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journal or publication title	The science reports of the Tohoku University. Ser. 8, Physics and astronomy
volume	4
number	1
page range	98-125
year	1983-06-10
URL	http://hdl.handle.net/10097/25507

Reduction of Three-Nucleon Potentials into Irreducible Tensors and Computation of Their Matrix Elements with Respect to the Triton Wave Function

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(Received May 12, 1983)

Three-nucleon potentials due to 2π or $2(\pi+\rho)$ exchanges via Δ_{33}, and those due to 2π exchange with all effects of S and P wave πN scatterings are reduced into sums of irreducible tensors in coordinate space. General expressions for the matrix elements of these three-nucleon potentials with respect to the triton wave function are presented.

Keywords: Three-nucleon potentials. Irreducible tensor representation. Perturbation matrix elements with respect to the triton wave function.

§1. Introduction

There have been a number of calculations on the triton bound state based on the Faddeev equation, either in momentum space or in coordinate space[1,2,3], using realistic two-nucleon interactions such as the Reid soft-core potential[4] and other equivalent potentials. It is well-known by now that all these realistic nuclear forces underbind the triton by approximately 1 to 1.5 MeV. There is also a well-known discrepancy between the theoretical electromagnetic form factors and the experimental data. Efforts have been made to reconcile these difficulties as due to three-body interactions and other meson degrees of freedom, but with a partial success so far. The discrepancy in the binding energy had been thought to be explained by the contribution from the Fujita-Miyazawa(FM) three-nucleon interaction[5] arising from the process of two-pion exchanges via Δ_{33} resonance among three nucleons[6). However, the calculations by Hajduk and Sauer[7) and by Harper, Kim and Tubis[8) based on the Faddeev equation in which the N-Δ channels are explicitly taken into account showed that there was a repulsive effect (of the size approximately 0.5 MeV) in the triton bound state due to a dispersive effect of two-nucleon potentials. We could think of other sources of repulsive effect, such as contributions from ρ-meson exchanges. It became therefore necessary to look for other sources of attraction not included in the FM force.

In the nuclear matter calculations of Ueda, Sawada and Takagi[9), the contributions from the two-pion exchange three-nucleon potentials were investigated, in which all effects of S and P wave πN scattering were taken into account with the constraint of the PCAC condition. Their result showed that the contribution from the πN intermediate states other than Δ_{33} was attractive. In addition, they found that the three-nucleon potential due to σ-meson exchange, called as the SS-F coupling potential in their paper, gave a considerable attraction at the normal density.

Based on these considerations, one of the present authors(T.U.) derived the two-pion exchange three-nucleon($\pi\pi$-F) interaction in which all the S and P wave effects were contained with the PCAC condition as was done in ref. 9 but in addition the treatment was extended so as to include the spin-flip and the isospin-flip parts of the off πN scattering amplitudes[10). This was different from the two-pion exchange three-nucleon potential of Coon et al.[11) and McKeller et al[12). These authors used Adler's PCAC condition with the Weinberg condition, which is based on the theoretical hypothesis of the current algebra and requires the knowledge of the σ term which is largely unknown to us at present. T.U. did not use this condition. Nor did he use the approximation of setting the Lorentz invariant quantity V equal to zero as was done in ref. 12. As a result, he found some differences in the potential from the one in ref. 12, though both were qualitatively the same. In addition, T.U. also derived the two ($\pi+\rho$) exchange three nucleon potential via Δ_{33} (2($\pi+\rho$)-Δ) in static approximation. This provides means to investigate the repulsive contribution due to the ρ-exchange.

In this paper, we present the general expressions of the three-body perturbation matrix elements of the FM, $\pi\pi$-F, and 2($\pi+\rho$)-Δ three-nucleon forces in coordinate space with respect to the triton bound state wave functions resulting from the Faddeev equation. A similar perturbative calculation in momentum space has been reported by Muslim, Kim and Ueda[13), which yielded a rather negligibly small contributions of the above mentioned three-body forces. The three-body matrix elements in coordinate space, however, turns out to be very complicated, having both strongly repulsive parts and strongly attractive parts interwoven. If the finding of Muslim et al.[13) is true, then we expect that there are severe cancellations among various contributions from attractive parts and repulsive parts, which point has not been clarified by their work, and on which we hope to be able to cast some light by the use of coordinate representation. There has recently been a report by Coelho, Das and Fabre de la Ripelle[14], in which the Faddeev equation with a three-body potential was solved exactly in hyperspherical representation. In our perturbative calculation, we hope to investigate on this point also. Further, the algebra presented here is directly usable in our future work in treating the Faddeev equation exactly with the three-nucleon potentials based on our iterative approach[3]. Motivated by these considerations, we present in this paper the detailed derivations of the three-body matrix elements.

In section 2, the matrix elements with respect to our triton wave functions are described briefly. In section 3, we discuss the reduction of the three-nucleon interactions into sums of products of isoscalar irreducible tensors and scalar products of spin and spatial irreducible tensors. The general expressions for the matrix elements are presented in section 4. There arises a question of regularization of potentials associated with their use in a three (or more) -body system. This is described in section 5. In Appendix A, we present a summary on the FM, $\pi\pi$-F and $2(\pi+\rho)$-Δ three-nucleon potentials we specialize in this note. Appendices B to E contain detailed derivations of formulas presented in the text.

§2. Triton Wave Function and the Perturbation Expression

For the purpose of the perturbation calculation, it is convenient to express the triton wave function in terms of a single partition, say (12,3) of three nucleons. The relative coordinates in this partition are

$$\vec{x} = \vec{r}_{21} = \vec{r}_1 - \vec{r}_2 \tag{2.1}$$

and

$$\vec{y} = \vec{r}_3 - \frac{\vec{r}_1+\vec{r}_2}{2} \tag{2.2}$$

In our method of solving the Faddeev equation[3,15], we expand the triton wave function in terms of a complete set of products $F_\alpha(p,y)|\alpha(12,3)\rangle$, where $F_\alpha(p,y)$ is the normalized spherical Bessel function

$$F_\alpha(p,y) = \sqrt{\frac{2}{\pi}}\, p\, j_\ell(py) \tag{2.3}$$

and $|\alpha(12,3)\rangle$ is the normalized spin-isospin-angular function

$$|\alpha(12,3)\rangle = |(I\tfrac{1}{2})TM_T(12,3)\rangle|(LS)J,(\ell\tfrac{1}{2})j;\ J_0M_0(12,3)\rangle \tag{2.4}$$

with $T = \frac{1}{2}$. $|\alpha(12,3)\rangle$ is antisymmetric with respect to the pair (1,2). The orthonormality relation is

$$\int_0^\infty y^2 dy\ F_\alpha(p,y)\langle\alpha(12,3)|\alpha'(12,3)\rangle F_{\alpha'}(p',y) = \delta_{\alpha\alpha'}\delta(p-p') \qquad (2.5)$$

After solving the Faddeev equation, we obtain the antisymmetrized triton wave function in the following form.

$$\Psi(\vec{x},\vec{y}) = \sum_\alpha \int_0^\infty dp\ F_\alpha(p,y)|\alpha(12,3)\rangle\theta_\alpha(q,x) \qquad (2.6)$$

where $\Theta_\alpha(q,x)$ is the radial wave function in which the pair (1,2) has the energy $-\frac{\hbar^2}{M}q^2$ and the third particle has the energy $\frac{3\hbar^2}{4M}p^2$. The relation between p and q is

$$|E(^3H)| = \frac{3\hbar^2}{4M}p^2 + \frac{\hbar^2}{M}q^2 \qquad (2.7)$$

We write the three-nucleon interaction $V^{(3)}$ as

$$V^{(3)} = V(12,3) + V(23,1) + V(31,2) \qquad (2.8)$$

where, for example, V(12,3) contains the contribution to the three-body force in which a pion is exchanged between 1 and 2 via 3. Due to the symmetry property of $\Psi(\vec{x},\vec{y})$, we can express the first order perturbation energy of $V^{(3)}$ in the following way.

$$\langle\Psi|V^{(3)}|\Psi\rangle = 3\int d^3x \int d^3y\ \Psi(\vec{x},\vec{y})^+ V(12,3)\Psi(\vec{x},\vec{y}) \qquad (2.9)$$

Introducing

$$\rho_\alpha(x,y) = \int_0^\infty dp\ F_\alpha(p,y)\Theta_\alpha(q,x) \ , \qquad (2.10)$$

we find for $\Psi(x,y)$ of Eq.(2.6)

$$\Psi(\vec{x},\vec{y}) = \sum_\alpha |\alpha(12,3)\rangle\rho_\alpha(x,y) \ , \qquad (2.11)$$

and hence

$$\langle\Psi|V^{(3)}|\Psi\rangle = 3\sum_\alpha \sum_{\alpha'} \int_0^\infty x^2 dx \int_0^\infty y^2 dy\ \rho_\alpha(x,y)\langle\alpha(12,3)|V(12,3)|\alpha'(12,3)\rangle \times\rho_{\alpha'}(x,y) \qquad (2.12)$$

§3. The Three-Body Forces and the Reduction to Irreducible Tensors

Each component in Eq.(2.8) is expressed as[10)]

$$V(12,3) = \sum_{N=1}^{5} V_N(12,3) \tag{3.1}$$

where

$$V_1(12,3) = (\vec{\tau}_1\cdot\vec{\tau}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2)f_1(12,3) \tag{3.2}$$

$$V_2(12,3) = (\vec{\tau}_1\cdot\vec{\tau}_2)[(\vec{\sigma}_1\cdot\hat{r}_{23})(\vec{\sigma}_1\cdot\hat{r}_{23})f_2^{(1)}(12,3)+(\vec{\sigma}_1\cdot\hat{r}_{13})(\vec{\sigma}_2\cdot\hat{r}_{13})f_2^{(2)}(12,3)$$
$$+(\vec{\sigma}_2\cdot\hat{r}_{23})(\vec{\sigma}_1\cdot\hat{r}_{23})f_2^{(2)}(21,3)] \tag{3.3}$$

$$V_3(12,3) = -(\vec{\tau}_3\cdot i\vec{\tau}_1\times\vec{\tau}_2)(\vec{\sigma}_3\cdot i\vec{\sigma}_1\times\vec{\sigma}_2)f_3(12,3) \tag{3.4}$$

$$V_4(12,3) = -(\vec{\tau}_3\cdot i\vec{\tau}_1\times\vec{\tau}_2)[(\vec{\sigma}_2\cdot\hat{r}_{23})(\vec{\sigma}_3\cdot i\vec{\sigma}_1\times\hat{r}_{23})f_4(12,3)$$
$$-(\vec{\sigma}_1\cdot\hat{r}_{13})(\vec{\sigma}_3\cdot i\vec{\sigma}_2\times\hat{r}_{13})f_4(21,3)] \tag{3.5}$$

$$V_5(12,3) = -(\vec{\tau}_3\cdot i\vec{\tau}_1\times\vec{\tau}_2)(\vec{\sigma}_1\cdot\hat{r}_{13})(\vec{\sigma}_2\cdot\hat{r}_{23})(\vec{\sigma}_3\cdot i\hat{r}_{13}\times\hat{r}_{23})f_5(12,3) \tag{3.6}$$

There are other terms but they have much smaller coupling constants compared to those listed above, and hence we ignore them in this report. In these expressions, $\vec{\sigma}_i$ and $\vec{\tau}_i$ are the spin and the isospin operators of the i th nucleon, and $\hat{r}_{ij}$ is the unit vector along $\vec{r}_{ij}=\vec{r}_i-\vec{r}_j$. The functions $f_n(ij,k)$ are functions of r_{ik} , r_{jk} and $(\hat{r}_{ik}\cdot\hat{r}_{jk})$. Explicit forms of them are given in Appendix A together with the values of coupling constants.

To facilitate the Racah algebra, we introduce the following irreducible tensors of rank F in spin, isospin and spatial coordinates[17)].

$$\underset{\sim}{T}^{(K,n;F)}(ij,k) = [[\underset{\sim}{\tau}^{(1)}(i)\otimes\underset{\sim}{\tau}^{(1)}(j)]^{(K)}\otimes\underset{\sim}{\tau}^{(n)}(k)]^{(F)} \quad (n=0,\text{ or }1) \tag{3.7}$$

$$\underset{\sim}{S}^{(K,n;F)}(ij,k) = [[\underset{\sim}{\sigma}^{(1)}(i)\otimes\underset{\sim}{\sigma}^{(1)}(j)]^{(K)}\otimes\underset{\sim}{\sigma}^{(n)}(k)]^{(F)} \quad (n=0,\text{ or }1) \tag{3.8}$$

$$\underset{\sim}{U}^{(L,L';F)}(ij,k) = [\underset{\sim}{Y}^{(L)}(\hat{r}_{ik})\otimes\underset{\sim}{Y}^{(L')}(\hat{r}_{jk})]^{(F)} \tag{3.9}$$

Here, $\underset{\sim}{\tau}^{(1)}(i)$ and $\underset{\sim}{\sigma}^{(1)}(i)$ are the irreducible tensor operators of rank 1 composed of the isospin and the spin operators of the i th nucleon, and $\underset{\sim}{\sigma}^{(0)}(i)$

and $\tau^{(0)}(i)$ are equal to 1. Also, $\underset{\sim}{Y}^{(L)}(\hat{r})$ is the irreducible tensor of rank L composed of the spherical harmonics $Y_L^M(\hat{r})$ defined by Eqs.(B.4) and (B.27) in Appendix B. In the following, quantities without partition symbol, such as $\underset{\sim}{T}^{(K,n;F)}$, refer to partition (12,3). As shown there, we can express V_N(N=1 to 5) of Eqs.(3.2) to (3.6) as follows:

$$V_1 = 3\underset{\sim}{T}^{(0,0;0)} \quad \underset{\sim}{S}^{(0,0;0)} \quad f_1(12,3) \tag{3.10}$$

$$V_2 = -\sqrt{3}\ \underset{\sim}{T}^{(0,0;0)} \sum_{K=0}^{2} (-)^K (\underset{\sim}{S}^{(K,0;K)} \cdot \{\underset{\sim}{U}^{(1,1;K)}(12,3) f_2^{(1)}(12,3)$$

$$+ \underset{\sim}{U}^{(1,1;K)}(11,3) f_2^{(2)}(12,3) + \underset{\sim}{U}^{(1,1;K)}(22,3) f_2^{(2)}(21,3)\}) \tag{3.11}$$

$$V_3 = -6\underset{\sim}{T}^{(1,1;0)} \quad \underset{\sim}{S}^{(1,1;0)} \quad f_3(12,3) \tag{3.12}$$

$$V_4 = 6\underset{\sim}{T}^{(1,1;0)} \sum_{K} \sum_{K'} (-)^{K+K'} \hat{K}' \left\{\begin{matrix} 1 & 1 & k \\ K' & 1 & 1 \end{matrix}\right\} (\underset{\sim}{S}^{(K',1;K)} \cdot \{\underset{\sim}{U}^{(1,1;K)}(22,3) f_4(12,3)$$

$$-(-)^{K'} \underset{\sim}{U}^{(1,1;K)}(11,3) f_4(21,3)\}) \tag{3.13}$$

$$V_5 = \sqrt{12}\ \underset{\sim}{T}^{(1,1;0)} \sum_{K} (-)^K \sum_{F} (-)^{K-F+1} (\underset{\sim}{S}^{(K,1;F)} \cdot \underset{\sim}{W}^{(K;F)}) f_5(12,3) \tag{3.14}$$

where, for V_5 , we have defined

$$\underset{\sim}{W}^{(K;F)} = 4\pi\sqrt{3}\ \hat{K} \sum_{\xi=0,2} \sum_{\xi'=0,2} \frac{\langle 1010|\xi 0\rangle}{A_\xi} \frac{\langle 1010|\xi' 0\rangle}{A_{\xi'}} \left\{\begin{matrix} 1 & 1 & K \\ 1 & 1 & 1 \\ \xi & \xi' & F \end{matrix}\right\} \underset{\sim}{U}^{(\xi\xi';F)}(12,3) \tag{3.15}$$

where

$$A_0 = \sqrt{4\pi} \quad , \quad \text{and} \quad A_2 = \sqrt{8\pi/15} \tag{3.16}$$

§4. Matrix Elements

From Eqs.(3.10) to (3.15), we see that the matrix elements in Eq.(2,12) are all of the following form.

$$\langle\alpha|V_N|\alpha'\rangle = C_N \tau_N^{II'} \sum_{\text{terms}} \langle (LS)J, (\ell\tfrac{1}{2})j; J_0 M_0|$$

$$\times (\underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})} \cdot \underset{\sim}{U}^{(\xi,\xi';\bar{F})}_{(ij;3)}) f|(L'S')J', (\ell'\tfrac{1}{2})j'; J_0 M_0\rangle \tag{4.1}$$

where $(i,j)=(1,1),(2,2)$ or $(1,2)$ and $\sum_{\text{terms}}$ indicates sums over various terms in Eqs.(3.10) to (3.15). We have also introduced, for N=1 and 2,

$$\tau_N^{II'} = \langle\alpha|\underset{\sim}{T}^{(0,0;0)}|\alpha'\rangle$$

$$= \frac{1}{\sqrt{2T+1}} \langle(I\tfrac{1}{2})T\,\|\underset{\sim}{T}^{(0,0;0)}\|\,(I'\tfrac{1}{2})T\rangle \quad , \qquad (N=1,2) \qquad (4.2)$$

and for N=3,4 and 5

$$\tau_N^{II'} = \langle\alpha|\underset{\sim}{T}^{(1,1;0)}|\alpha'\rangle$$

$$= \frac{1}{\sqrt{2T+1}} \langle(I\tfrac{1}{2})T\,\|\underset{\sim}{T}^{(1,1;0)}\|\,(I'\tfrac{1}{2})T\rangle \quad , \qquad (N=3,4,5) \qquad (4.3)$$

Explicit formulas of $\tau_N^{II'}$ are given in the following subsections. The forms of the operator $\underset{\sim}{U}^{(\xi,\xi';\bar{F})}(ij;3)$ and the function f can be identified by comparing Eq.(4.1) with Eqs.(3.10) to (3.15).

To calculate the spin-angular matrix elements in Eq.(4.1), we transform from the jj-coupling scheme in Eq.(2.4) to the LS-coupling scheme.

$$|(LS)J,(\ell\tfrac{1}{2})j;J_0M_0\rangle = \sum_{L_0S_0} N_\alpha^{(L_0S_0)}|\bar{\alpha}\rangle \qquad (4.4)$$

where the spin-angular state in the LS coupling scheme is denoted by

$$|\bar{\alpha}\rangle = |(L\ell)L_0 \ , \ (S\tfrac{1}{2})S_0;J_0M_0\rangle \qquad (4.5)$$

and the transformation coefficient $N_\alpha^{(L_0S_0)}$ is given by

$$N_\alpha^{(L_0S_0)} = \hat{J}\hat{j}\hat{L}_0\hat{S}_0 \begin{Bmatrix} L & \ell & L_0 \\ S & \frac{1}{2} & S_0 \\ J & j & J_0 \end{Bmatrix} \quad , \qquad (4.6)$$

where we have introduced the convention $\hat{A} = \sqrt{2A+1}$. Thus we find

$$\langle(LS)J,(\ell\tfrac{1}{2})j;J_0M_0|(\underset{\sim}{S}^{(\bar{K},\bar{n}:\bar{F})}\cdot\underset{\sim}{U}^{(\xi\xi';\bar{F})}_{(ji;3)})f|(L'S')J',(\ell'\tfrac{1}{2})j';J_0M_0\rangle$$

$$= \sum_{L_0S_0}\sum_{L_0'S_0'} N_\alpha^{(L_0S_0)}N_{\alpha'}^{(L_0'S_0')}\langle\bar{\alpha}|(\underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})}\cdot\underset{\sim}{U}^{(\xi,\xi';\bar{F})}_{(ij;3)}f)|\bar{\alpha}'\rangle \qquad (4.7)$$

where (see Eq.(C.91) of reference 17)

$$\langle\bar{\alpha}|(\underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})}\cdot\underset{\sim}{U}^{(\xi,\xi';\bar{F})}_{(ij;3)}f)|\bar{\alpha}'\rangle$$

$$= (-)^{J_0+S_0+L_0'}\begin{Bmatrix} L_0 & \bar{F} & L_0' \\ S_0' & J_0 & S_0 \end{Bmatrix}\langle(S\tfrac{1}{2})S_0\|\underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})}\|(S'\tfrac{1}{2})S_0'\rangle$$

$$\times \langle(L\ell)L_0\|\underset{\sim}{U}^{(\xi,\xi';\bar{F})}_{(ij;3)}f\|(L'\ell')L_0'\rangle \tag{4.8}$$

Substituting this into Eq.(4.7), we obtain for the matrix element (4.1) the following expression

$$\langle\alpha|V_N|\alpha'\rangle = C_N\tau_N^{II'}\sum_{L_0S_0}\sum_{L_0'S_0'}N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')}\sum_{\text{terms}}\begin{Bmatrix} L_0 & \bar{F} & L_0' \\ S_0' & J_0 & S_0 \end{Bmatrix}$$

$$\times \langle(S\tfrac{1}{2})S_0\|\underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})}\|(S'\tfrac{1}{2})S_0'\rangle\langle(L\ell)L_0\|\underset{\sim}{U}^{(\xi\xi';\bar{F})}_{(ij;3)}\;f\;\|(L'\ell')L_0'\rangle \tag{4.9}$$

where we have defined

$$N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')} = N_{\alpha}^{(L_0S_0)}N_{\alpha'}^{(L_0'S_0')}(-)^{J_0+S_0+L_0'} \tag{4.10}$$

The spatial matrix element in Eq.(4.9) involves a four dimensional angular integral over $\hat{x}$ and $\hat{y}$. This integral can be reduced to a one dimensional integral over the cosine of the angle between $\vec{x}$ and $\vec{y}$ by transforming from an arbitrary chosen space-fixed reference frame used so far to a body-fixed reference frame, in which the z-axis is chosen along the vector $\vec{x}$ and the xz plane on the plane of the three particles. As shown in Appendix C, we find

$$\langle(L\ell)L_0\|\underset{\sim}{U}^{(\xi,\xi';\bar{F})}_{(ij;3)}f\|(L'\ell')L_0'\rangle$$

$$= \sum_n\sum_d \mathbb{K}^{n,d}_{\alpha\alpha';\bar{F}}\sum_\lambda\begin{pmatrix} n & \bar{F} & d \\ \lambda & -\lambda & 0 \end{pmatrix}\int_{-1}^{1}du\;\mathring{U}^{(\xi,\xi';\bar{F})}_{\lambda(ij;3)}\mathring{f}\;Y_n^{-\lambda}(\mathring{\hat{y}}) \tag{4.11}$$

where

$$\mathbb{K}^{n,d}_{\alpha\alpha';\bar{F}} = \sqrt{\pi}\;\hat{L}\hat{L}'\hat{L}_0\hat{L}_0'\hat{\ell}\hat{\ell}'(-)^{L'-\ell'}\hat{n}\begin{pmatrix} \ell & \ell' & n \\ 0 & 0 & 0 \end{pmatrix}\hat{d}^2\begin{pmatrix} L & L' & d \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} \ell & \ell' & n \\ L_0 & L_0' & \bar{F} \\ L & L' & d \end{Bmatrix} \tag{4.12}$$

In Eq.(4.11), u=cos θ, and the quantities $\mathring{\underset{\sim}{U}}^{(\xi,\xi';\bar{F})}_{(ij;3)}$, $\mathring{f}$ and $\mathring{\hat{y}}$ refer to the body-fixed reference frame difined above.

In Appendix D, we show that, starting from Eq.(4.11) we can derive the following expression with the help of Moshinsky's formula[16).]

$$\langle (L\ell)L_0 \| \underset{\sim}{U}^{(\xi,\xi';\bar{F})}_{(ij;3)} f \| (L'\ell')L_0' \rangle$$

$$= \sum_{\substack{a=0 \\ (b=\xi-a)}}^{\xi} \sum_{\substack{a'=0 \\ (b'=\xi'-a')}}^{\xi'} x^{a+a'} y^{b+b'} p_i^{a} p_j^{a'} \sum_h R^{(aa',\xi\xi',h)}_{(\alpha\alpha',\bar{F})} \int_{-1}^{1} du \frac{\mathring{f}}{r_{i3}^{\xi} r_{j3}^{\xi'}} P_h(u) \tag{4.13}$$

where $p_1 = -\frac{1}{2}$, $p_2 = \frac{1}{2}$ (see Eq.(D.3)), and

$$R^{(aa',\xi\xi',h)}_{(\alpha\alpha',\bar{F})} = \frac{1}{8\pi} \Gamma^{\xi\xi',aa'}_{\bar{F}} \hat{h}^2 \sum_c \hat{c}^2 (-)^{b-b'} \begin{pmatrix} b & b' & c \\ 0 & 0 & 0 \end{pmatrix} \sum_g (-)^g \hat{g}^2 \begin{pmatrix} a & a' & g \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \xi & \xi' & \bar{F} \\ b & b' & c \\ a & a' & g \end{Bmatrix}$$

$$\times \sum_{n,d} \frac{1}{\sqrt{\pi}} \mathbb{K}^{n,d}_{\alpha\alpha',\bar{F}} (-)^n \hat{n} \begin{pmatrix} g & h & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c & n & h \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} g & h & d \\ n & \bar{F} & c \end{Bmatrix} \quad . \tag{4.14}$$

Here, $\mathbb{K}^{n,d}_{\alpha\alpha',\bar{F}}$ is given by Eq.(4.12), and $\Gamma^{\xi\xi',aa'}_{\bar{F}}$ by Eq.(D.9)

$$\Gamma^{\xi\xi',aa'}_{\bar{F}} = A_{\xi} A_{\xi'} \binom{2\xi+1}{2a}^{1/2} \binom{2\xi'+1}{2a'}^{1/2} \hat{b}\hat{b}'\hat{\xi}\hat{\xi}'\hat{\bar{F}} \tag{4.15}$$

with A_{ξ} defined by

$$A_0 = \sqrt{4\pi} \,, \qquad A_1 = \sqrt{\frac{4\pi}{3}} \,, \qquad A_2 = \sqrt{\frac{8\pi}{15}} \tag{4.16}$$

For the spin matrix element in Eq.(4.9), we write for simplicity

$$Z^{(\bar{K},\bar{n};\bar{F})}_{(SS_0,S'S_0')} = \langle (S\tfrac{1}{2})S_0 \| \underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})} \| (S'\tfrac{1}{2})S_0' \rangle \tag{4.17}$$

As shown in Appendix E, this is given by

$$Z^{(\bar{K},\bar{n};\bar{F})}_{(SS_0,S'S_0')} = \hat{S}_0 \hat{\bar{F}} \hat{S}_0' \begin{Bmatrix} S' & \frac{1}{2} & S_0' \\ \bar{K} & \bar{n} & \bar{F} \\ S & \frac{1}{2} & S_0 \end{Bmatrix} 6\hat{S}\hat{\bar{K}}\hat{S}' \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S' \\ 1 & 1 & \bar{K} \\ \frac{1}{2} & \frac{1}{2} & S \end{Bmatrix} \times \begin{cases} \sqrt{2}, & \text{for } \bar{n}=0 \\ \sqrt{6}, & \text{for } \bar{n}=1 \end{cases} \tag{4.18}$$

As special cases of Eq.(4.18), we obtain for the isospin matrix elements

$$\tau^{II'}_N = \frac{1}{\sqrt{2}} Z^{(0,0;0)}_{(I\frac{1}{2},I'\frac{1}{2})} \equiv \tau^{II'}_A \,, \qquad \text{for N=1 and 2} \tag{4.19a}$$

$$= \frac{1}{\sqrt{2}} Z^{(1,1;0)}_{(I\frac{1}{2},I'\frac{1}{2})} \equiv \tau^{II'}_B \,, \qquad \text{for N=3,4 and 5} \tag{4.19b}$$

where we have introduced the notation $\tau_A^{II'}$ and $\tau_B^{II'}$. The values of these matrix elements are given in Table 1.

I \ I'	0	1
0	$\sqrt{3}$	0
1	0	$-1/\sqrt{3}$

$\tau_A^{II'}$

I \ I'	0	1
0	0	$-\sqrt{2}$
1	$\sqrt{2}$	0

$\tau_B^{II'}$

Table 1. The matrix elements $\tau_A^{II'}$ and $\tau_B^{II'}$ of Eq.(4.19).

Substituting Eqs.(4.13) and (4.17) into Eq.(4.9), we find for the matrix element

$$V_{N,\alpha\alpha'} = \langle\alpha|V_N|\alpha'\rangle$$

$$= c_N\tau_N^{II'} \sum_{L_0S_0} \sum_{L_0'S_0'} N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')}$$

$$\times \sum_{\text{terms}} \begin{Bmatrix} L_0 & \bar{F} & L_0' \\ S_0' & J_0 & S_0 \end{Bmatrix} Z_{(SS_0,S'S_0')}^{(\bar{K},\bar{n};\bar{F})}$$

$$\times \sum_{\substack{a=0 \\ (b=\xi-a)}}^{\xi} \sum_{\substack{a'=0 \\ (b'=\xi'-a')}}^{\xi'} x^{a+a'} y^{b+b'} p_i^a p_j^{a'} \sum_h R_{(\alpha\alpha',\bar{F})}^{(aa',\xi\xi',h)} \int_{-1}^{1} du \frac{\mathring{f}}{r_{i3}^{\xi} r_{j3}^{\xi'}} P_h(u) \tag{4.20}$$

In this expression, $\sum_{\text{terms}}$ means the summation over all terms that appear in Eqs.(3.10) to (3.15) when they are expressed in the form of Eq.(4.1). We shall identify $\underset{\sim}{U}_{(ij;3)}^{(\xi,\xi';\bar{F})}$, f, and $\underset{\sim}{S}^{(\bar{K},\bar{n};\bar{F})}$ for each case below.

(i) <u>N=1</u>

Comparing Eq.(4.1) with Eq.(3.10), we find $C_1=3$, $\bar{K}=\bar{n}=\bar{F}=\xi=\xi'=0$, $\underset{\sim}{U}^{(0,0;0)}=1$, and $f=f_1(12,3)$. Hence, we also have a=a'=b=b'=0, and

$$R_{(\alpha\alpha',0)}^{(00,00,h)} = \frac{1}{\sqrt{4\pi}} \mathbb{K}_{\alpha\alpha',0}^{h,h} (-)^h \tag{4.21}$$

Then, we find

$$V_{1,\alpha\alpha'} = \sum_h D_{(\alpha\alpha',h)}^{(1)} \cdot \int_{-1}^{1} du\, \mathring{f}_1(12,3) P_h(u) \tag{4.22}$$

where

$$D_{(\alpha\alpha',h)}^{(1)} = \frac{3}{2} \tau_A^{II'} \sum_{\substack{L_0S_0 \\ L_0'S_0'}} \delta_{L_0L_0'} \delta_{S_0S_0'} \delta_{SS'} N_{\alpha\alpha'}^{(L_0S_0,L_0S_0)}$$

$$\times \begin{Bmatrix} L_0 & 0 & L_0 \\ S_0 & J_0 & S_0 \end{Bmatrix} \frac{(-)^h}{\sqrt{\pi}} \mathbb{K}_{\alpha\alpha',0}^{hh} Z_{(SS_0,SS_0)}^{(0,0;0)} \tag{4.23}$$

(ii) N=2

By comparing Eq.(4.1) with Eq.(3.11), we have $C_2=-\sqrt{3}$, $\bar{n}=0$, $\bar{K}=\bar{F}=K$, $\xi=\xi'=1$. Thus, Eq.(4.20) becomes

$$V_{2,\alpha\alpha'} = \sum_{\substack{a=0\\(b=1-a)}}^{1} \sum_{\substack{a'=0\\(b'=1-a')}}^{1} x^{a+a'} y^{b+b'} \sum_h \{p_1^a p_2^{a'} \int_{-1}^{1} du \frac{\mathring{f}_2^{(1)}(12,3)}{r_{13}r_{23}} P_h(u)$$
$$+ p_1^{a+a'} \int_{-1}^{1} du \frac{\mathring{f}_2^{(2)}(12,3)}{r_{13}^2} P_h(u) + p_2^{a+a'} \int_{-1}^{1} du \frac{\mathring{f}_2^{(2)}(21,3)}{r_{23}^2} P_h(u)\}$$
$$\times D^{(2)}_{(\alpha\alpha',aa',h)} \qquad (4.24)$$

where

$$D^{(2)}_{(\alpha\alpha',aa',h)} = -\sqrt{3}\,\tau_A^{II'} \sum_{L_0S_0} \sum_{L_0'S_0'} N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')} \sum_K (-)^K \begin{Bmatrix} L_0 & K & L_0' \\ S_0' & J_0 & S_0 \end{Bmatrix}$$
$$\times Z^{(K,0;K)}_{(SS_0,S'S_0')} \cdot R^{(aa',11,h)}_{(\alpha\alpha',K)} \qquad (4.25)$$

(iii) N=3

Comparing Eqs.(3.12) and (4.1), we find $C_3=-6$, $\bar{K}=1$, $\bar{n}=1$, $\bar{F}=0$, $\xi=\xi'=0$, $\mathring{\underset{\sim}{U}}^{(\xi\xi',\bar{F})}=1$, and $f=f_3(12,3)$. As for N=1, $R^{(aa',\xi\xi',h)}_{(\alpha\alpha',\bar{F})}$ becomes $R^{(00,00,h)}_{(\alpha\alpha',0)}$ of Eq.(4.21). Thus, we find from Eqs.(4.20) and (4.21)

$$V_{3,\alpha\alpha'} = \sum_n D^{(3)}_{(\alpha\alpha',n)} \int_{-1}^{1} du\, P_n(u)\, \mathring{f}_3(12,3) \,, \qquad (4.26)$$

where

$$D^{(3)}_{(\alpha\alpha',n)} = -3\tau_B^{II'} \sum_{L_0S_0} \sum_{L_0'S_0'} N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')} \delta_{L_0L_0'} \delta_{S_0S_0'} \begin{Bmatrix} L_0 & 0 & L_0 \\ S_0 & J_0 & S_0 \end{Bmatrix}$$
$$\times Z^{(1,1;0)}_{(SS_0,S'S_0)} \frac{(-)^n}{\sqrt{\pi}} \mathbb{K}^{n,n}_{\alpha\alpha';0} \qquad (4.27)$$

(iv) N=4

From Eq.(3.13), we find for Eq.(4.1), $\bar{K}=K'$, $\bar{n}=1$, $\bar{F}=K$, $\xi=\xi'=1$. Thus, Eq.(4.20) becomes

$$V_{4,\alpha\alpha'} = \sum_{\substack{a=0\\(b=1-a)}}^{1} \sum_{\substack{a'=0\\(b'=1-a')}}^{1} x^{a+a'} y^{b+b'} \sum_h \{ p_2^{a+a'} \int_{-1}^{1} du\, P_h(u)$$

$$\times \frac{\overset{\circ}{f}_4(12,3)}{r_{23}^2} D^{(4A)}_{(\alpha\alpha',aa',h)} - p_1^{a+a'} \int_{-1}^{1} du\, P_h(u) \frac{\overset{\circ}{f}_4(21,3)}{r_{13}^2} D^{(4B)}_{(\alpha\alpha',aa',h)} \} \tag{4.28}$$

where

$$\begin{pmatrix} D^{(4A)}_{(\alpha\alpha',aa',h)} \\ \\ D^{(4B)}_{(\alpha\alpha',aa',h)} \end{pmatrix} = 6\tau_B^{II'} \sum_{L_0S_0} \sum_{L_0'S_0'} N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')} \sum_k \begin{Bmatrix} L_0 & K & L_0' \\ S_0' & J_0 & S_0 \end{Bmatrix} \sum_{K'} \begin{pmatrix} (-)^{K'} \\ \\ 1 \end{pmatrix} \hat{K}' \begin{Bmatrix} 1 & 1 & K \\ K' & 1 & 1 \end{Bmatrix}$$

$$\times Z^{(K',1;K)}_{(SS_0,S'S_0')} \cdot R^{(aa',11,h)}_{(\alpha\alpha',K)} \tag{4.29}$$

(v) N=5

From Eqs.(3.14) and (3.15), we find for Eq.(4.1), $\bar{K}=K$, $\bar{n}=1$, $\bar{F}=F$, $\xi=0$ or 2, $\xi'=0$ or 2. Thus, from Eq.(4.20) we have

$$V_{5,\alpha\alpha'} = \sum_{\xi=0,2} \sum_{\xi'=0,2} \sum_{\substack{a=0\\(b=\xi-a)}}^{\xi} \sum_{\substack{a'=0\\(b'=\xi'-a')}}^{\xi'} x^{a+a'} y^{b+b'} \sum_h \int_{-1}^{1} du\, P_h(u)$$

$$\times \frac{\overset{\circ}{f}_5(12,3)}{r_{13}^{\xi} r_{23}^{\xi'}} D^{(5)}_{(\alpha\alpha',aa',\xi\xi',h)} \tag{4.30}$$

with

$$D^{(5)}_{(\alpha\alpha',aa',\xi\xi',h)} = 6\tau_B^{II'} 4\pi \frac{\langle 1010|\xi 0\rangle \langle 1010|\xi' 0\rangle}{A_\xi A_{\xi'}} \sum_{L_0S_0} \sum_{L_0'S_0'} N_{\alpha\alpha'}^{(L_0S_0,L_0'S_0')}$$

$$\times \sum_K \hat{K} \sum_F (-)^{1-F} Z^{(K,1;F)}_{(SS_0,S'S_0')} \begin{Bmatrix} L_0 & F & L_0' \\ S_0' & J_0 & S_0 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & K \\ 1 & 1 & 1 \\ \xi & \xi' & F \end{Bmatrix} p_1^a p_2^{a'} R^{(aa',\xi\xi',h)}_{(\alpha\alpha',F)} \tag{4.31}$$

§5. Regularization

The regularization of the Yukawa function normally required to accomodate meson-nucleon vertex form factors is discussed in Appendix A. This problem can be treated with the introduction of one or two cutoff masses per meson.

The problem of regularization we wish to discuss in this section is of entirely different origin to this. It is required by the u-integrations in Eq.(4.20) in the three-body matrix elements due to the use of Moshinsky's formula. This problem, however, exists whether we use Moshinsky's formula or not, only that it is made explicit by the formula. Furthermore, this problem will persist in more-than-three body problems in exactly the same degree but not worse than we encounter here.

The integral in question takes the following form:

$$\int_{-1}^{1} du\ P_h(u)\ \frac{\overset{\circ}{f}}{r_{i3}^{\xi} r_{j3}^{\xi'}} \qquad (i,j=1 \text{ or } 2) \tag{5.1}$$

When $y \to \frac{x}{2}$ and $\theta \to 0$ (or π), r_{13} (or r_{23}) approaches zero. In order to have finite values for the u-integration, $\overset{\circ}{f}$, and hence various Yukawa functions in it, must be appropriately regularized. For $\xi=\xi'=0$, the usual regularization is sufficient for this purpose. However, since ξ and ξ' can take values up to 2 (which is due to the fact that the three-body forces of Eqs.(3.2) to (3.6) contain spatial tensors composed of $\vec{r}_{13}$ and/or $\vec{r}_{23}$ of rank up to 2), Eq.(5.1) demands a much severe cutoff. In fact, inspection of u-integrations in detail (in Eqs.(4.24), (4.28) and (4.30)) shows that we need to introduce four cutoff masses if we were to use the regularization procedure of Eq.(A.13). Even if we did introduce four cutoff masses, the choice of heavier cutoff masses must be quite arbitrary. Then, one might as well introduce a simpler procedure. The method we propose is as follows.

We have certain confidence in the cutoff masses already present in the OBE potentials, in that they determine the two-nucleon interactions for the region, say $r \geq 1$ fm. Therefore, whatever OBEP we employ, we wish to preserve its r-dependence beyond the OBE region, or othrewise we shall destroy the fit to the two-nucleon data. For inner region, however, let us assume Gaussian forms for all functions of r_{12}, r_{13} or r_{23} that appear in the three-body matrix elements. Take, for instance, the $\pi\pi$-F coupling constants in Table A of Appendix A, and let us use two cutoff masses (κ_2 and κ_3 in Eq.(A.13)). With the conditions

$$a_1 + a_2 + a_3 = 0 \tag{5.2}$$

and

$$a_1\kappa_1^2 + a_2\kappa_2^2 + a_3\kappa_3^2 = 0 \tag{5.3}$$

with $a_1=\kappa_1=1$, we are going to change the inner part $r \leq R_c$ by assuming a Gaussian form $Ce^{-\Lambda r^2}$ for each one of $J^{(2)}(r)$, $Z^{(2)}(r)$, $J'(r)/r$, $J^{(2)'}(r)$, $Z^{(2)}(r)/r$ and $Z^{(2)}(r)/r^2$ that appear in the matrix elements. The constants C and Λ are determined so that the values and the first derivatives of $J^{(2)}(r)$ etc. at $r=R_c$ are correctly reproduced. The point $r=R_c$ where the outer (regularized Yukawa) form is joined smoothly to the inner Gaussian form is treated as a parameter. This procedure guarantees the convergence of the u-integral of Eq.(5.1). In other words, instead of introducing two extra heavier cutoff masses to each and all mesons, we introduce just one common parameter R_c. Beside the simplification, this procedure also guarantees the same r-dependence for $r \geq R_c$ as the original force, whereas the introduction of cutoff masses will inevitably affect the outer region to some extent. The result may or may not depend on the choice of R_c. That, we shall have to see.

Appendix A Functional Forms and Coupling Constants of the Three-Body Force

For the Fujita-Miyazawa (FM) force and the $\pi\pi$-F force, the functions f_1 to f_5 of Eqs.(3.2) to (3.6) take the following forms[10].

$$f_1(12,3) = G_\pi(-\tfrac{1}{9})D_3^+\{Z^{(2)}(r_{13})-J^{(2)}(r_{13})\}\cdot\{Z^{(2)}(r_{23})-J^{(2)}(r_{23})\} \tag{A.1}$$

$$f_2^{(1)}(12,3) = G_\pi[D_1^+\cdot J'(r_{13})J'(r_{23})-D_2^+\{J^{(2)'}(r_{13})J'(r_{23})+J'(r_{13})J^{(2)'}(r_{23})\} -D_3^+\cdot(\hat{r}_{13}\cdot\hat{r}_{23})Z^{(2)}(r_{13})Z^{(2)}(r_{23})] \tag{A.2}$$

$$f_2^{(2)}(12,3) = G_\pi(-\tfrac{1}{3})D_3^+\cdot Z^{(2)}(r_{13})\{-Z^{(2)}(r_{23})+J^{(2)}(r_{23})\} \tag{A.3}$$

$$f_3(12,3) = G_\pi \tfrac{1}{9} E^-\{Z^{(2)}(r_{13})-J^{(2)}(r_{13})\}\cdot\{Z^{(2)}(r_{23})-J^{(2)}(r_{23})\} \tag{A.4}$$

$$f_4(12,3) = G_\pi \tfrac{1}{3} E^-\{-Z^{(2)}(r_{13})Z^{(2)}(r_{23})+J^{(2)}(r_{13})Z^{(2)}(r_{23})\} \tag{A.5}$$

$$f_5(12,3) = G_\pi\cdot E^- Z^{(2)}(r_{13})Z^{(2)}(r_{23}) \tag{A.6}$$

The overall factor G_π is given by

$$G_\pi = -\frac{g_\pi^2}{4\pi}\frac{1}{4\pi}\left(\frac{1}{2M}\right)^2 m_\pi c^2 = -0.8818 \text{ MeV} \tag{A.7}$$

The coupling constants D_1^+, D_2^+, D_3^+ and E^- are given in Table A. The functions $J'(r)$, $J^{(2)}(r)$, $J^{(2)'}$ and $Z^{(2)}(r)$ are defined by

$$J'(r) = -\sum_i a_i(\frac{1}{r} + \kappa_i)Y_i(r) \quad (A.8)$$

$$J^{(2)}(r) = \sum_i a_i\kappa_i^2 Y_i(r) \quad (A.9)$$

$$J^{(2)'}(r) = -\sum_i a_i\kappa_i^2(\frac{1}{r} + \kappa_i)Y_i(r) \quad (A.10)$$

$$Z^{(2)}(r) = \sum_i a_i\kappa_i^2\{1+\frac{3}{\kappa_i r}+\frac{3}{(\kappa_i r)^2}\}Y_i(r), \quad (A.11)$$

	ππF	FM
D_1^+	-0.91±0.14	0.
D_2^+	-0.76±0.12	0.
D_3^+	-2.36±0.10	-1.02
E^-	-0.92±0.015	-0.26

Table A. Coupling constants D_1^+, D_2^+, D_3^+ and E^- in Eqs.(A.1) to (A.6)[10] in units of the pion mass.

where

$$Y_i(r) = e^{-\kappa_i r}/r \quad (A.12)$$

with r given in units of $\mu_\pi^{-1}=\hbar/m_\pi C$ = 1.420 fm, $\kappa_1=1$ and $\kappa_i=\Lambda_i/m_\pi$ for $i\geq 2$ ($m_\pi C^2$=138.7 MeV). Also, $a_1=1$. The coefficients a_i and the cutoff masses Λ_i are introduced for the purpose of regularizing the Yukawa function. For n cutoff masses $\kappa_2,\kappa_3,\ldots\kappa_{n+1}$, we inpose n conditions

$$\sum_{i=1}^{n} a_i\kappa_i^{2N} = 0 , \qquad N = 0,1,2,\ldots,n-1 \quad (A.13)$$

with $a_1=1$ and $\kappa_1=1$. The question of cutoffs, however, has to do with the convergence of angular integrals in the three-body matrix elements also. This is discussed in section 5.

For the two(π+ρ)-Δ three-body force, the functions f_1 to f_5 are given as follows[10].

$$f_1(12,3) = -\frac{4}{9}\{(y_2-z_2)(\bar{y}_1-\bar{z}_1)+(y_1-z_1)(\bar{y}_2-\bar{z}_2)\} \quad (A.14)$$

$$f_2^{(1)}(12,3) = -4(\hat{r}_{13}\cdot\hat{r}_{23})\{z_2\bar{z}_1+z_1\bar{z}_2\} \quad (A.15)$$

$$f_2^{(2)}(12,3) = -\frac{4}{3}\{(y_2-z_2)\bar{z}_1+z_1(\bar{y}_2-\bar{z}_2)\} \quad (A.16)$$

$$f_2^{(2)}(21,3) = -\frac{4}{3}\{z_2(\bar{y}_1-\bar{z}_1)+(y_1-z_1)\bar{z}_2\} \quad (A.17)$$

$$f_3(12,3) = \frac{1}{9}\{(y_2-z_2)(\bar{y}_1-\bar{z}_1)+(y_1-z_1)(\bar{y}_2-\bar{z}_2)\} \quad (A.18)$$

$$f_4(12,3) = \frac{1}{3}\{z_2(\bar{y}_1-\bar{z}_1)+(y_1-z_1)\bar{z}_2\} \tag{A.19}$$

$$f_4(21,3) = \frac{1}{3}\{(y_2-z_2)\bar{z}_1+z_1(\bar{y}_2-\bar{z}_2)\} \tag{A.20}$$

$$f_5(12,3) = z_2\bar{z}_1 + z_1\bar{z}_2 \tag{A.21}$$

where $y_i=y(r_{3i})$ and $z_i=z(r_{3i})$. Similarly for $\bar{y}_i$ and $\bar{z}_i$. The functions $y(r)$, $z(r)$, $\bar{y}(r)$ and $\bar{z}(r)$ are defined by

$$y(r) = \frac{1}{3}\frac{f_\pi f_\pi^*}{4\pi}\frac{1}{m_\pi^2}Y_{m_\pi}^{(2)}(r) + \frac{2}{3}\frac{f_\rho f_\rho^*}{4\pi}\frac{1}{m_\rho^2}Y_{m_\rho}^{(2)}(r) , \tag{A.22}$$

$$z(r) = \frac{1}{3}\frac{f_\pi f_\pi^*}{4\pi}\frac{1}{m_\pi^2}Z_{m_\pi}^{(2)}(r) - \frac{1}{3}\frac{f_\rho f_\rho^*}{4\pi}\frac{1}{m_\rho^2}Z_{m_\rho}^{(2)}(r) , \tag{A.23}$$

$$\bar{y}(r) = \frac{1}{3}\frac{f_\pi f_\pi^*}{4\pi}\frac{1}{m_\pi^2}\delta_\pi\bar{Y}_{m_\pi}^{(2)}(r) + \frac{2}{3}\frac{f_\rho f_\rho^*}{4\pi}\frac{1}{m_\rho^2}\delta_\rho\bar{Y}_{m_\rho}^{(2)}(r) , \tag{A.24}$$

$$\bar{z}(r) = \frac{1}{3}\frac{f_\pi f_\pi^*}{4\pi}\frac{1}{m_\pi^2}\delta_\pi\bar{Z}_{m_\pi}^{(2)}(r) - \frac{1}{3}\frac{f_\rho f_\rho^*}{4\pi}\frac{1}{m_\rho^2}\delta_\rho\bar{Z}_{m_\rho}^{(2)}(r) , \tag{A.25}$$

where

$$Y_\mu^{(2)}(r) = \mu^2 Y_\mu(r) - \sum_{j=1}^{N}\omega_j^{(\mu)}\Omega_j^2\, Y_{\Omega_j}(r) \tag{A.26}$$

$$Z_\mu^{(2)}(r) = \mu^2 Z_\mu(r) - \sum_{j=1}^{N}\omega_j^{(\mu)}\Omega_j^2\, Z_{\Omega_j}(r) \tag{A.27}$$

$$\bar{Y}_\mu^{(2)}(r) = \mu^2 Y_\mu(r) - \sum_{j=1}^{N+1}\bar{\omega}_j^{(\mu)}\Omega_j^2\, Y_{\Omega_j}(r) \tag{A.28}$$

$$\bar{Z}_\mu^{(2)}(r) = \mu^2 Z_\mu(r) - \sum_{j=1}^{N+1}\bar{\omega}_j^{(\mu)}\Omega_j^2\, Z_{\Omega_j}(r) \tag{A.29}$$

The coefficients $\omega_j^{(\mu)}$ and $\bar{\omega}_j^{(\mu)}$ are to be determined by the conditions (A.13). Explicitly, they are given by

$$\omega_j^{(\mu)} = \prod_{\substack{i=1\\ i\neq j}}^{N}{}' \frac{-\mu^2+\Omega_i^2}{-\Omega_j^2+\Omega_i^2} \tag{A.30}$$

and similarly for $\bar{\omega}_j^{(\mu)}$ with N+1 instead of N. The function $Y_\mu(r)$ is given by Eq.(A.12) with $\kappa_i=\mu$, and

$$Z_\mu(r) = (1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2})\ Y_\mu(r) \quad . \tag{A.31}$$

Also, in Eqs.(A.24), (A.25), (A.28) and (A.29),

$$\Omega^2_{N+1} = 2m_\Delta(m_\Delta - m) \tag{A.32}$$

and

$$\delta_{\substack{\pi \\ \rho}} = 2m_\Delta/\{2m_\Delta(m_\Delta - m) - m^2_{\substack{\pi \\ \rho}}\} \tag{A.33}$$

where m_Δ is the Δ mass (1235 MeV) and m is the nucleon mass.
The coupling constants are given as follows: From Ueda-Green I model[18],

$$\frac{f_\pi^2}{4\pi} = 0.0777 \quad , \qquad \frac{f_\rho^2}{4\pi} = 4.26 \quad . \tag{A.34}$$

From the Δ decay width Γ=115 MeV

$$\frac{f_\pi^{*2}}{4\pi} = 0.28 \tag{A.35}$$

The value of f_ρ^* is obtained from the quark model[19]. There are two alternative choices:

$$\frac{f_\rho^{*2}}{4\pi} = 14.9 \quad \text{and} \quad 12.3 \quad . \tag{A.36}$$

Appendix B Irreducible Tensor Representation of V_N

We define the irreducible tensor of rank 1 composed of a vector $\vec{A}=(A_x, A_y, A_z)$ by[17]

$$A_1^{(1)} = -\frac{1}{\sqrt{2}}(A_x + iA_y) \quad , \quad A_0^{(1)} = A_z \quad , \quad A_{-1}^{(1)} = \frac{1}{\sqrt{2}}(A_x - iA_y) \quad , \tag{B.1}$$

and write $\underset{\sim}{A}^{(1)} = (A_1^{(1)}, A_0^{(1)}, A_{-1}^{(1)})$. Applied to $\vec{\tau}$ and $\vec{r}$, we have

$$\tau_1^{(1)} = -\frac{1}{\sqrt{2}}(\tau_x + i\tau_y) \quad , \quad \tau_0^{(1)} = \tau_z \quad , \quad \tau_{-1}^{(1)} = \frac{1}{\sqrt{2}}(\tau_x - i\tau_y) \tag{B.2}$$

$$Y_1^{(1)} = -\frac{1}{\sqrt{2}}(\frac{x}{r} + i\frac{y}{r}) \quad , \quad Y_0^{(1)} = \frac{z}{r} \quad , \quad Y_{-1}^{(1)} = \frac{1}{\sqrt{2}}(\frac{x}{r} - i\frac{y}{r}) \tag{B.3}$$

$\underset{\sim}{Y}^{(1)}$ is related to the spherical harmonics $Y_1^m(\theta,\phi)$ by

$$Y_m^{(1)}(\hat{r}) = \sqrt{\frac{4\pi}{3}}\, Y_1^m(\theta,\phi) \tag{B.4}$$

Let $\underset{\sim}{T}^{(k_1)}$ and $\underset{\sim}{U}^{(k_2)}$ be two irreducible tensors of rank k_1 and k_2, respectively. Then, the irreducible tensor product $\underset{\sim}{V}^{(K)}=[\underset{\sim}{T}^{(k_1)}\otimes\underset{\sim}{U}^{(k_2)}]^{(K)}$ has the components[17)]

$$V_Q^{(K)} = \sum_{q_1q_2} \langle k_1q_1,k_2q_2|KQ\rangle T_{q_1}^{(k_1)}\ U_{q_2}^{(k_2)} \tag{B.5}$$

Inserting the values of the Clebsh-Gordon coefficients $\langle k_1q_1,k_2q_2|KQ\rangle$, we find from Eq.(3.7)

$$\underset{\sim}{T}^{(0,0;0)}(12,3) = -\frac{1}{\sqrt{3}}(\vec{\tau}_1\cdot\vec{\tau}_2) \tag{B.6}$$

Using Eq.(B.5), we can show easily that the rank 1 tensor $\underset{\sim}{C}^{(1)}$ corresponding to $\vec{C}=i(\vec{A}\times\vec{B})$ is

$$\underset{\sim}{C}^{(1)} = \sqrt{2}[\underset{\sim}{A}^{(1)}\times\underset{\sim}{B}^{(1)}]^{(1)} \tag{B.7}$$

Then, it is easily seen that

$$\underset{\sim}{T}^{(1,1;0)}(12,3) = -\frac{1}{\sqrt{6}}\, i(\vec{\tau}_1\times\vec{\tau}_2)\cdot\vec{\tau}_3 \tag{B.8}$$

Similarly to Eqs.(B.6) and (B.8), we have

$$\underset{\sim}{S}^{(0,0;0)}(12,3) = -\frac{1}{\sqrt{3}}\, i(\vec{\sigma}_1\cdot\vec{\sigma}_2) \tag{B.9}$$

$$\underset{\sim}{S}^{(1,1;0)}(12,3) = -\frac{1}{\sqrt{6}}\, i(\vec{\sigma}_1\times\vec{\sigma}_2)\cdot\vec{\sigma}_3 \tag{B.10}$$

The scalar product of $\underset{\sim}{T}^{(k)}$ and $\underset{\sim}{U}^{(k)}$ is defined by[17)]

$$(\underset{\sim}{T}^{(k)}\cdot\underset{\sim}{U}^{(k)}) = \sum_Q(-)^Q T_Q^{(k)}\ U_{-Q}^{(k)} \tag{B.11}$$

In particular, $(\underset{\sim}{T}^{(1)}\cdot\underset{\sim}{U}^{(1)})=(\vec{T}\cdot\vec{U})$. Using Eqs.(B.5) and (B.11) we find

$$\sum_{K=0}^{2}(-)^K(\underset{\sim}{S}^{(K,0;K)}(12,3)\cdot\underset{\sim}{U}^{(1,1;K)}(12,3)) = (\vec{\sigma}_1\cdot\hat{r}_{13})(\vec{\sigma}_2\cdot\hat{r}_{23}) \tag{B.12}$$

Also, using Eqs.(B.5), (B.11) and (B.7) we obtain

$$\sqrt{2}\sum_{K=0}^{2}(-)^{K}(\underset{\sim}{S}^{(1,1;K)}(31,2)\cdot\underset{\sim}{U}^{(1,1;K)}(22,3)) = (i[\vec{\sigma}_3\times\vec{\sigma}_1]\cdot\hat{r}_{23})(\vec{\sigma}_2\cdot\hat{r}_{23})$$

$$= (\vec{\sigma}_3\cdot i\vec{\sigma}_1\times\hat{r}_{23})(\vec{\sigma}_2\cdot\hat{r}_{23}) \qquad \text{(B.13)}$$

By changing the order of coupling in Eq.(3.8), we can show that

$$\underset{\sim}{S}^{(K,1;F)}(31,2) = \sum_{K'}(-)^{K'}\hat{K}\,\hat{K}'\begin{Bmatrix}K & 1 & F\\ K' & 1 & 1\end{Bmatrix}\underset{\sim}{S}^{(K',1;F)}(12,3) \qquad \text{(B.14)}$$

where $\hat{K}=\sqrt{2K+1}$. Thus, from Eqs.(B.13) and (B.14), we also have

$$(\vec{\sigma}_3\cdot i\vec{\sigma}_1\times\hat{r}_{23})(\vec{\sigma}_2\cdot\hat{r}_{23}) = \sqrt{6}\sum_{K}\sum_{K'}(-)^{K+K'}\hat{K}'\begin{Bmatrix}1 & 1 & K\\ K' & 1 & 1\end{Bmatrix}(\underset{\sim}{S}^{(K',1;K)}(12,3)\cdot\underset{\sim}{U}^{(1,1;K)}(22,3)) \qquad \text{(B.15)}$$

Equation (B.7) tells us that the rank 1 irreducible tensor corresponding to $i(\hat{r}_{13}\times\hat{r}_{23})$ is $\sqrt{2}[\underset{\sim}{Y}^{(1)}(\hat{r}_{13})\otimes\underset{\sim}{Y}^{(1)}(\hat{r}_{23})]^{(1)}=\sqrt{2}\underset{\sim}{U}^{(1,1;1)}(12,3)$. Thus

$$(\vec{\sigma}_3\cdot i\hat{r}_{13}\times\hat{r}_{23}) = \sqrt{2}(\underset{\sim}{\sigma}^{(1)}(3)\cdot\underset{\sim}{U}^{(1,1;1)}(12,3)) \qquad \text{(B.16)}$$

Using Eqs.(B.12) and (B.16) we find

$$(\vec{\sigma}_1\cdot\hat{r}_{13})(\vec{\sigma}_2\cdot\hat{r}_{23})(\vec{\sigma}_3\cdot i\hat{r}_{13}\times\hat{r}_{23}) = \sqrt{2}\sum_{K=0}^{2}(-)^{K}(\underset{\sim}{S}^{(K,0;K)}(12,3)\cdot\underset{\sim}{U}^{(1,1;K)}(12,3))$$

$$\times\ (\underset{\sim}{\sigma}^{(1)}(3)\cdot\underset{\sim}{U}^{(1,1;1)}(12,3)) \qquad \text{(B.17)}$$

On the other hand, we can show by Eq.(B.5) that

$$\sum_{K}(-)^{K}\sum_{F}(-)^{K+F+1}(\underset{\sim}{S}^{(K,1;F)}(12,3)\cdot[\underset{\sim}{U}^{(1,1;K)}(12,3)\otimes\underset{\sim}{U}^{(1,1;1)}(12,3)]^{(F)})$$

$$= \sum_{K}(-)^{K}(\underset{\sim}{S}^{(K,0;K)}(12,3)\cdot\underset{\sim}{U}^{(1,1;K)}(12,3))(\underset{\sim}{\sigma}^{(1)}(3)\cdot\underset{\sim}{U}^{(1,1;1)}(12,3)) \qquad \text{(B.18)}$$

Thus, introducing the notation

$$\underset{\sim}{W}^{(K;F)}(12,3) = [\underset{\sim}{U}^{(1,1;K)}(12,3)\otimes\underset{\sim}{U}^{(1,1;1)}(12,3)]^{(F)} \qquad \text{(B.19)}$$

we find from Eqs.(B.17) and (B.18)

$$(\vec{\sigma}_1\cdot\hat{r}_{13})(\vec{\sigma}_2\cdot\hat{r}_{23})(\vec{\sigma}_3\cdot i\hat{r}_{13}\times\hat{r}_{23})=\sqrt{2}\sum_{K}(-)^{K}\sum_{F}(-)^{K+F+1}(\underset{\sim}{S}^{(K,1;F)}(12,3)\cdot\underset{\sim}{W}^{(K;F)}(12,3)) \qquad \text{(B.20)}$$

Next we show that $\underset{\sim}{W}^{(K;F)}(12,3)$ as defined by Eq.(B.19) can also be written as

$$\underset{\sim}{W}^{(K;F)}(12,3) = 4\pi\sqrt{3}\sum_{\xi=0,2}\sum_{\xi'=0,2}\frac{\langle 1010|\xi 0\rangle}{A_\xi}\frac{\langle 1010|\xi' 0\rangle}{A_{\xi'}}\hat{K}\begin{Bmatrix}1 & 1 & K\\ 1 & 1 & 1\\ \xi & \xi' & F\end{Bmatrix}\underset{\sim}{U}^{(\xi,\xi';F)}(12,3) \tag{B.21}$$

where A_ξ is defined later by Eq.(B.28)'. Using Eq.(B.19) with Eq.(B.5) we can write

$$W_\lambda^{(K;F)}(12,3) = \sum_{a,b}\langle K a 1 b|F\lambda\rangle U_a^{(1,1;K)}(12,3)\,U_b^{(1,1;1)}(12,3)$$

$$= \sum_{ab}\langle K a 1 b|F\lambda\rangle\sum_{qq'}\langle 1q1q'|Ka\rangle Y_q^{(1)}(\hat{r}_{13})Y_{q'}^{(1)}(\hat{r}_{23})\sum_{mm'}\langle 1m1m'|1b\rangle$$

$$\times\, Y_m^{(1)}(\hat{r}_{13})Y_{m'}^{(1)}(\hat{r}_{23})$$

$$= \sum_{ab}\langle K a 1 b|F\lambda\rangle\sum_{qq'}\langle 1q1q'|Ka\rangle\sum_{mm'}\langle 1m1m'|1b\rangle\left(\frac{4\pi}{3}\right)^2\sum_{\xi M}\frac{3}{\sqrt{4\pi}\hat{\xi}}\langle 1010|\xi 0\rangle$$

$$\times\langle 1q1m|\xi M\rangle Y_\xi^M(\hat{r}_{13})\sum_{\xi' M'}\frac{3}{\sqrt{4\pi}\hat{\xi}'}\langle 1010|\xi' 0\rangle\langle 1q'1m'|\xi' M'\rangle Y_{\xi'}^{M'}(\hat{r}_{23}) \tag{B.22}$$

where we have used Eq.(B.4) to convert $Y_m^{(1)}(\hat{r})$ to $Y_1^m(\hat{r})$. Now, we avail ourselves of the following formula[17).]

$$\sum_{JM}\hat{J}^2\begin{pmatrix}J_{12} & J_{34} & J\\ M_{12} & M_{34} & M\end{pmatrix}\begin{pmatrix}J_{13} & J_{24} & J\\ M_{13} & M_{24} & M\end{pmatrix}\begin{Bmatrix}j_1 & j_2 & J_{12}\\ j_3 & j_4 & J_{34}\\ J_{13} & J_{24} & J\end{Bmatrix}$$

$$= \sum_{m_1m_2m_3m_4}\begin{pmatrix}j_1 & j_2 & J_{12}\\ m_1 & m_2 & M_{12}\end{pmatrix}\begin{pmatrix}j_3 & j_4 & J_{34}\\ m_3 & m_4 & M_{34}\end{pmatrix}\begin{pmatrix}j_1 & j_3 & J_{13}\\ m_1 & m_3 & M_{13}\end{pmatrix}\begin{pmatrix}j_2 & j_4 & J_{24}\\ m_2 & m_4 & M_{24}\end{pmatrix} \tag{B.23}$$

The sum over (q,q',m,m') in Eq.(B.22) then becomes

$$\sum_{qq'mm'}\langle 1q1q'|Ka\rangle\langle 1m1m'|1b\rangle\langle 1q1m|\xi M\rangle\langle 1q'1m'|\xi' M'\rangle$$

$$= (-)^{-a-b-M-M'}\hat{K}\sqrt{3}\hat{\xi}\hat{\xi}'\sum_{q\omega}\hat{g}^2\begin{pmatrix}K & 1 & g\\ -a & -b & \omega\end{pmatrix}\begin{pmatrix}\xi & \xi' & g\\ -M & -M' & \omega\end{pmatrix}\begin{Bmatrix}1 & 1 & K\\ 1 & 1 & 1\\ \xi & \xi' & g\end{Bmatrix} \tag{B.24}$$

On substituting Eq.(B.24) into Eq.(B.22), the sum over a and b becomes

$$\sum_{a,b} \langle Ka1b|F\lambda\rangle (-)^{-a-b} \begin{pmatrix} K & 1 & g \\ -a-b & \omega \end{pmatrix} = \frac{(-)^F}{\hat{F}} \delta_{F,g} \delta_{\lambda,\omega} \tag{B.25}$$

Thus, with Eqs.(B.24) and (B.25), we find for Eq.(B.22) the following expression.

$$W_\lambda^{(K;F)}(12,3) = 4\pi\sqrt{3} \sum_{\xi\xi'} \langle 1010|\xi 0\rangle\langle 1010|\xi' 0\rangle \hat{K}\hat{F}(-)^F \begin{Bmatrix} 1 & 1 & K \\ 1 & 1 & 1 \\ \xi & \xi' & F \end{Bmatrix} (-)^{\xi+\xi'+F}$$

$$\times \sum_{MM'} (-)^{-M-M'} \begin{pmatrix} \xi & \xi' & F \\ M & M' & -\lambda \end{pmatrix} Y_\xi^M(\hat{r}_{13}) Y_{\xi'}^{M'}(\hat{r}_{23}) \tag{B.26}$$

We now define the irreducible tensors of rank 0 and 2 by

$$\underset{\sim}{Y}^{(0)}(\Omega) = 1 \quad \text{and} \quad \underset{\sim}{Y}^{(2)}(\Omega) = [Y^{(1)}(\Omega) \otimes Y^{(1)}(\Omega)]^{(2)} \quad . \tag{B.27}$$

Using Eq.(B.4), we find altogether the following relations between the irreducible tensor $Y_M^{(\xi)}(\Omega)$ and the spherical harmonics $Y_\xi^M(\Omega)$ for ξ=0,1 and 2:

$$Y_M^{(\xi)}(\Omega) = A_\xi \, Y_\xi^M(\Omega) \quad , \tag{B.28}$$

where

$$A_0 = \sqrt{4\pi} \quad , \quad A_1 = \sqrt{\frac{4\pi}{3}} \quad , \quad \text{and} \quad A_2 = \sqrt{\frac{8\pi}{15}} \tag{B.28'}$$

Therefore, Eq.(B.26) can be written as

$$W_\lambda^{(K;F)}(12,3) = 4\pi\sqrt{3} \sum_{\xi\xi'} \frac{\langle 1010|\xi 0\rangle\langle 1010|\xi' 0\rangle}{A_\xi \qquad A_{\xi'}} \hat{K} \begin{Bmatrix} 1 & 1 & K \\ 1 & 1 & 1 \\ \xi & \xi' & F \end{Bmatrix} \sum_{MM'} \langle \xi M \xi' M'|F\lambda\rangle$$

$$\times Y_M^{(\xi)}(\hat{r}_{13}) Y_{M'}^{(\xi')}(\hat{r}_{23})$$

$$= 4\pi\sqrt{3} \sum_{\xi\xi'} \frac{\langle 1010|\xi 0\rangle\langle 1010|\xi' 0\rangle}{A_\xi \qquad A_{\xi'}} \hat{K} \begin{Bmatrix} 1 & 1 & K \\ 1 & 1 & 1 \\ \xi & \xi' & F \end{Bmatrix} [Y^{(\xi)}(\hat{r}_{13}) \otimes Y^{(\xi')}(\hat{r}_{23})]_\lambda^{(F)} \tag{B.29}$$

which, by Eq.(3.9), proves Eq.(B.21).

Now we are ready to write down V_n of Eqs.(3.2) to (3.6) in terms of the irreducible tensors defined above. Using Eqs.(B.6) and (B.9) we find for V_1 of Eq.(3.2)

$$V_1(12,3) = 3 \, \underset{\sim}{T}^{(0,0;0)}(12,3) \, \underset{\sim}{S}^{(0,0;0)}(12,3) \, f_1(12,3) \quad . \tag{B.30}$$

From Eqs.(B.6) and (B.12), we obtain for V_2 of Eq.(3.3)

$$V_2(12,3) = -\sqrt{3}\ \underset{\sim}{T}^{(0,0;0)}(12,3) \sum_{K=0}^{2} (-)^K (\underset{\sim}{S}^{(K,0;K)}(12,3)\cdot\{\underset{\sim}{U}^{(1,1;K)}(12,3) f_2^{(1)}(12,3)$$

$$+ \underset{\sim}{U}^{(1,1;K)}(11,3) f_2^{(2)}(12,3) + \underset{\sim}{U}^{(1,1;K)}(22,3) f_2^{(2)}(21,3)\}) \ . \qquad (B.31)$$

Using Eqs.(B.8) and (B.10), we find for V_3 of Eq.(3.4)

$$V_3(12,3) = -6\ \underset{\sim}{T}^{(1,1;0)}(12,3)\ \underset{\sim}{S}^{(1,1;0)}(12,3)\ f_3(12,3) \ . \qquad (B.32)$$

For V_4 of Eq.(3.5), we can use Eqs.(B.8) and (B.15) to find

$$V_4(12,3) = 6\ \underset{\sim}{T}^{(1,1;0)}(12,3) \sum_K \sum_{K'} (-)^{K+K'} \hat{K}' \left\{\begin{matrix} 1 & 1 & K \\ K' & 1 & 1 \end{matrix}\right\} (\underset{\sim}{S}^{(K',1;K)}(12,3)$$

$$\cdot \{\underset{\sim}{U}^{(1,1;K)}(22,3) f_4(12,3) - (-)^{K'} \underset{\sim}{U}^{(1,1;K)}(11,3) f_4(21,3)\}) \ . \qquad (B.33)$$

Finally, for V_5 of Eq.(3.6), we use Eqs.(B.8) and (B.20) to obtain

$$V_5(12,3) = \sqrt{12}\ \underset{\sim}{T}^{(1,1;0)}(12,3) \sum_K (-)^K \sum_F (-)^{K-F+1} (\underset{\sim}{S}^{(K,1;F)}(12,3)\cdot \underset{\sim}{W}^{(K;F)}(12,3))$$

$$\times f_5(12,3) \qquad (B.34)$$

where $\underset{\sim}{W}^{(K;F)}(12,3)$ is given by Eq.(B.21).

Appendix C Spatial Matrix Elements in a Body Fixed Reference Frame

By the Wigner-Eckart theorem, we have for the spatial matrix element in Eq.(4.8)

$$<(L\ell)L_0(12,3)\,||\,\underset{\sim}{U}^{(\xi,\xi';\bar{F})}(ij;3) f\,||\,(L'\ell')L_0'(12,3)>$$

$$= \hat{L}_0 (-)^{2\bar{F}} <(L\ell)L_0K_0\,|U_c^{(\xi,\xi';\bar{F})}(ij;3) f\,|\,(L'\ell')L_0'K_0'>/<L_0'K_0'\bar{F}c\,|\,L_0K_0> \qquad (C.1)$$

Transforming to the body-fixed reference frame in which the z-axis is along $\vec{x}$ and the xz plane is on the plane of $\vec{x}$ and $\vec{y}$, we utilize the following transformations.

$$Y_L^M(\hat{x}) = D_{0M}^{(L)}(\omega)\frac{\hat{L}}{\sqrt{4\pi}} \ , \qquad (C.2)$$

$$Y_\ell^m(\hat{y}) = \sum_k D_{km}^{(\ell)}(\omega)\; Y_\ell^k(\mathring{\hat{y}}) \quad . \tag{C.3}$$

Here, ω is the Euler angles of the transformation, and the quantities with the symbol $\circ$ over them refer to the body-fixed reference frame. We thus have

$$|(L\ell)L_0K_0> = \sum_{Mm} <LM\ell m|L_0K_0> D_{0M}^{(L)}(\omega)\; \frac{\hat{L}}{\sqrt{4\pi}} \sum_k D_{km}^{(\ell)}(\omega) Y_\ell^k(\mathring{\hat{y}}) \quad . \tag{C.4}$$

Further,

$$U_c^{(\xi,\xi';\bar{F})}(ij;3)\,f = \sum_d D_{dc}^{(\bar{F})}(\omega)\; \mathring{U}_d^{(\xi,\xi';\bar{F})}\; \mathring{f} \tag{C.5}$$

Substituting Eqs.(C.4) and (C.5) into Eq.(C.1) we find

$$<(L\ell)L_0(12,3)\,||\,\underset{\sim}{U}^{(\xi,\xi';\bar{F})}(ij;3)\,f\,||\,(L'\ell')L_0'(12,3)>$$

$$= \frac{\hat{L}_0(-)^{2\bar{F}}}{<L_0'K_0'\bar{F}c|L_0K_0>} \sum_{Mm} \sum_{M'm'} <LM\ell m|L_0K_0><L'M'\ell'm'|L_0'K_0'> \frac{\hat{L}\hat{L}'}{4\pi}$$

$$\times \int d\omega\; D_{0M}^{(L)*}(\omega) \sum_{kk'd} D_{km}^{(\ell)*}(\omega) D_{dc}^{(\bar{F})}(\omega) D_{k'm'}^{(\ell')}(\omega) D_{0M'}^{(L')}(\omega) \int_{-1}^{1} d(\cos\theta)$$

$$\times\; Y_\ell^{k*}(\mathring{\hat{y}})\, \mathring{U}_d^{(\xi,\xi';\bar{F})}(ij;3)\, \mathring{f}\; Y_{\ell'}^{k'}(\mathring{\hat{y}}) \quad . \tag{C.6}$$

On account of the following three relations[17)]

$$\sum_{Mm} <LM\ell m|L_0K_0> D_{0,M}^{(L)}(\omega) D_{k,m}^{(\ell)}(\omega) = <L0\ell k|L_0k> D_{kK_0}^{(L_0)}(\omega) \quad , \tag{C.7}$$

$$D_{d,c}^{(\bar{F})}(\omega) D_{k',K_0'}^{(L_0')}(\omega) = \sum_{L_0''} <L_0'k'\bar{F}d|L_0'',k'+d><L_0'K_0'\bar{F}c|L_0'',K_0'+c> D_{k'+d,K_0'+c}^{(L_0'')}(\omega) \quad , \tag{C.8}$$

and

$$\int d\omega\; D_{k,K_0}^{(L_0)*}(\omega) D_{k'+d,K_0'+c}^{(L_0'')}(\omega) = \frac{8\pi^2}{\hat{L}_0^2}\, \delta_{k,k'+d}\, \delta_{K_0,K_0'+c}\, \delta_{L_0,L_0''} \quad , \tag{C.9}$$

Eq.(C.6) becomes

$$<(L\ell)L_0(12,3)\,||\,\underset{\sim}{U}^{(\xi,\xi';\bar{F})}(ij;3)\,f\,||\,(L'\ell')L_0'(12,3)>$$

$$= \frac{8\pi^2}{\hat{L}_0}\frac{\hat{L}\hat{L}'}{4\pi}\sum_{kk'd}<L0\ell k|L_0k><L'0\ell'k'|L_0'k'><L_0'k'\bar{F}d|L_0k>\delta_{k,k'+d}$$
$$\times \int_{-1}^{1} d(\cos\theta) Y_\ell^{k*}(\mathring{\hat{y}})\mathring{U}_d^{(\xi,\xi';\bar{F})}(ij;3)\mathring{f}Y_{\ell'}^{k'}(\mathring{\hat{y}})$$
$$= \frac{8\pi^2}{\hat{L}_0}\frac{\hat{L}\hat{L}'}{4\pi}\sum_{d,n}\sum_\nu \frac{\hat{\ell}\hat{\ell}'}{\sqrt{4\pi}\,\hat{n}}<\ell 0\ell'0|n0>\int_{-1}^{1} d(\cos\theta) Y_n^\nu(\mathring{\hat{y}})\mathring{U}_d^{(\xi,\xi';\bar{F})}(ij;3)\mathring{f}$$
$$\times \sum_{kk'}(-)^k\delta_{k,k'+d}<L0\ell k|L_0k><L'0\ell'k'|L_0'k'><L_0'k'\bar{F}d|L_0k><\ell-k\ell'k'|n\nu> \quad (C.10)$$

Using Eq.(B.23), the last sum over k and k' in Eq.(C.10) becomes

$$\sum_{kk'}(-)^k<L0\ell k|L_0k><L'0\ell'k'|L_0'k'><L_0'k'\bar{F}d|L_0k><\ell-k\ell'k'|n\nu>$$
$$= (-)^{L'+\ell}\hat{L}_0^2\hat{L}_0'\hat{n}\sum_g \hat{g}^2\begin{pmatrix} n & \bar{F} & g\\ -\nu & -d & 0\end{pmatrix}\begin{pmatrix} L & L' & g\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} \ell & \ell' & n\\ L_0 & L_0' & \bar{F}\\ L & L' & g\end{Bmatrix} \quad (C.11)$$

Using Eq.(C.11) in Eq.(C.10), we find

$$<(L\ell)L_0(12,3)\,||\,\underset{\sim}{U}^{(\xi,\xi';\bar{F})}(ij;3)f\,||\,(L'\ell')L_0'(12,3)>$$
$$= \sum_n\sum_g \mathbb{K}_{\alpha\alpha';\bar{F}}^{n,g}\sum_\nu\begin{pmatrix} n & \bar{F} & g\\ \nu & -\nu & 0\end{pmatrix}\int_{-1}^{1} d(\cos\theta)\mathring{U}_\nu^{(\xi,\xi';\bar{F})}(ij;3)\cdot\mathring{f}\cdot Y_n^{-\nu}(\mathring{\hat{y}}), \quad (C.12)$$

where

$$\mathbb{K}_{\alpha\alpha';\bar{F}}^{n,g} = \sqrt{\pi}\hat{L}\hat{L}'\hat{L}_0\hat{L}_0'\hat{\ell}\hat{\ell}'(-)^{L'-\ell'}\hat{n}\begin{pmatrix} \ell & \ell' & n\\ 0 & 0 & 0\end{pmatrix}\hat{g}^2\begin{pmatrix} L & L' & g\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} \ell & \ell' & n\\ L_0 & L_0' & \bar{F}\\ L & L' & g\end{Bmatrix}. \quad (C.13)$$

Appendix D The Matrix Elements and Moshinsky's Formula

We reduce $\mathring{U}_\lambda^{(\xi,\xi';\bar{F})}(ij;3)$ by the help of Moshinsky's formula[16] in this appendix. From the definition (3.9) and by Eq.(B.28)

$$\mathring{U}_\lambda^{(\xi,\xi';\bar{F})}(ij;3) = [\underset{\sim}{Y}^{(\xi)}(\hat{r}_{i3})\otimes\underset{\sim}{Y}^{(\xi')}(\hat{r}_{j3})]_\lambda^{(\bar{F})} = \sum_{MM'}<\xi M\xi'M'|\bar{F}\lambda>Y_M^{(\xi)}(\hat{r}_{i3})Y_{M'}^{(\xi')}(\hat{r}_{j3})$$
$$= A_\xi A_{\xi'}\sum_{MM'}<\xi M\xi'M'|\bar{F}\lambda>Y_\xi^M(\hat{r}_{i3})Y_{\xi'}^{M'}(\hat{r}_{j3}) \quad (D.1)$$

The vectors $\vec{r}_{13}$ and $\vec{r}_{23}$ are given by (see Fig. 1)

$$\vec{r}_{13} = \vec{r}_3 - \vec{r}_1 = \vec{y} - \frac{\vec{x}}{2}$$
$$\vec{r}_{23} = \vec{r}_3 - \vec{r}_2 = \vec{y} + \frac{\vec{x}}{2} \qquad \text{(D.2)}$$

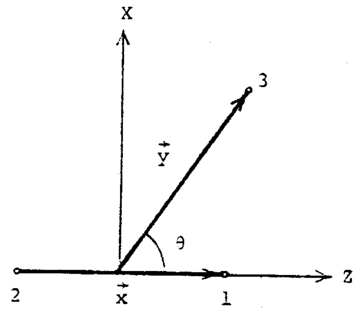

Fig. 1 Body-fixed reference frame

We write these vectors as

$$\vec{r}_{i3} = p_i \vec{x} + \vec{y} \;, \qquad (p_1 = -\frac{1}{2} \;,\; p_2 = \frac{1}{2}) \qquad \text{(D.3)}$$

Then, Moshinsky's formula[16] as applied here in the body-fixed reference frame becomes (for i=1 and 2)

$$Y_\xi^M(\hat{r}_{i3}) = \frac{1}{r_{i3}^\xi} \sum_{\substack{a=0 \\ (b=\xi-a)}}^{\xi} \binom{2\xi+1}{2a}^{1/2} (p_i x)^a y^b \langle a0bM|\xi M\rangle Y_b^M(\mathring{y}) \qquad \text{(D.4)}$$

where we have taken advantage of the fact that $\hat{x}$ is along the body-fixed Z-axis. With the help of Eq.(D.4), Eq.(D.1) becomes

$$\mathring{U}_\lambda^{(\xi,\xi';\bar{F})}(ij;3) = A_\xi A_{\xi'} \frac{1}{r_{i3}^\xi} \frac{1}{r_{j3}^{\xi'}} \sum_{\substack{a=0 \\ (b=\xi-a)}}^{\xi} \sum_{\substack{a'=0 \\ (b'=\xi'-a')}}^{\xi'} \binom{2\xi+1}{2a}^{1/2} \binom{2\xi'+1}{2a'}^{1/2}$$
$$\times\, p_i^a p_j^{a'} x^{a+a'} y^{b+b'} \sum_{MM'} \langle \xi M \xi' M'|\bar{F}\lambda\rangle \langle a0bM|\xi M\rangle \langle a'0b'M'|\xi' M'\rangle$$
$$\times\, Y_b^M(\mathring{y}) Y_{b'}^{M'}(\mathring{y}) \qquad \text{(D.5)}$$

With the relations[17)]

$$Y_b^M(\mathring{y}) Y_{b'}^{M'}(\mathring{y}) = \sum_{c\gamma} \frac{1}{\sqrt{4\pi}} \frac{\hat{b}\hat{b}'}{\hat{c}} \langle b0b'0|c0\rangle \langle bMb'M'|c\gamma\rangle Y_c^\gamma(\mathring{y}) \qquad \text{(D.6)}$$

and

$$\sum_{MM'} \langle \xi M \xi' M'|\bar{F}\lambda\rangle \langle a0bM|\xi M\rangle \langle a'0b'M'|\xi' M'\rangle \langle bMb'M'|c\gamma\rangle$$
$$= \delta_{\gamma,\lambda} (-)^{\xi+a-b+\xi'+a'-b'} \hat{\bar{F}}\hat{\xi}\hat{\xi}'\hat{c} (-)^{c+\lambda} \sum_g \hat{g}^2 \begin{pmatrix} \bar{F} & C & g \\ -\lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} a & a' & g \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \xi & \xi' & \bar{F} \\ b & b' & c \\ a & a' & g \end{Bmatrix} \qquad \text{(D.7)}$$

which can be obtained from Eq.(B.23), we find for Eq.(D.5)

$$\mathring{U}_\lambda^{(\xi\xi';\bar{F})}(ij;3) = \frac{1}{\sqrt{4\pi}} \frac{1}{r_{i3}^\xi r_{j3}^{\xi'}} \sum_{\substack{a=0 \\ (b=\xi-a)}}^{\xi} \sum_{\substack{a'=0 \\ (b'=\xi'-a')}}^{\xi'} x^{a+a'} y^{b+b'} \Gamma_{\bar{F}}^{\xi\xi',aa'} p_i^a p_j^{a'}$$

$$\times\sum_{c}(-)^{c}\langle b0b'0|c0\rangle\sum_{g}\hat{g}^{2}(-)^{\lambda}\begin{pmatrix}\bar{F} & c & g\\ -\lambda & \lambda & 0\end{pmatrix}\begin{pmatrix}a & a' & g\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\xi & \xi' & \bar{F}\\ b & b' & c\\ a & a' & g\end{Bmatrix} Y_{c}^{\lambda}(\mathring{\hat{y}}) \qquad (D.8)$$

where

$$\Gamma_{\bar{F}}^{\xi\xi',aa'} = A_{\xi}A_{\xi'}\left(\frac{2\xi+1}{2a}\right)^{1/2}\left(\frac{2\xi'+1}{2a'}\right)^{1/2}\hat{b}\hat{b}'\hat{\xi}\hat{\xi}'\hat{\bar{F}} \qquad (D.9)$$

When Eq.(D.8) is substituted in Eq.(4.11), the sum over λ becomes

$$\sum_{\lambda}\begin{pmatrix}n & \bar{F} & d\\ \lambda & -\lambda & 0\end{pmatrix}(-)^{\lambda}\begin{pmatrix}\bar{F} & c & g\\ -\lambda & \lambda & 0\end{pmatrix}Y_{c}^{\lambda}(\mathring{\hat{y}})Y_{n}^{-\lambda}(\mathring{\hat{y}})$$

$$=\sum_{\lambda}\begin{pmatrix}n & \bar{F} & d\\ \lambda & -\lambda & 0\end{pmatrix}(-)^{\lambda}\begin{pmatrix}\bar{F} & c & g\\ -\lambda & \lambda & 0\end{pmatrix}\sum_{h}\frac{\hat{c}\hat{n}\hat{h}}{\sqrt{4\pi}}\begin{pmatrix}c & n & h\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}c & n & h\\ \lambda & -\lambda & 0\end{pmatrix}Y_{h}^{0}(\mathring{\hat{y}})$$

$$=\frac{1}{4\pi}(-)^{g+n}\hat{c}\hat{n}\sum_{h}\hat{h}^{2}\begin{pmatrix}g & h & d\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}c & n & h\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}g & h & d\\ n & \bar{F} & c\end{Bmatrix}P_{h}(u) \qquad (D.10)$$

where $u=\cos\theta_{x\hat{y}}$. Therefore, we obtain for Eq.(4.11)

$$\langle(L\ell)L_{0}(12,3)\,||\,\underset{\sim}{U}^{(\xi,\xi';\bar{F})}(ij;3)f\,||\,(L'\ell')L_{0}'(12,3)\rangle$$

$$=\sum_{n,d}\mathbb{K}_{\alpha\alpha';\bar{F}}^{n,d}\sum_{\lambda}\begin{pmatrix}n & \bar{F} & d\\ \lambda & -\lambda & 0\end{pmatrix}\int_{-1}^{1}du\mathring{U}_{\lambda}^{(\xi,\xi';\bar{F})}(ij;3)Y_{n}^{-\lambda}(\mathring{\hat{y}})\mathring{f}$$

$$=\sum_{\substack{a=0\\(b=\xi-a)}}^{\xi}\sum_{\substack{a'=0\\(b'=\xi'-a')}}^{\xi'}x^{a+a'}y^{b+b'}p_{i}^{a}p_{j}^{a'}\sum_{h}R_{(\alpha\alpha',\bar{F})}^{(aa',\xi\xi',h)}\int_{-1}^{1}P_{h}(u)\frac{\mathring{f}}{r_{i3}^{\xi}r_{j3}^{\xi'}}du \qquad (D.11)$$

where we have defined

$$R_{(\alpha\alpha',\bar{F})}^{(aa',\xi\xi',h)} = \frac{1}{(4\pi)^{3/2}}\Gamma_{\bar{F}}^{\xi\xi',aa'}\sum_{c}(-)^{c}\langle b0b'0|c0\rangle\sum_{g}\hat{g}^{2}\begin{pmatrix}a & a' & g\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\xi & \xi' & \bar{F}\\ b & b' & c\\ a & a' & g\end{Bmatrix}$$

$$\times\ (-)^{g}\sum_{n,d}\mathbb{K}_{\alpha\alpha';\bar{F}}^{n,d}(-)^{n}\hat{c}\hat{n}\hat{h}^{2}\begin{pmatrix}g & h & d\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}c & n & h\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}g & h & d\\ n & \bar{F} & c\end{Bmatrix} \qquad (D.12)$$

in which $\Gamma_{\bar{F}}^{\xi\xi',aa'}$ and $\mathbb{K}_{\alpha\alpha';\bar{F}}^{n,d}$ are defined by Eqs.(D.9) and (4.12), respectively.

Appendix E Derivation of Eq.(4.18)

From Eqs.(4.17) and (3.8), we find

$$Z^{(\bar{K},\bar{n};\bar{F})}_{(SS_0,S'S_0')} = \hat{S}_0\hat{\bar{F}}\hat{S}_0' \begin{Bmatrix} S' & \frac{1}{2} & S_0' \\ \bar{K} & \bar{n} & \bar{F} \\ S & \frac{1}{2} & S_0 \end{Bmatrix} \langle S(1,2) \| [\underset{\sim}{\sigma}^{(1)}(1)\otimes\underset{\sim}{\sigma}^{(1)}(2)]^{(\bar{K})} \| S'(1,2)\rangle$$

$$\times \langle\tfrac{1}{2} \| \underset{\sim}{\sigma}^{(\bar{n})} \| \tfrac{1}{2}\rangle \ . \tag{E.1}$$

For $\bar{n}=0$, the last reduced matrix element becomes

$$\langle\tfrac{1}{2} \| \underset{\sim}{\sigma}^{(\bar{n})} \| \tfrac{1}{2}\rangle = \langle\tfrac{1}{2} \| 1 \| \tfrac{1}{2}\rangle = \sqrt{2} \ . \tag{E.2}$$

For $\bar{n}=1$,

$$\langle\tfrac{1}{2} \| \underset{\sim}{\sigma}^{(\bar{n})} \| \tfrac{1}{2}\rangle = \langle\tfrac{1}{2} \| \underset{\sim}{\sigma}^{(1)} \| \tfrac{1}{2}\rangle = \sqrt{2}\langle\tfrac{1}{2}\ \tfrac{1}{2}|\sigma_z|\tfrac{1}{2}\ \tfrac{1}{2}\rangle/\langle\tfrac{1}{2}\ \tfrac{1}{2}\ 10|\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \sqrt{6} \tag{E.3}$$

Further,

$$\langle S(1,2) \| [\underset{\sim}{\sigma}^{(1)}(1)\otimes\underset{\sim}{\sigma}^{(1)}(2)]^{(\bar{K})} \| S'(1,2)\rangle = \hat{S}\hat{\bar{K}}\hat{S}' \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S' \\ 1 & 1 & \bar{K} \\ \frac{1}{2} & \frac{1}{2} & S \end{Bmatrix} \langle\tfrac{1}{2} \| \underset{\sim}{\sigma}^{(1)} \| \tfrac{1}{2}\rangle^2 \ . \tag{E.4}$$

Thus, combining Eqs.(E.1) to (E.4), we obtain

$$Z^{(\bar{K},\bar{n};\bar{F})}_{(SS_0,S'S_0')} = \hat{S}_0\hat{\bar{F}}\hat{S}_0' \begin{Bmatrix} S' & \frac{1}{2} & S_0' \\ \bar{K} & \bar{n} & \bar{F} \\ S & \frac{1}{2} & S_0 \end{Bmatrix} 6\hat{S}\hat{\bar{K}}\hat{S}' \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S' \\ 1 & 1 & \bar{K} \\ \frac{1}{2} & \frac{1}{2} & S \end{Bmatrix} \times \begin{cases} \sqrt{2} \ , & \text{for } \bar{n}=0, \\ \sqrt{6} \ , & \text{for } \bar{n}=1 \ . \end{cases} \tag{E.5}$$

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