3$ of temporal intervals ΔT_j on the vertical part of $\mathcal{S}_{\mathbf{step}}$.

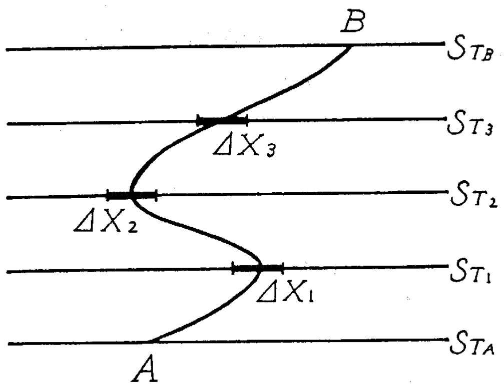

Fig. 7

A path passes through the cylindrical set $\Delta\vec{X}_3$ made up of spatial intervals ΔX_j on surfaces $\mathcal{S}_{T_j}$. The usual "time slicing" definition of the sum over paths is capable of summing e^{iS} over the paths specified by the cylindrical set $\Delta\vec{X}_3$, namely, all possible paths which start from A, move forward in time to pass through the set $\Delta\vec{X}_3$ and arrive at B; this sum is expressed as (III · 1 · 2) with $n = 3$. By contrast, the set $\Delta\vec{T}_3$ in Fig. 6 is not a cylindrical set because it is not expressible as a set of spatial intervals on surfaces of constant time.

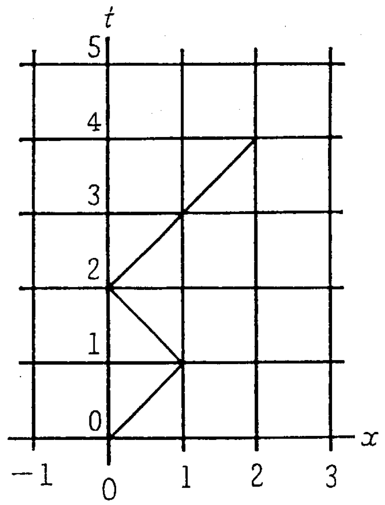

Fig. 8

A random walk on a spacetime lattice $[x, t]$. A 'particle' starting from [0,0] moves one step in 'time' to [-1,1] with a 'probability' 1/2 or to [1,1] with the same 'probability'. When $x + t$ is odd there are in general many 'paths' of t steps which move forward in 'time' to link $[0, 0]$ to $[x, t]$. The zigzag diagram in the figure is an example of such 'paths'. The sum of $(1/2)^t$ over all such 'paths' gives the 'probability' $u[x, t; 0, 0]$ that the position of the 'particle' at 'time' t is x, provided that it started from $[0, 0]$.

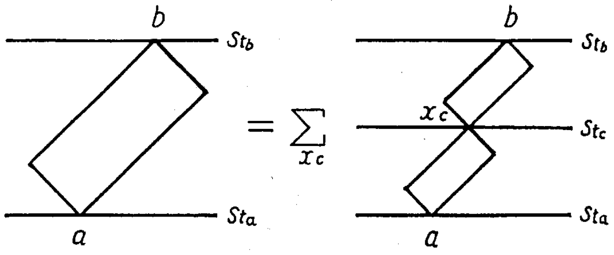

Fig. 9

A classification of 'paths'. The diagram on the left-hand side stands for the 'paths' defining $u[b;a]$, namely, all possible 'paths' which start from a, move forward in 'time' and arrive at b. Since all the 'paths' intersect s_{tc} once and only once, they can be classified with respect to intersection x_c of a 'path' with s_{tc}. The diagram on the right-hand side stands for the 'paths' defining $u[b;c]u[c;a]$; the sum of such 'paths' over all intersections recovers the 'paths' on the left-hand side.

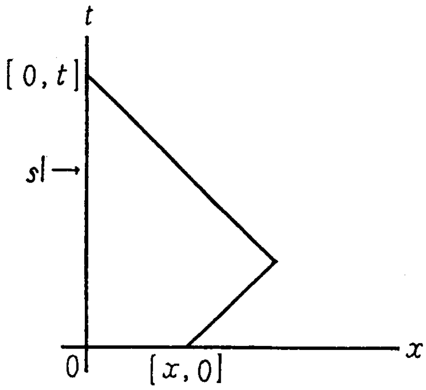

Fig. 10

The 'paths' which define $f[x,t]$. The bent line is the abbreviation of all possible 'paths' which start from $[x,0]$, move forward in 'time' never intersecting $s|$ of $x = 0$ before 'time' t, and arrive at $[0,t]$ on $s|$. (Note that 'paths' are allowed to touch $s|$ even before 'time' t.) The sum of $(1/2)^t$ over all such 'paths' defines $f[x,t]$.

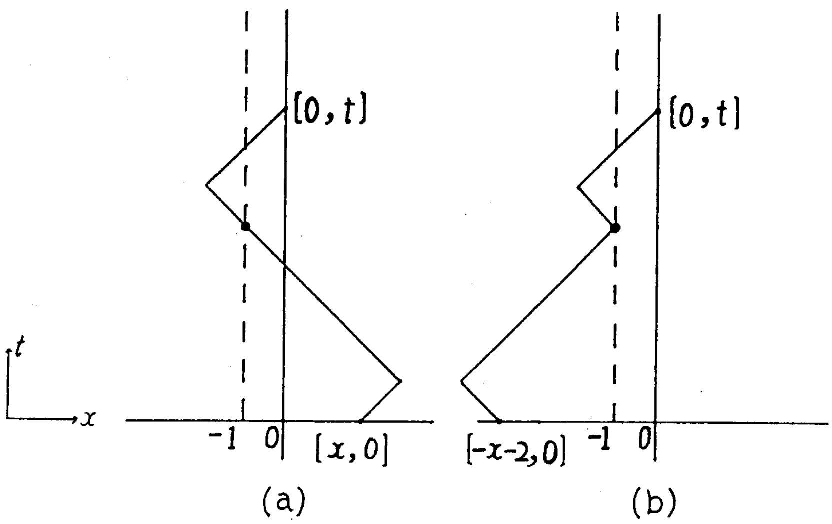

Fig. 11

(a)An example of 'paths' which invade the region $x < 0$ on the way from $[x,0]$ to $[0,t]$. This 'path' can be divided into two parts: a partial 'path' from $[x,0]$ to the first hitting of $x = -1$ and a partial 'path' from the first hitting to $[0,t]$. (b)The partial 'path' in (a) from $[x,0]$ to the first hitting is mirror reflected with respect to $x = -1$ and the partial 'path' from the first hitting to $[0,t]$ is unchanged. The resultant 'path' is of equal weight with the original 'path' in (a). In this way, a 'path' invading the region $x < 0$ is mirror reflected before hitting $x = -1$ to give a corresponding 'path' of equal weight which connects $[-x-2,0]$ and $[0,t]$. This correspondence is one-to-one.

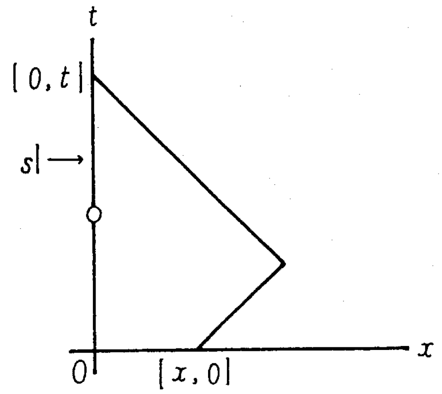

Fig. 12

The 'paths' which define $\bar{f}[x,t]$. The bent line is the abbreviation of all possible 'paths' which start from $[x,0]$, move forward in 'time' never intersecting nor touching $s|$ of $x = 0$ before 'time' t, and arrive at $[0,t]$ on $s|$. The sum of $(1/2)^t$ over all such 'paths' defines $\bar{f}[x,t]$. In contrast to the case of $f[x,t]$ defined in Fig. 10, 'paths' are not allowed to touch $s|$; this restriction is denoted by the mark o attached to $s|$.

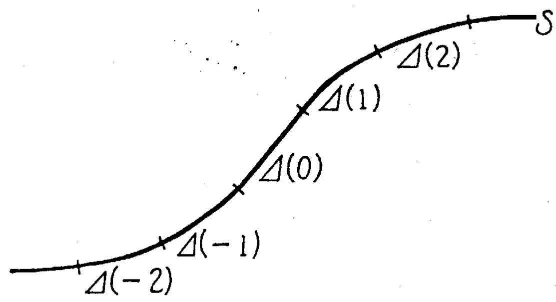

Fig. 13

A general surface $\mathcal{S}$ divided into a countable set of non-overlapping domains $\{\Delta(l) \mid l = 0, \pm1, \pm2, \cdots\}$. The figure shows the (1 + 1)-dimensional case. The sample of ESI such that the particle intersects $\mathcal{S}$ at the domains $\Delta(-1)$, $\Delta(0)$ and $\Delta(2)$ is denoted by $\Delta(\vec{l}_3)$ with $\vec{l}_3 = (-1, 0, 2)$. A general sample of ESI is denoted by $\Delta(\vec{l}_n)$.

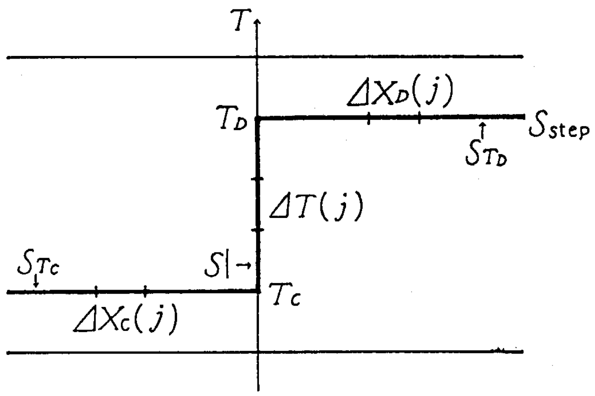

Fig. 14

The steplike surface $\mathcal{S}_{\mathbf{step}}$ made up of three surfaces $\mathcal{S}_{T_C}$ $(X_C < 0)$, $\mathcal{S}_{T_D}$ $(X_D > 0)$ and $\mathcal{S}|$, where $\mathcal{S}|$ stands for the surface of $X = 0$ for $T_C < T < T_D$. A domain $\Delta(j)$ on $\mathcal{S}_{\mathbf{step}}$ is $\Delta X_C(j)$ on $\mathcal{S}_{T_C}$, $\Delta T(j)$ on $\mathcal{S}|$ or $\Delta X_D(j)$ on $\mathcal{S}_{T_D}$.

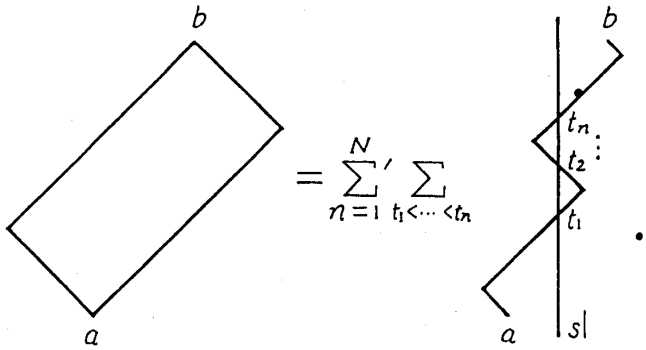

Fig. 15

Classification of the 'paths' defining $u[b;a]$ according to how many times (n) and at what locations $(t_1 < \cdots < t_n)$ they intersect the 'surface' $s|$ of $x = 0$.

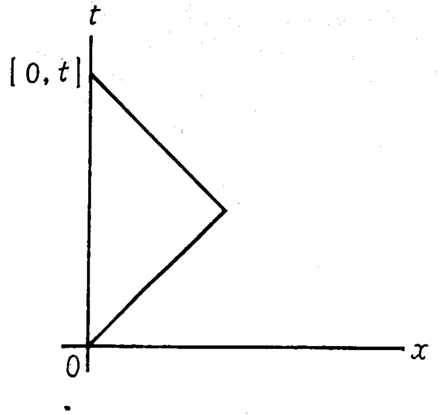

Fig. 16

The 'paths' which define $g[t] \equiv f[0,t]$. This is a special case of Fig. 10.

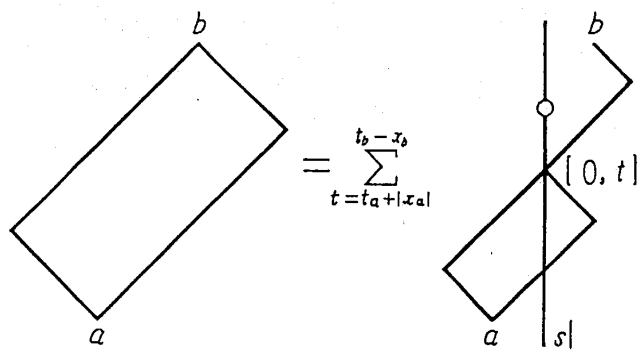

Fig. 17

Classification of the 'paths' defining $u[b;a]$ with respect to the 'time' t of the last hitting of $s|$. The diagram on the right-hand side is the abbreviation of the 'paths' whose last hitting of $s|$ is at $[0,t]$.

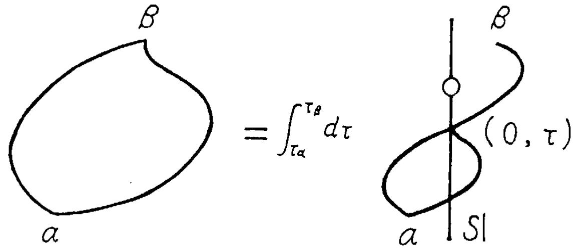

Fig. 18

The diffusion limit of Fig. 17 defines the classification of the 'paths' defining $\Phi_E(\beta;\alpha)$ with respect to the 'time' τ of the last hitting of the surface $\mathcal{S}|$ of $X=0$.

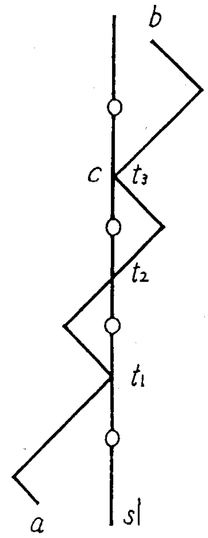

Fig. 19

Decomposition of u_3 for ESII. The zigzag diagram connecting a and b is an example of 'paths' contributing to $u_3[b;t_3,t_2,t_1;a]$. This diagram can be decomposed into two partial 'paths': One is from a to $c=[0,t_3]$ and the other from c to b. The former contributes to $u_2[c;t_2,t_1;a]$, and the latter to $\bar{f}[x_b,t_b-t_3]$. Since this separation is possible for all the 'paths' defining $u_3[b;t_3,t_2,t_1;a]$, Eq. (IV · 2 · 4) holds with $n=2$.

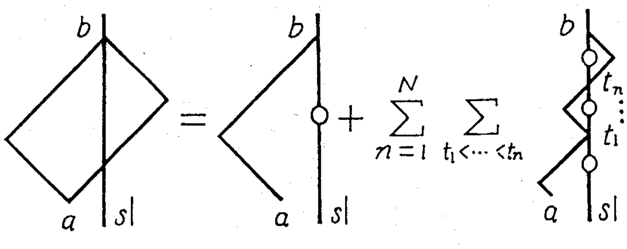

Fig. 20

The case of the final point b lying on $s|$. There exist those 'paths' which start from a and arrive at b without hitting (i.e., intersecting or touching) $s|$ before 'time' t_b; the first diagram on the right-hand side is the abbreviation of all such 'paths'. The second diagram (zigzag one) is the abbreviation of all possible 'paths' which start from a, hit $s|$ at $t_1 < \cdots < t_n$ before 'time' t_b and arrive at b on $s|$.

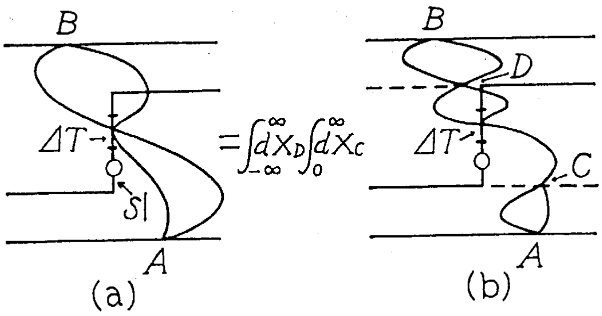

Fig. 22

(a) Examples of paths contributing to sum (IV · 3 · 7). The mark ∘ attached to $\mathcal{S}|$ denotes the restriction that 'paths' are not allowed to hit $\mathcal{S}|$ before $T \in \Delta T$. (b) Sum (IV · 3 · 7) can be decomposed into a sum over paths from A to a point C, a sum over paths from C to D whose first hitting of $\mathcal{S}|$ lies in ΔT, a sum over paths from D to B, and integrations over $X_C > 0$ and $-\infty < X_D < \infty$. The first and the third sums over paths give $\Phi(C; A)$ and $\Phi(B; D)$, respectively. Therefore, sum (IV · 3 · 7) can be rewritten as (IV · 3 · 8).

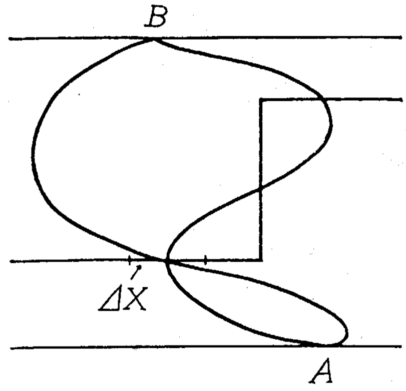

Fig. 21

Typical paths contributing to the sum (IV · 3 · 5).

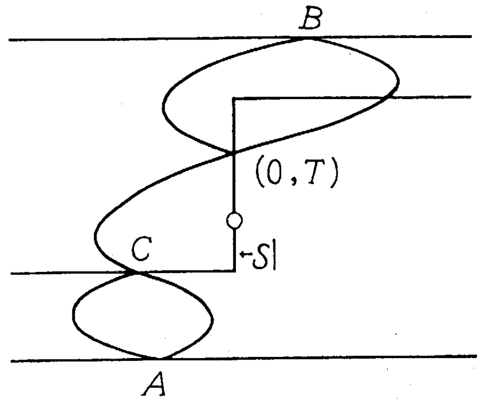

Fig. 23

Typical paths which contribute to $\Phi(B; \text{Yes}; A)$. The case $X_C < 0$ is shown. Paths are unrestricted from A to C, restricted so that they do not hit $\mathcal{S}|$ before time T on the way from C to $(0, T)$ and unrestricted again from $(0, T)$ to B. The unrestricted paths are associated with the usual propagators in the potential (IV·3·1); the amplitude for the restricted paths is $iF(X_C, iT)$. Since we do not specify the values of X_C and T for $\Phi(B; \text{Yes}; A)$, they are integrated over all possible values on the right-hand side of Eq. (V · 1 · 6).

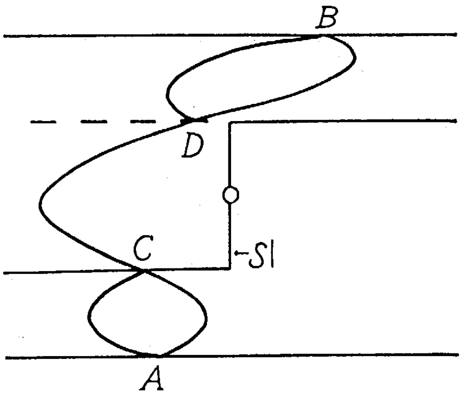

Fig. 24

Typical paths which contribute to $\Phi(B; \text{No}; A)$. The case of $X_C, X_D < 0$ is shown. The other case is $X_C, X_D > 0$. Paths are unrestricted from A to C, restricted so that they never hit $\mathcal{S}|$ on the way from C to D, and unrestricted again from D to B. The amplitude for the restricted paths is given by formula (III · 3 · 67).

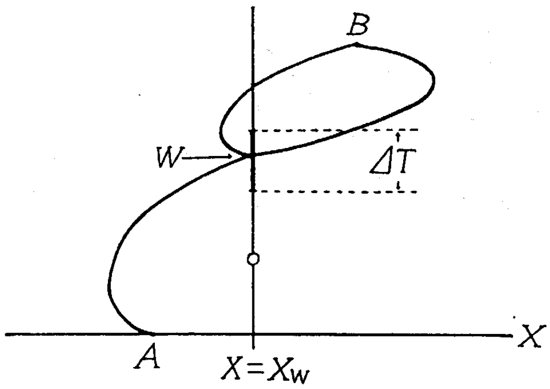

Fig. 25

Typical paths contributing to sum (V · 2 · 4). Paths from A to $W = (X_W, T)$ ($T \in \Delta T$) are restricted to $X < X_W$, restriction which is denoted by the mark o attached to the "wall" of $X = X_W$. The sum of e^{iS} over all the paths from A to W thus restricted yields the "restricted propagator" $F(W; A)$ in Eq. (V · 2 · 4). Paths are not restricted from W to B; the sum over all such paths gives the usual propagator $\Phi(B; W)$.

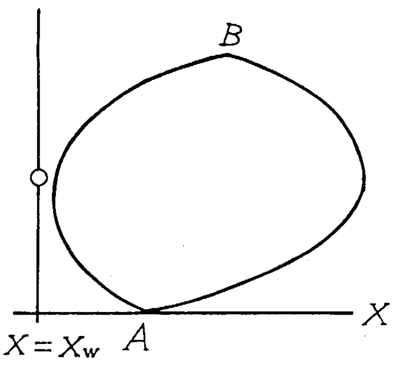

Fig. 26

Typical paths which never hit the wall at $X = X_W$ on the way from A to B lying on the same side of the wall. The case $X_A, X_B > X_W$ is shown. Paths are restricted to the half space $X > X_W$. The sum of e^{iS} over all such paths gives the right-hand side of formula (V · 2 · 10).

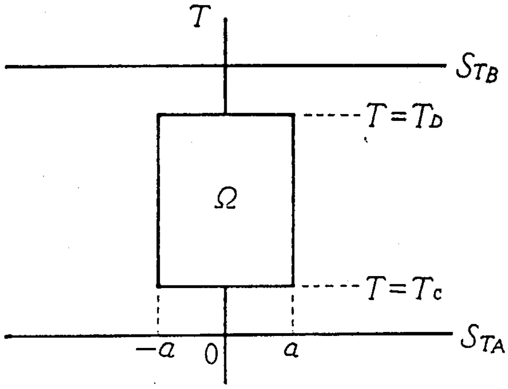

Fig. 27

Spacetime domain Ω bounded by $\mathcal{S}_{T_A}$ and $\mathcal{S}_{T_B}$. In the figure $\mathcal{S}_{T_A}$ is identified with the spatial axis.

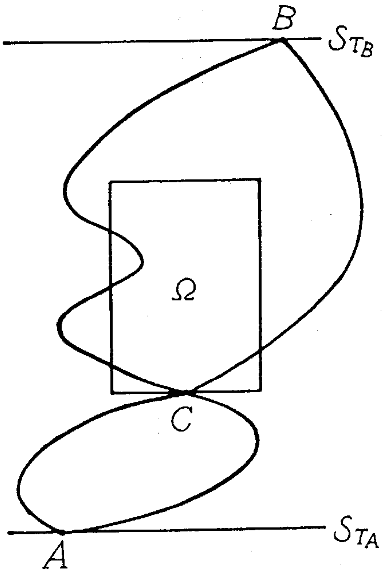

Fig. 28

Typical paths which intersect the bottom of Ω.

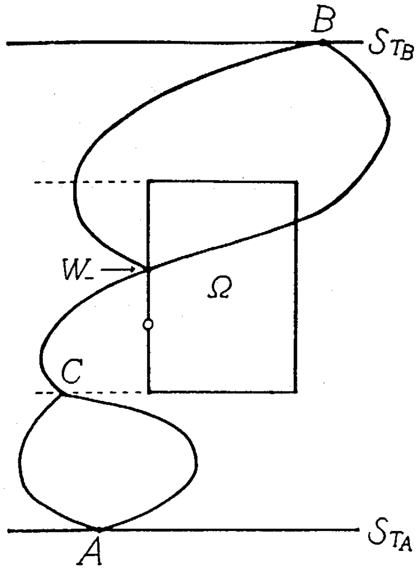

Fig. 29

Typical paths which pass through $X_C < -a$, hit the left wall of Ω at $W_- = (-a, T)$ and then arrive at B. Paths are restricted to $X < -a$ from C to W_-. The partial paths from C to B are of the type of Fig. 25.

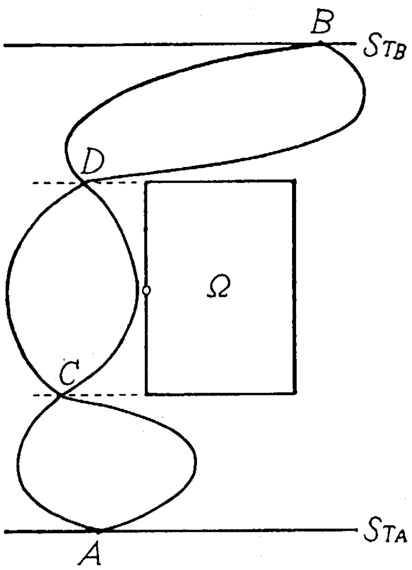

Fig. 30

Typical paths which pass through $X_C, X_D < -a$ and never hit the left wall on the way from A to B. The partial paths from C to D are of the type of Fig. 26.

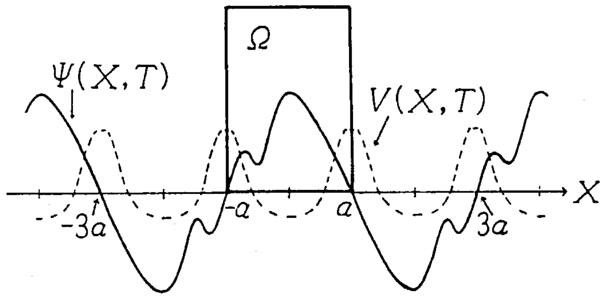

Fig. 31

The situation in which spacetime probabilities can be defined for {Yes, No}. The broken and the solid curve show the typical form of the potential between T_C and T_D and the typical form of the Schrödinger's wave function at time T_C, respectively.

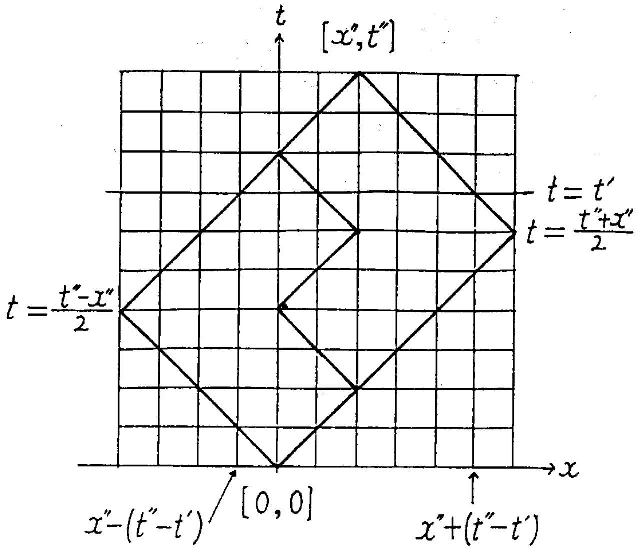

Fig. 32

Lattice paths on the Euclidean lattice.

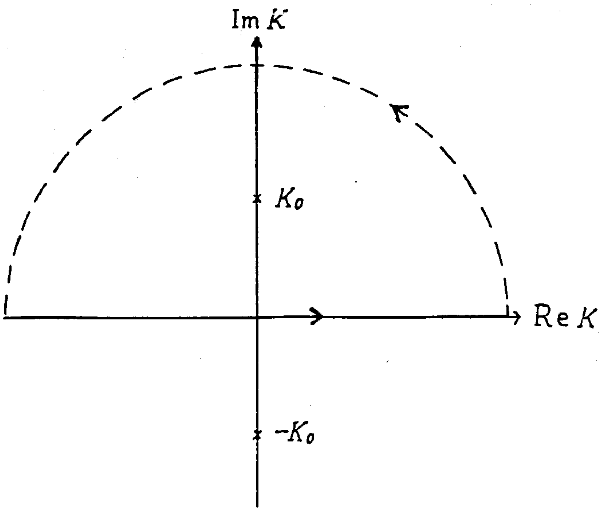

Fig. 34

Contour for the integral (B·8).

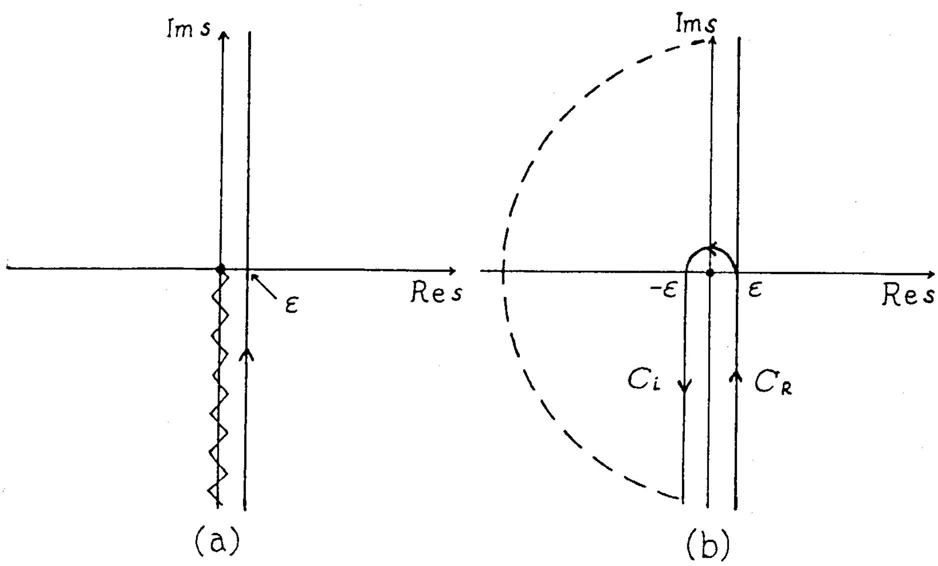

Fig. 33

(a) Contour for the integral (B·2).
(b) Deformation of the contour. The vertical line running from ϵ to $\epsilon + i\infty$ in the upper right half plane can be closed by an infinite semicircle in the left half plane with the vertical line running from $-\epsilon$ to $-i\infty - \epsilon$, which we denote by C_L, and with the semicircle of radius ϵ which detours the branch point $s = 0$ anticlockwise from $s = \epsilon$ to $-\epsilon$. Since the infinite semicircle and the infinitesimal semicircle contribute nothing to the integral, the contour is consequently deformed into $C_R + C_L$, where C_R is the vertical line running from $-i\infty + \epsilon$ to ϵ.