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Resonance Phenomenon in Classical Cepheids

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To investigate resonance phenomenon in classical cepheids, the non-linear radial oscillation of stars is studied based on the assumption that the non-adiabatic perturbation is expressed in terms of van der Pol's type damping. Two- and three-wave resonance in this system is applied to classical cepheids to describe their bump and double-mode behavior. The phase of bump and the depression of amplitude are explained for bump cepheids. The double-periodicity is shown by the enhancement of the third overtone in three-wave resonance. Non-linear effect on resonant period is also discussed briefly.

Keywords: Variable stars, Classical cepheids, Bump cepheids, Double-mode cepheids, Resonance theory.

§1. Introduction

The double-mode cepheid is still a problem of the pulsation theory of variable stars. However Stellingwerf[1] once succeeded in showing the possibility of double-periodicity by non-linear hydrodynamical calculation, their period-ratio has not been solved yet (see reviews by J.P. Cox[2] and A.N. Cox[3]). The simultaneous excitation of two modes has been supposed as the result of resonance[4] but the non-linear simulation by Simon et al.[5] does not succeed in showing the double-mode behavior for likely resonant model envelopes. The calculated period-ratio differs from the observed one, and moreover the resonance distance presented by the linear theory is not barely equal to zero for models with evolutionary mass, even if chemically inhomogeneous envelopes are assumed.[6] Concerning with the resonance problem the bump cepheid is also interesting because their bumps are likely evidence for the resonance between the fundamental mode and the second overtone mode as pointed out by Simon and Schmidt.[7]

The resonance theory in oscillation gives several interesting results, so it seems necessarily to study the multiple-periodicity in the pulsating stars in view of the resonance. In the present paper, we use revised expressions presenting the stellar radial oscillation in our previous paper.[6] Van der Pol's type damping force is assumed to describe the self-excitation following

Krogdahl.[8] The effect of mode-coupling on the amplitude of oscillation is analysed as well as the phase-difference between enhanced modes. The effect of higher order terms upon the period ratio of maximum resonance is also discussed. It seems very important to study the stellar pulsation by using the resonance analysis.

§2. Equations for Coupling Oscillation

2-1. Coupling coefficients

In our previous paper,[6] we derived non-linear and non-adiabatic equation of stellar radial pulsation as follows:

$$\dddot{r} = \frac{2r\dot{r}}{\rho_0 a^2}\frac{\partial P}{\partial a} - \frac{r^2}{\rho_0 a^2}\frac{\partial}{\partial a}(\dot{P}) + \frac{2g_0 a^2 \dot{r}}{r^3} . \tag{1}$$

Equation (2) of our previous paper, the equation of continuity, should be written up to the second order of displacement $\zeta(a,t) = r(a,t) - a$ as follows:

$$\rho/\rho_0 = 1 - \mathrm{div}\,\zeta - 2(\zeta/a)\mathrm{div}\,\zeta + 3(\zeta/a)^2 + (\mathrm{div}\,\zeta)^2 . \tag{2}$$

The operator div means $a^{-2}(\partial/\partial a)a^2$ through the present paper. We used the expression neglecting the second order terms of the equation of continuity. For this reason, equation (6) in our previous paper has to be revised. In the adiabatic case the pressure is written in the following form.

$$P/P_0 = 1 - \gamma\,\mathrm{div}\,\zeta + 3\gamma(\zeta/a)^2 - 2\gamma(\zeta/a)\mathrm{div}\,\zeta + (1/2)\gamma(\gamma+1)(\mathrm{div}\,\zeta)^2 . \tag{3}$$

Using this equation we have

$$\begin{aligned}\dddot{\zeta} - (1/\rho_0)\frac{\partial}{\partial a}(\gamma P_0 \mathrm{div}\,\dot{\zeta}) - 4g_0\dot{\zeta}/a = {} & -(1/\rho_0)\frac{\partial}{\partial a}[(\gamma+1)\gamma P_0 \mathrm{div}\,\zeta\,\mathrm{div}\,\dot{\zeta}] \\ & + (4/\rho_0)[(\zeta/a)\frac{\partial}{\partial a}(\gamma P_0 \mathrm{div}\,\dot{\zeta}) + (\dot{\zeta}/a)\frac{\partial}{\partial a}(\gamma P_0 \mathrm{div}\,\zeta)] \\ & + (2/\rho_0)[\frac{\partial}{\partial a}(\zeta/a)\gamma P_0 \mathrm{div}\,\dot{\zeta} + \frac{\partial}{\partial a}(\dot{\zeta}/a)\gamma P_0 \mathrm{div}\,\zeta] \\ & - (6/\rho_0)[\frac{\partial}{\partial a}(\gamma P_0 \zeta\dot{\zeta}/a^2) - 4g_0\zeta\dot{\zeta}/a^2 ,\end{aligned} \tag{4}$$

instead of equation (6) of the previous paper. Non-adiabatic terms are ignored in this expression.

In the present paper, we describe detail to derive the coupling coefficient because of its importance although it seems not so complicated. We put here that $f(\zeta,a)$ is the right hand side of equation (4). In the case of $f(\zeta,a) \equiv 0$, equation (4) is Sturm-Liouville's differential equation. Putting ξ_m the eigenfunction of order m of this case, we set the solution ζ of equation (4) in the following form:

$$\zeta(a,t) = \sum_m q_m(t)\xi_m(a). \tag{5}$$

We used that expression in the previous paper although it is not exact. We use it again in the present paper. Then we can expand $f(\zeta,a)$ by function ξ_m.

$$\begin{aligned} f(a) = \sum_m\sum_n \dot{q}_m q_n\{&-(1/\rho_0)\frac{d}{da}[(\gamma+1)\gamma P_0 \mathrm{div}\,\xi_m \mathrm{div}\,\xi_n] \\ &+ (4/\rho_0)[(\xi_m/a)\frac{d}{da}(\gamma P_0 \mathrm{div}\,\xi_n) + (\xi_n/a)\frac{d}{da}(\gamma P_0 \mathrm{div}\,\xi_m)] \\ &+ (2/\rho_0)[\frac{d}{da}(\xi_m/a)\gamma P_0 \mathrm{div}\,\xi_n + \frac{d}{da}(\xi_n/a)\gamma P_0 \mathrm{div}\,\xi_m] \\ &- (6/\rho_0)\frac{d}{da}(\gamma P_0 \xi_m\xi_n/a^2) - 4g_0\xi_m\xi_n/a^2\}. \end{aligned} \tag{6}$$

Taking into account equation (6), we can derive the following equation for each q_s.

$$\sum_s (\dddot{q}_s + \sigma_s^2\dot{q}_s)\xi_s = f(a), \tag{7}$$

where σ_s is an eigenvalue associated with the eigenfunction ξ_s. Multyiplying this equation by $4\pi a^2\rho_0\xi_s da$, integrating over the radius of the star, it follows from the condition of orthogonality that

$$\dddot{q}_s + \sigma_s^2\dot{q}_s = \sum_m\sum_n \sigma_s^2 C(s;m,n)\dot{q}_m q_n. \tag{8}$$

Here we define the coupling constant as follows:

$$\begin{aligned} C(s;m,n)\sigma_s^2\int_0^M \xi_s^2\,dm = \int_0^M \{&-(\xi_s/\rho_0)\frac{d}{da}[(\gamma+1)\gamma P_0 \mathrm{div}\,\xi_m \mathrm{div}\,\xi_n] \\ &+ (4\xi_s/\rho_0)[(\xi_m/a)\frac{d}{da}(\gamma P_0 \mathrm{div}\,\xi_n) + (\xi_n/a)\frac{d}{da}(\gamma P_0 \mathrm{div}\,\xi_m)] \\ &+ (2\xi_s/\rho_0)[\frac{d}{da}(\xi_m/a)\gamma P_0 \mathrm{div}\,\xi_n + \frac{d}{da}(\xi_n/a)\gamma P_0 \mathrm{div}\,\xi_m] \\ &- (6\xi_s/\rho_0)\frac{d}{da}(\gamma P_0\xi_m\xi_n/a^2) - 4g_0\xi_s\xi_m\xi_n/a^2\}dm, \end{aligned} \tag{9}$$

where

$$dm = 4\pi a^2\rho_0 da. \tag{10}$$

The integral $\sigma_s^2\int_0^M \xi_s^2\,dm$ is the oscillatory moment of inertia. Using the partial integration taking into account the condition that $P_0 = 0$ at the surface and

$\xi = 0$ at the centre, we have

$$C(s;m,n)\sigma_s^2 \int_0^M \xi_s^2\, dm = \int_0^M \{(\gamma+1)\gamma(P_0/\rho_0)\operatorname{div}\xi_m \operatorname{div}\xi_n \operatorname{div}\xi_s$$

$$- 2\gamma(P_0/\rho_0)[\operatorname{div}\xi_m \operatorname{div}(\xi_n\xi_s/a) + \operatorname{div}\xi_n \operatorname{div}(\xi_s\xi_m/a)$$

$$+ \operatorname{div}\xi_s \operatorname{div}(\xi_m\xi_n/a)] - 4g_0\xi_m\xi_n\xi_s/a^2\}dm$$

$$= \int_0^M \{(\gamma+1)\gamma(P_0/\rho_0)\operatorname{div}\xi_m \operatorname{div}\xi_n \operatorname{div}\xi_s$$

$$- 2\gamma(P_0/\rho_0)[(\xi_m/a)\operatorname{div}\xi_n(2\operatorname{div}\xi_s - 3\xi_s/a)$$

$$+ (\xi_n/a)\operatorname{div}\xi_s(2\operatorname{div}\xi_m - 3\xi_m/a)$$

$$+ (\xi_s/a)\operatorname{div}\xi_m(2\operatorname{div}\xi_n - 3\xi_n/a)]$$

$$- 4g_0\xi_m\xi_n\xi_s/a^2\}dm. \qquad (11)$$

The exponent of the denominator of the last term in the integrand of the right hand side of equation (9) in our previous paper should be read as 2. It was caused from a carelessness in reading the proofs. The change in the equation of continuity does not affect this exponent. The expansion or contraction of a layer in a star caused from the oscillations of mode m and n affects the oscillation of another mode of order s. C(s/m,n) is used to analyse the resonance of stellar radial oscillation through the present paper.

2-2. Two-wave coupling

In the present sub-section we consider two-wave coupling by using the coupling coefficient defined above. We have the following two equations including damping term K.

$$\ddot{q}_i + \sigma_i^2\,(1-(1/2)C(i;i,i)q_i - C(i;i,j)q_j)q_i - K_i\dot{q}_i = (1/2)\sigma_i^2\, C(i;j,j)q_j^2\,, \qquad (12a)$$

$$\ddot{q}_j + \sigma_j^2\,(1-(1/2)C(j;j,j)q_j - C(j;i,j)q_i)q_j - K_j\dot{q}_j = (1/2)\sigma_j^2\, C(j;i,i)q_i^2\,. \qquad (12b)$$

Although the expression of damping term is subject to the conclusion of careful study on non-linear and non-adiabatic behavior of outer envelopes of stars, we choose here van der Pol's one following Krogdahl.[8] It is probable that the damping force could concern with the total amplitude of oscillations when two or more modes are enhanced simultaneously, because the result of non-linear calculation showed that the instability seems to be independent on the mode of pulsation.[9] We choose, however, here van der Pol's type force which depends on the amplitude of only its own mode, for simplicity. So we have

$$K_i = \mu_i(1-\alpha_i^2 q_i^2), \tag{13a}$$

$$K_j = \mu_j(1-\alpha_j^2 q_j^2). \tag{13b}$$

Our interests are focussed to the problem of two-wave coupling for the case that the higher mode, say, j-mode, has frequency near superharmonic one of the lower mode, say i-mode. In this case, some terms of equations (12a,b) can be neglected, and we have

$$\ddot{q}_i + \sigma_i^2 q_i - \sigma_i^2 C(i;i,j)q_i q_j - K_i\dot{q}_i = 0, \tag{14a}$$

$$\ddot{q}_j + \sigma_j^2 q_j - K_j\dot{q}_j = (1/2)\sigma_j^2 C(j;i,i)q_i^2 . \tag{14b}$$

Putting q_i and q_j as follows:

$$q_i = a_i\cos\omega_i t + b_i\cos((\omega_j-\omega_i)t+\phi_i), \tag{15a}$$

$$q_j = a_j\cos\omega_j t + b_j\cos(2\omega_i t+\phi_j), \tag{15b}$$

and comparing the amplitude of $\cos\omega_i t$, $\sin\omega_j t$, $\cos((\omega_j-\omega_i)t+\phi_i)$, $\sin((\omega_j-\omega_i)t + \phi_i)$, $\cos\omega_j t$, $\sin\omega_j t$, $\cos(2\omega_i t+\phi_j)$ and $\sin(2\omega_i t+\phi_j)$ in equations (15a,b), we have equations for a_i, b_i, a_j and b_j, including ω_i, ω_j, ϕ_i and ϕ_j.

$$(\sigma_i^2-\omega_i^2)a_i = (1/2)\sigma_i^2 C(i;i,j)(a_i b_j\cos\phi_j - b_i a_j\cos\phi_i), \tag{16a}$$

$$-\mu_i\omega_i a_i[1-\alpha_i^2(a_i^2/4+b_i^2/2)] = (1/2)\sigma_i^2 C(i;i,j)(a_i b_j\sin\phi_j - b_i a_j\sin\phi_i), \tag{16b}$$

$$[\sigma_i^2-(\omega_j-\omega_i)^2]b_i = (1/2)\sigma_i^2 C(i;i,j)a_i a_j\cos\phi_i , \tag{17a}$$

$$\mu_i b_i(\omega_j-\omega_i)[1-\alpha_i^2(a_i^2/2+b_i^2/4)] = (1/2)\sigma_i^2 C(i;i,j)a_i a_j\sin\phi_i . \tag{17b}$$

$$(\sigma_j^2-\omega_j^2)a_j = (1/2)\sigma_j^2 C(j;i,i)a_i b_i\cos\phi_i, \tag{18a}$$

$$-\mu_j a_j\omega_j[1-\alpha_j^2(a_j^2/4+b_j^2/2] = (1/2)\sigma_j^2 C(j;i,i)a_i b_i\sin\phi_i, \tag{18b}$$

$$(\sigma_j^2-4\omega_i^2)b_j = (1/4)\sigma_j^2 C(j;i,i)a_i^2\cos\phi_j, \tag{19a}$$

$$2\mu_j b_j\omega_i[1-\alpha_j^2(a_j^2/2+b_j^2/4)] = (1/4)\sigma_j^2 C(j;i,i)a_i^2\sin\phi_j. \tag{19b}$$

The order of magnitude of the maximum value for a_i and b_i is α_i^{-1} and that for a_j and b_j, α_j^{-1}. So equations (16a)-(19b) have two important parameters, i.e.

$$A_{iij} = \sigma_i^2 \alpha_i C(i;i,j)/(\mu_i \omega_i \alpha_i \alpha_j), \tag{20a}$$

$$A_{jii} = \sigma_j^2 \alpha_j C(j;i,i)/(\mu_j \omega_j \alpha_i^2). \tag{20b}$$

Parameter A_{jii} denotes efficiency to excite the forced oscillation of j-mode by the principal mode. In the case $A_{jii} \gg 1$, we can expect that the forced oscillation might be enhanced strongly in the vicinity of resonance centre and the free oscillation of j-mode might be strongly suppressed. In the case $A_{iij} \gg 1$, the feed back of higher mode on the principal mode is weak, and so we obtain $b_i \approx 0$ from equations (17a,b).

Near the resonance centre we can generally expect that $b_i = 0$ and $a_j = 0$ because of the synchronization. Eliminating $\sin\phi_j$ in equations (16b) and (19b), we can derive the following expression in this case.

$$-(\mu_i/\sigma_i^2)C(j;i,i)a_i^2[1-\alpha_i^2(a_i^2/4)] = (4\mu_j/\sigma_j^2)C(i;i,j)b_j^2[1-\alpha_j^2(b_j^2/4)]. \tag{21}$$

Then we have

$$(\mu_i/\sigma_i^2)\alpha_i^2 C(j;i,i)(a_i^2 - 2/\alpha_i^2)^2 + (4\mu_j/\sigma_j^2)\alpha_j^2 C(i;i,j)(b_j^2 - 2/\alpha_j^2)^2$$
$$= (4\mu_i/\sigma_i^2)\alpha_i^{-2}C(j;i,i) + (16\mu_j/\sigma_j^2)\alpha_j^{-2}C(i;i,j). \tag{22}$$

Because C(i;i,j) and C(j;i,i) have the same sign, the following expression is derived from equation (21).

$$(a_i^2 - 4/\alpha_i^2)(b_j^2 - 4/\alpha_j^2) < 0. \tag{23}$$

The meaning of this expression is that the increase of the amplitude b_j from the limit-cycle value $2/\alpha_j$ should be compensated by the decrease of the principal amplitude a_i from $2/\alpha_i$ and vice versa. Equation (22) expresses an ellipse on the $a_i^2 - b_j^2$ plane, and then the maximum and minimum values of b_j^2 are found to be as follows:

$$b_j^2 = \frac{2}{\alpha_j^2}[1 \pm \{1 + \frac{\mu_i \sigma_j^2 \alpha_j^2 C(j;i,i)}{4\mu_j \sigma_i^2 \alpha_i^2 C(i;i,j)}\}^{1/2}], \tag{24}$$

as $a_i^2 = 2/\alpha_i^2$. The amplitude of resonant oscillation in j-mode will increase while α_i is small and/or C(j;i,i)/C(i;j,j) large. The fact that b_j has its maximum is a feature characterized in the resonance of self-exciting system.

In the case of $\sigma_j = 2\sigma_i$ and $\omega_i = \sigma_i$, ϕ_j is necessarily equal to $-\pi/2$. So we have a relation from equations (16b) and (19b) as follows:

$$(b_j^2 - 4/\alpha_j^2)b_j = \sigma_j/(2\mu_j\alpha_j^2)C(j;i,i)[(2\sigma_i/(\mu_i\alpha_i^2)C(i;i,j)b_j + 4/\alpha_i^2]. \tag{25}$$

A root of this equation is greater than $2/\alpha_j$ because the right hand side is positive with $b_j > 0$. Therefore the suppression of a_j is always kept at the resonance centre.

2-3. Three-wave coupling

Three-wave coupling is formulated like as the two-wave case studied above. In general, we have

$$\ddot{q}_i + \sigma_i^2(1 - (1/2)C(i;i,i)q_i - C(i;i,j)q_j - C(i;i,k)q_k)q_i - K_i\dot{q}_i$$
$$= \sigma_i^2((1/2)C(i;j,j)q_j^2 + C(i;j,k)q_iq_k + (1/2)C(i;k,k)q_k^2), \tag{26a}$$

$$\ddot{q}_j + \sigma_j^2(1 - C(j;i,j)q_i - (1/2)C(j;j,j)q_j - C(j;j,k)q_k)q_j - K_j\dot{q}_j$$
$$= \sigma_j^2((1/2)C(j;i,i)q_i^2 + C(j;i,k)q_iq_k + (1/2)C(j;k,k)q_k^2), \tag{26b}$$

$$\ddot{q}_k + \sigma_k^2(1 - C(k;i,k)q_i - C(k;j,k)q_j - (1/2)C(k;k,k)q_k)q_k - K_k\dot{q}_k$$
$$= \sigma_k^2((k/2)C(k;i,i)q_i^2 + C(k;i,j)q_iq_j + (1/2)C(k;j,j)q_j^2). \tag{26c}$$

In this case our interests are focussed to the problem of three-wave coupling in the vicinity of three-wave resonance, i.e. $\sigma_k \approx \sigma_i + \sigma_j$. Thus, some higher terms can be neglected and we have

$$\ddot{q}_i + \sigma_i^2 q_i - K_i\dot{q}_i = \sigma_i^2 C(i;j,k)q_jq_k, \tag{27a}$$

$$\ddot{q}_j + \sigma_j^2 q_j - K_j\dot{q}_j = \sigma_j^2 C(j;i,k)q_iq_k, \tag{27b}$$

$$\ddot{q}_k + \sigma_k^2 q_k - K_k\dot{q}_k = \sigma_k^2 C(k;i,j)q_iq_j. \tag{27c}$$

The damping term K_k is chosen as K_i and K_j. Then we have

$$K_k = \mu_k(1 - \alpha_k^2 q_k^2). \tag{28}$$

Solutions of equations (27a)-(27c) are obtained as the first approximation by expressing q_i, q_j and q_k as follows:

$$q_i = a_i\cos\omega_i t + b_i\cos((\omega_k - \omega_j)t + \phi_i), \tag{29a}$$

$$q_j = a_j\cos\omega_j t + b_j\cos((\omega_k - \omega_i)t + \phi_j), \tag{29b}$$

$$q_k = a_k \cos \omega_k t + b_k \cos((\omega_j + \omega_i)t + \phi_k). \tag{29c}$$

The first terms of the right hand side of equations (29a)-(29c) denote free oscillations and the second ones forced oscillations respectively. Comparing coefficient of each mode, we can derive the following equations for a_i, a_j, a_k, b_i, b_j and b_k including ω_i, ω_j, ω_k, ϕ_i, ϕ_j and ϕ_k.

$$[\sigma_i^2 - \omega_i^2]a_i = (1/2)\sigma_i^2 C(i;j,k)(a_j b_k \cos\phi_k + a_k b_j \cos\phi_j), \tag{30a}$$

$$\mu_i \omega_i a_i [1 - \alpha_i^2 (a_i^2/4 + b_i^2/2)] = (1/2)\sigma_i^2 C(i;j,k)(-a_j b_k \sin\phi_k + a_k b_j \sin\phi_j), \tag{30b}$$

$$[\sigma_i^2 - (\omega_k - \omega_j)^2]b_i = (1/2)\sigma_i^2 C(i;j,k) a_j a_k \cos\phi_i, \tag{31a}$$

$$\mu_i (\omega_k - \omega_j) b_i [1 - \alpha_i^2 (a_i^2/2 + b_i^2/4)] = (1/2)\sigma_i^2 C(i;j,k) a_j a_k \sin\phi_i, \tag{31b}$$

$$[\sigma_j^2 - \omega_j^2]a_j = (1/2)\sigma_j^2 C(j;i,k)(a_i b_k \cos\phi_k + a_k b_i \cos\phi_i), \tag{32a}$$

$$\mu_j \omega_j a_j [1 - \alpha_j^2 (a_j^2/4 + b_j^2/2)] = (1/2)\sigma_j^2 C(j;i,k)(-a_i b_k \sin\phi_k + a_k b_i \sin\phi_i), \tag{32b}$$

$$[\sigma_j^2 - (\omega_k - \omega_i)^2]b_j = (1/2)\sigma_j^2 C(j;i,k) a_i a_k \cos\phi_j, \tag{33a}$$

$$\mu_j (\omega_k - \omega_i)^2 b_j [1 - \alpha_j^2 (a_j^2/2 + b_j^2/4)] = (1/2)\sigma_j^2 C(j;i,k) a_i a_k \sin\phi_j, \tag{33b}$$

$$[\sigma_k^2 - \omega_k^2]a_k = (1/2)\sigma_k^2 C(k;i,j)(a_j b_i \cos\phi_i + a_i b_j \cos\phi_j), \tag{34a}$$

$$\mu_k \omega_k a_k [1 - \alpha_k^2 (a_k^2/4 + b_k^2/2)] = (1/2)\sigma_k^2 C(k;i,j)(-a_j b_i \sin\phi_i - a_i b_j \sin\phi_j), \tag{34b}$$

$$[\sigma_k^2 - (\omega_i + \omega_j)^2]b_k = (1/2)\sigma_k^2 C(k;i,j) a_i a_j \cos\phi_k, \tag{35a}$$

$$\mu_k (\omega_i + \omega_j) b_k [1 - \alpha_k^2 (a_k^2/2 + b_k^2/4)] = (1/2)\sigma_k^2 C(k;i,j) a_i a_j \sin\phi_k. \tag{35b}$$

Equations (30a)-(35b) have three important parameters, i.e.

$$A_{ijk} = \sigma_i^2 \alpha_i C(i;j,k)/(\mu_i \omega_i \alpha_j \alpha_k), \tag{36a}$$

$$A_{jik} = \sigma_j^2 \alpha_j C(j;i,k)/(\mu_j \omega_j \alpha_i \alpha_k), \tag{36b}$$

$$A_{kij} = \sigma_k^2 \alpha_k C(k;i,j)/(\mu_k \omega_k \alpha_i \alpha_j). \tag{36c}$$

These parameters determine strength of the influence of the other two modes on each oscillator. When oscillatory moments of inertia satisfy the following relations,

$$\sigma_k^2 \int_0^M \xi_k^2 \, dm \gg \sigma_j^2 \int_0^M \xi_j^2 \, dm \text{ and } \sigma_i^2 \int_0^M \xi_i^2 \, dm, \tag{37}$$

coupling coefficients should be as follows:

$$C(k;i,j) \gg C(j;i,k) \text{ and } C(i;j,k), \tag{38}$$

because of symmetric properties in expression of the coupling coefficient. Although we have no exact knowledge on α_i, α_j and α_k defined above, we may suppose that relation (37) among coupling coefficients derives $A_{kij} \gg A_{ijk} \approx A_{jik}$. If α_k is sufficiently small compared with α_i and α_j, we may also suppose the strong coupling on k-mode oscillator from i- and j-mode oscillators. In this case a_k could be suppressed at first by the synchronization with increasing b_k. Then b_i and b_j diminish simultaneously. The scheme of resonance is that the free oscillations of i- and j-modes enhance the forced one of k-mode for the case $\omega_k \approx \omega_i + \omega_j$. Eliminating $\sin \phi_k$ in equations (30b), (32b) and (35b), we derive equations as follows:

$$\mu_i\omega_i\alpha_i^2/(\sigma_i^2 C(i;j,k))(a_i^2 - 2/\alpha_i^2)^2 - \mu_j\omega_j\alpha_j^2/(\sigma_j^2 C(j;i,k))(a_j^2 - 2/\alpha_j^2)^2$$

$$= 4\mu_i\omega_i/(\sigma_i^2 \alpha_i^2 C(i;j,k)) - 4\mu_j\omega_j/(\sigma_j^2 \alpha_j^2 C(j;i,k)), \tag{39a}$$

$$\mu_i\omega_i\alpha_i^2/(\sigma_i^2 C(i;j,k))(a_i^2 - 2/\alpha_i^2)^2 + \mu_k(\omega_i + \omega_j)\alpha_k^2/(\sigma_k^2 C(k;i,j)(b_k^2 - 2/\alpha_k^2)^2$$

$$= 4\mu_i\omega_i/(\sigma_i^2 \alpha_i^2 C(i;j,k)) + 4\mu_k(\omega_i + \omega_j)/(\sigma_k^2 \alpha_k^2 C(k;i,j)). \tag{39b}$$

Equation (39a) indicates the hyperbola on the $a_i^2 - a_j^2$ plane and equation (39b) presents the ellipse on the $a_i^2 - b_k^2$ plane. Equation (39a) means that the decrease of a_i from the limit-cycle amplitude $2/\alpha_i$ is necessarily accompanied by the decrease of the amplitude of another driving mode a_j from $2/\alpha_j$. We can derive an expression similar to equation (39b) as follows:

$$\mu_k(\omega_i + \omega_j)\alpha_k^2/(\sigma_k^2 C(k;i,j))b_k^2(b_k^2 - 4/\alpha_k^2)$$

$$= \mu_i\omega_i\alpha_i^2/(\sigma_i^2 C(i;j,k))a_i^2(4/\alpha_i^2 - a_i^2), \tag{40}$$

from which the following relation between a_i^2 and b_k^2 is found.

$$(b_k^2 - 4/\alpha_k^2)(a_i^2 - 4/\alpha_i^2) < 0. \tag{41}$$

This expresses that the increase of b_k should be accompanied by the decrease of a_i like as the result in the two-wave coupling. The scheme described here differs from that stated in our previous paper.[6] We presumed the existence of unstable free oscillations of i- and k-mode and expected the enhancement of forced oscillation b_j of the frequency $\omega_k - \omega_i$ by the three-wave resonance. This may be appear in the case of $A_{ijk} \ll A_{jik} \sim A_{kij}$. It depends on the relation among these three parameters which scheme works in actual star. These equations will be more precisely investigated on models of classical cepheids in later section.

We must mention here an interesting relation about frequencies. In the synchronized case, we can derive the following equations by eliminating $\cos\phi_k$ in equations (30a), (32a) and (35a).

$$a_i^2\,\omega_i^2/(C(i;j,k)\sigma_i^2) - a_j^2\,\omega_j^2/(C(j;i,k)\sigma_j^2) = a_i^2/C(i;j,k) - a_j^2/C(j;i,k), \quad (42a)$$

and

$$a_i^2\,\omega_i^2/(C(i;j,k)\sigma_i^2) - b_k^2\,(\omega_i+\omega_j)^2/(C(k;i,j)\sigma_k^2)$$

$$= a_i^2/C(i;j,k) - b_k^2/C(k;i,j). \quad (42b)$$

These express the hyperbola on the ω_i-ω_j plane and the ω_i-$(\omega_i+\omega_j)$ plane respectively. The crossing point of two curves determines ω_i and ω_j with given a_i, a_j and b_k. Both gradients $d\omega_j/d\omega_i$ and $d(\omega_i+\omega_j)/d\omega_i$ are rather steep in the case that $C(i;j,k)a_j^2 \ll C(j;i,k)a_i^2$ and $C(i;j,k)b_k^2 \ll C(k;i,j)a_i^2$. So ω_j has the tendency to leave from σ_j as ω_i remains to be close to σ_i. Because equation (42b) passes the point

$$\omega_i^2/\sigma_i^2 = (1-C(i;j,k)b_k^2/(C(k;i,j)a_i^2))/(1-C(i;j,k)b_k^2/(C(k;i,j)\sigma_k^2\,a_i^2))$$

with $\omega_j = 0$. Curves of equations (42a) and (42b) on the ω_i-ω_j plane crosses close to the point $\omega_i = \sigma_i$ with $\omega_j = \sigma_j$ in the case as follows:

$$1 - C(i;j,k)a_j^2/(C(j;i,k)a_i^2) < \frac{1 - C(i;j,k)b_k^2/(C(k;i,j)a_i^2)}{1 - C(i;j,k)b_k^2/(C(k;i,j)a_i^2\,\sigma_k^2)}. \quad (43)$$

This is almost generally satisfied by the condition

$$a_j^2/C(j;i,k) \gg b_k^2/C(k;i,j). \quad (44)$$

Then we have the frequency of j-mode ω_j which is greater than σ_j while $\sigma_k < \sigma_j + \sigma_i$ and vice versa. The condition (44) is important because it means that the strongly enhanced b_k is realized with the frequency near $\sigma_i + \sigma_j$ only in the case that $C(k;i,j) \gg C(j;i,k)$.

§3. Effect of Higher Order Terms

3-1. Single-mode oscillation

We treat the effect of higher order terms in resonance problem in the present section. In the first place we consider a single-mode case by using the approximation discussed in previous section. Then we have

$$\ddot{q}_s + \sigma_s^2 (1 - C(s;s,s)q_s)q_s - K_s\dot{q}_s = 0. \tag{45}$$

Van der Pol's type damping force is also chosen. So we have

$$K_s = \mu_s(1 - \alpha_s^2 q_s^2). \tag{46}$$

μ_s is the negative damping-constant in the linear approximation. μ_s/σ_s for classical cepheids is estimated at $10^{-2} \sim 10^{-3}$ by summarizing recent numerical calculation.[9]

The differential equation including van der Pol's type term is well studied in the oscillation theory. Krogdahl[8] studied differential equation similar to equation (45). The asymmetry appeared in the solution of the equation having van der Pol's term is not similar to the variation observed in real cepheids. Krogdahl's calculation have confirmed the general trends of the solution of this equation. As suggested by Ledoux and Walraven,[10] Krogdahl's asymmetry is not significant in the case $\mu_s/\sigma_s \ll 1$. So we may use van der Pol's type term for studying the asymmetry of cepheid variables. C(s;s,s) is, in general, positive for the polytropic gas spheres (see Appendix). This leads the asymmetry of solution agrees with the result of non-linear study of pulsation of gas spheres and also with observed one.

We put here the solution of equation (45) as follows:

$$q_s = a_0 + a_1\cos\omega t + a_2\cos(2\omega t + \psi). \tag{47}$$

Then we have

$$\omega^2 = \sigma_s^2 [1 - (1/2)C(s;s,s)(2a_0 + a_2\cos\psi)], \tag{48a}$$

$$-\mu_s\omega a_1 [1 - \alpha_s^2 a_1^2/4] = (1/2)\sigma_s^2 C(s;s,s)a_2\sin\psi, \tag{48b}$$

$$a_0 = (1/4)C(s;s,s)a_1^2, \tag{49}$$

$$[(-4\omega^2 + \sigma_s^2)/\sigma_s^2 - C(s;s,s)a_0]a_1$$

$$= (1/4)C(s;s,s)a_1^2\cos\psi - (\mu_s\omega\alpha_s^2/\sigma_s^2)a_0a_1^2\sin\psi, \tag{50a}$$

$$(2\mu_s\omega/\sigma_s^2)a_2[1 - \alpha_s^2 a_1^2/2]$$

$$= (1/4)C(s;s,s)a_1^2 \sin\psi + (\mu_s\omega\alpha_s^2/\sigma_s^2)a_0a_1^2 \cos\psi. \quad (50b)$$

Frequency ω is determined by equation (48a). Equation (49) gives a_0. The amplitude a_2 and phase ψ of harmonic mode are expressed by equations (50a,b). Equation (48b) determines the amplitude a_1, which we express with the stellar radius as a unit. So, a_0 and a_2 are of the second order, therefore, from equation (48b), we can derive an expression as follows:

$$a_1^2 \approx 4/\alpha_s^2 . \quad (51)$$

This relation is important to estimate the quantity α_s. On the assumption that a_1 is given by the approximation (51), we have that the left hand side of equation (50b) is nearly equal to $(-2\mu_s\omega/\sigma_s^2)a_2$. In general, the relation

$$|4\mu_s\omega\alpha_s^2 a_0/(\sigma_s^2 C(s;s,s))| << 1 \quad (52)$$

is likely satisfied, and for the case of $\omega \approx \sigma_s$, a_0, a_2 and ψ are given approximately in the following form.

$$a_0 \approx (1/4)C(s;s,s)a_1^2 , \quad (53a)$$

$$a_2 \approx (1/12)C(s;s,s)a_1^2 \quad (53b)$$

and

$$\psi \approx -\pi. \quad (53c)$$

Equations(48a)-(50b) are based on the similar idea to Eddington's approximation to study non-linear oscillation.[11] Kluyver[12] carried out numerical calculation by using this approximation for a polytropic gas sphere with $\gamma = 1.43$. Values of a_2/a_1^2 calculated from her table are close to -4.617. It corresponds to the value of 55.4 for C(s;s,s) in our study. The value of C(s;s,s) calculated by the equation (11) of present paper is 2.38 for this case. The difference between our value and hers indicates that between our method and hers. Equation (48a) gives the correction of frequency caused from the non-linear effect. Combined with equations (53a)-(53c), the period is found as follows:

$$\omega^2 = \sigma_s^2(1 - (5/24)C(s;s,s)^2a_1^2). \quad (54)$$

Non-linear period given by the present approximation is also larger than linear one in accordance with the conclusion confrimed in recent stellar pulsation theory. The increase in periods depends on the value of self-coupling coefficient C(s;s,s) and the amplitude a_1.

3-2. Two-wave coupling

We shall study here the effect of non-linear terms of the principal mode on two-wave coupling described in section 2-2. We assume that a_{i0}, a_{i2} and a_j are of second order in the following equations and b_i and b_j of first order because of its significance. Put oscillations as follows:

$$q_i = a_{i0} + a_i \cos\omega_i t + a_{i2}\cos(2\omega_i t + \psi) + b_i \cos((\omega_j - \omega_i)t + \phi_i), \tag{55a}$$

$$q_j = a_j \cos\omega_j t + b_j \cos(2\omega_i t + \phi_j). \tag{55b}$$

Insert these expressions to equations (12a,b). Then we have

$$[-\omega_i^2 + \sigma_i^2 (1 - (1/2)C(i;i,i)(2a_{i0} + a_{i2}\cos\psi))]a_i$$
$$= (1/2)\sigma_i^2 C(i;i,j)(a_i b_j \cos\phi_j + a_j b_i \cos\phi_i), \tag{56a}$$

$$[-\mu_i\omega_i[1 - \alpha_i^2 (a_i^2/4 + b_i^2/2)] - (1/2)\sigma_i^2 C(i;i,i)a_{i2}\sin\psi]a_i$$
$$= (1/2)\sigma_i^2 C(i;i,j)(a_i b_j \sin\phi_j - a_j b_i \sin\phi_i), \tag{56b}$$

$$a_{i0} = (1/4)[C(i;i,i)(a_i^2 + b_i^2) + C(i;j,j)(a_j^2 + b_j^2)], \tag{57}$$

$$[-4\omega_i^2 + \sigma_i^2]a_{i2} = (1/4)\sigma_i^2 C(i;i,i)a_i^2 \cos\psi, \tag{58a}$$

$$2\mu_i\omega_i a_{i2} = (1/4)\sigma_i^2 C(i;i,i)a_i^2 \sin\psi, \tag{58b}$$

$$[-(\omega_j - \omega_i)^2 + \sigma_i^2 (1 - C(i;i,i)a_{i0})]b_i = (1/2)\sigma_i^2 C(i;i,j)a_i a_j \cos\phi_i, \tag{59a}$$

$$\mu_i(\omega_j - \omega_i)b_i[1 - \alpha_i^2 (a_i^2/2 + b_i^2/4)] = (1/2)\sigma_i^2 C(i;i,j)a_i a_j \sin\phi_i, \tag{59b}$$

$$[-\omega_j^2 + \sigma_j^2 (1 - C(j;i,j)a_{i0})]a_j = (1/2)\sigma_j^2 C(j;i,i)a_i b_i \cos\phi_i, \tag{60a}$$

$$-\mu_j\omega_j a_j[1 - \alpha_j^2 (a_j^2/4 + b_j^2/2)] = (1/2)\sigma_j^2 C(j;i,i)a_i b_i \sin\phi_i, \tag{60b}$$

$$[-4\omega_i^2 + \sigma_j^2 (1 - C(j;i,j)a_{i0})]b_j = (1/4)\sigma_j^2 C(j;i,i)a_i^2 \cos\phi_j, \tag{61a}$$

$$2\mu_j\omega_i b_j[1 - \alpha_j^2 (a_j^2/2 + b_j^2/4)] = (1/4)\sigma_j^2 C(j;i,i)a_i^2 \sin\phi_j. \tag{61b}$$

Equation (60b) expresses the suppresion of a_j by the synchronization. In the case that the free oscillation (ω_j) is pulled in the resonant one ($2\omega_i$), the oscillation $(\omega_j - \omega_i)$ is pulled in the oscillation (ω_i) simultaneously, so we have

$$a_j = b_i = 0, \text{ and } \Psi \approx -\pi. \tag{62}$$

At the centre of resonance ϕ_j should be $-\pi/2$ and then frequency ω_i is

$$\omega_i^2 \approx \sigma_i^2 [1 - (1/2)C(i;i,i)(2a_{i0} - a_{i2})], \tag{63}$$

with

$$a_{i0} = (1/4)C(i;i,i)a_i^2 + (1/4)C(i;j,j)b_j^2. \tag{64}$$

Because the condition for the resonance centre is that the left hand side of equation (61a) is zero, the condition,

$$\sigma_j/\sigma_i = 2[1 - (1/4)C(i;i,i)(2a_{i0} - a_{i2}) + (1/2)C(j;i,j)a_{i0}], \tag{65}$$

is that for the resonance centre. The non-linear effect on the resonant period is of the same order to the non-linear effect on isolated oscillation.

3-3. Three-wave coupling

We choose here the following oscillation for our three-wave coupling.

$$q_i = a_{i0} + a_i \cos \omega_i t + a_{i2} \cos(2\omega_i t + \psi), \tag{66a}$$

$$q_j = a_j \cos \omega_j t, \tag{66b}$$

$$q_k = b_k \cos((\omega_j + \omega_i)t + \phi_k). \tag{66c}$$

In these equations, the free oscillation of k-mode is assumed as suppressed already and the resonant forced oscillations of i- and j-modes are disappeared too. Then we have

$$[-\omega_i^2 + \sigma_i^2 (1 - (1/2)C(i;i,i)(2a_{i0} + a_{i2}\cos\psi)]a_i$$

$$= (1/2)\sigma_i^2 C(i;j,k)a_j b_k \cos\phi_k, \tag{67a}$$

$$[-\mu_i\omega_i(1 - \alpha_i^2 a_i^2/4) - (1/2)\sigma_i^2 C(i;i,i)a_{i2}\sin\psi]a_i$$

$$= (1/2)\sigma_i^2 C(i;j,k)a_j b_k \sin\phi_k, \tag{67b}$$

$$a_{i0} = (1/4)[C(i;i,i)a_i^2 + C(i;j,j)a_j^2 + C(i;k,k)b_k^2], \tag{68}$$

$$[-4\omega_i^2 + \sigma_i^2]a_{i2} = (1/4)\sigma_i^2 C(i;i,i)a_i^2 \cos\psi, \tag{69a}$$

$$2\mu_i\omega_i a_{i2} = (1/4)\sigma_i^2 C(i;i,i)a_i^2 \sin\psi, \tag{69b}$$

$$[-\omega_j^2 + \sigma_j^2(1 - C(j;i,j)a_{i0})]a_j = (1/2)\sigma_j^2 C(j;i,k)a_i b_k \cos\phi_k, \tag{70a}$$

$$-\mu_j\omega_j a_j[1 - \alpha_j^2 a_j^2/4] = (1/2)\sigma_j^2 C(j;i,k)a_i b_k \sin\phi_k, \tag{70b}$$

$$[-(\omega_i + \omega_j)^2 + \sigma_k^2(1 - C(k;i,k)a_{i0})]b_k = (1/2)\sigma_k^2 C(k;i,j)a_i a_j \cos\phi_k, \tag{71a}$$

$$\mu_k(\omega_i + \omega_j)b_k[1 - \alpha_k^2 b_k^2/4] = (1/2)\sigma_k^2 C(k;i,j)a_i a_j \sin\phi_k. \tag{71b}$$

Following previous analysis we have three equations for frequencies.

$$\omega_i^2 = \sigma_i^2[1 - (1/2)C(i;i,i)(2a_{i0} - a_{i2})], \tag{72a}$$

$$\omega_j^2 = \sigma_j^2[1 - C(j;i,j)a_{i0}], \tag{72b}$$

$$(\omega_i + \omega_j)^2 = \sigma_k^2[1 - C(k;i,k)a_{i0}], \tag{72c}$$

where $\cos\phi_k = 0$ is assumed. The resonance condition,

$$(\sigma_i + \sigma_j)/\sigma_k = 1 + (1/2)[\sigma_i/(\sigma_i + \sigma_j)C(i;i,i)(a_{i0} - a_{i2}/2) + \sigma_j/(\sigma_i + \sigma_j)C(j;i,j)a_{i0} - C(k;i,k)a_{i0}], \tag{73}$$

is adequate in this case. Then the resonance distance for the non-linear resonance centre is as follows:

$$d(i+j;k) = 1 - (\sigma_i + \sigma_j)/\sigma_k = -(1/2)[\sigma_i/(\sigma_i + \sigma_j)C(i;i,i)(a_{i0} - a_{i2}/2) + \sigma_j/(\sigma_i + \sigma_j)C(j;i,j)a_{i0} - C(k;i,k)a_{i0}]. \tag{74}$$

The non-linear effect is not so great while coupling coefficients remain in small values.

§4. Application to Classical Cepheids

4-1. Coupling coefficients

For applying the resonance theory to classical cepheids, we must calculate coupling constants and estimate other parameters appearing in resonance equations. So we should construct model envelopes first. The theory of stellar interior is well studied for a few decade, so that the masses of classical cepheids are expected of those derived from the evolution theory. Recently studies on the mass of cepheids are summarized by A.N. Cox.[3] However it remains some problems concerning the accuracy of distance modulus and the scale for the effective temperature, it seems that the difference between evolutionary masses and those derived by using the pulsation theory is not so great for single-mode cepheids. In the case of bump and double-mode cepheids, the situation is quite different. For these stars masses derived from the standard evolution theory appear not to melt into the pulsation theory without any modification of physical or chemical properties in model envelopes.

Following a survey about the positions of possible resonance,[13] we choose the resonance between the fundamental mode (0) and the second overtone mode (2), and that among the fundamental mode (0), the first overtone (1) and the third overtone (3) in the present section. We can show two series of model envelopes at the cepheid instability strip. They are constructed by using Stellingwerf's opacity formula[1,14] for the chemical composition of $X = 0.7$ and $Z = 0.02$ through envelopes and the mixing-length theory for convection. The ratio of the mixing-length to the pressure scale-height is one. The characteristics and pulsation properties of models are tabulated in Table 1. The oscillatory moments of inertia calculated by adiabatic pulsation function are also tabulated in Table 2. The models of a series are constructed by using

Table 1. Characteristics and pulsation properties of model envelopes at the cepheid instability strip.

Model	1	2	3	4	1a	2a	3a	4a
$M/M_{\odot}$	5.176	6.714	8.710	11.30	3.1056	4.0284	5.226	6.780
$\log(L/L_{\odot})$	2.958	3.358	3.758	4.158	2.958	3.358	3.758	4.158
T_{eff} (K)	6155	5850	5561	5286	6155	5850	5561	5286
P_0 (in days)	2.1493	4.6394	10.1845	22.8645	2.9844	6.6009	14.9483	34.9923
σ_1/σ_0	1.312	1.342	1.374	1.427	1.368	1.403	1.479	1.635
σ_2/σ_0	1.626	1.673	1.775	1.954	1.756	1.892	2.094	2.393
σ_3/σ_0	1.956	2.062	2.232	2.484	2.205	2.401	2.700	3.161

Table 2. The oscillatory moment of inertia for model cepheids. (in unit of 10^{46} cm^{2}gr sec^{-2})

Model	1	2	3	4	1a	2a	3a	4a
$\sigma_0^2 I_0$	0.29	0.35	0.47	0.66	0.03	0.05	0.08	0.13
$\sigma_1^2 I_1$	0.09	0.14	0.25	0.60	0.01	0.03	0.09	0.28
$\sigma_2^2 I_2$	0.05	0.12	0.38	1.19	0.02	0.06	0.17	0.53
$\sigma_3^2 I_3$	0.05	0.18	0.55	1.81	0.02	0.08	0.29	1.07

evolutionary masses and those of another series are calculated by using 0.6 times masses. The resonances (0+0,2) and (0+1,3) are expressed by squares and triangles on each sequence in the period-period ratio diagram (Fig. 1). In accordance with the result of several investigators (e.g. see the review by A.N. Cox[3]), the evolutionary mass is too large to fit the observational data of double-mode cepheids expressed by crosses in Fig. 1.

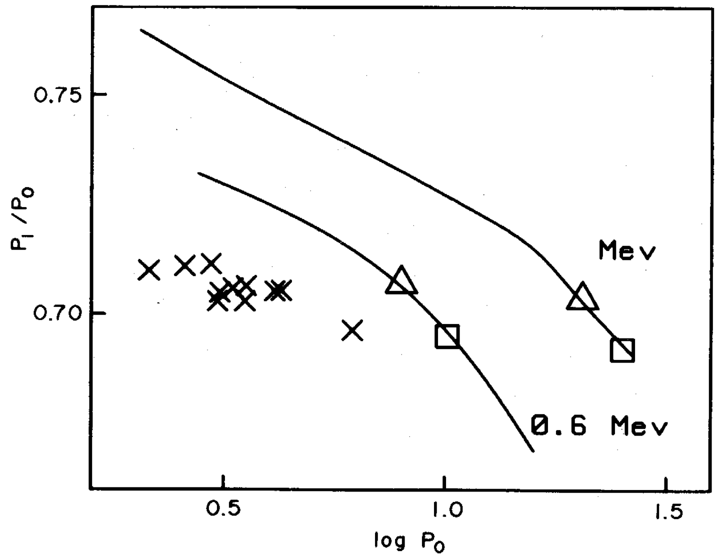

Fig. 1. Period-period ratio diagram for model cepheid envelopes. Squares and triangles indicate the superharmonic resonance (0+0,2) and the three-wave resonance (0+1,3), respectively. Crosses are double-mode cepheids.

The coupling coefficient is calculated by using equation (11) which is derived on the assumption (5) that the pulsation function in real cepheid is well expressed by the adiabatic eigenfunction. This seems to be checked much more carefully, but we used it in the following discussion. Results are tabulated in Tables 3 and 4.

Table 3. Coupling coefficients for two-wave resonance.

Model	1	2	3	4	1a	2a	3a	4a
C(0;0,2)	0.6	0.8	1.2	1.7	0.9	1.4	1.9	1.9
C(2;0,0)	3.2	2.3	1.4	0.9	1.6	1.1	0.8	0.5
C(0;0,0)	3.4	3.9	4.6	5.4	4.4	5.2	6.1	6.9
C(0;2,2)	1.3	2.4	4.9	9.7	3.7	7.3	13.5	23.3
C(2;0,2)	7.2	6.9	8.4	8.4	6.7	6.1	5.9	5.7

Table 4. Coupling coefficients for three-wave resonance.

Model	1	2	3	4	1a	2a	3a	4a
C(0;1,3)	0.4	0.8	1.5	2.5	1.1	1.8	2.3	1.9
C(1;0,3)	1.2	2.1	2.8	2.7	2.5	2.6	1.9	0.9
C(3;0,1)	2.2	1.6	1.3	0.9	1.4	1.1	0.6	0.2
C(1;0,1)	5.3	6.3	7.0	6.7	7.4	7.4	6.6	5.7
C(3;0,3)	6.9	6.2	5.9	5.4	6.5	6.1	5.6	5.3

4-2. Bump cepheids

Near the resonance centre of coupling (0+0,2), we may expect that the free oscillation of the second overtone mode is suppressed by the synchronization with the increase of amplitude of the forced oscillation, even if the second overtone is unstable. Then we can use the result of sections 2-2 and 3-2. Coupling coefficients between the fundamental mode and the second overtone are tabulated in Table 3. We may put $\mu_0/\sigma_0 = 10^{-3}$, and $\alpha_0 = 40$ which is chosen to normalize the amplitude of limit cycle as 0.05. In linear approximation, the maximum of b_2 which determines the height of bump is given by equation (24). So if the parameter $(\mu_2/\sigma_2)C(2;0,0)/C(0;0,2)$ is large enough, the second overtone which has the same period as the first harmonic of the fundamental mode prevails in oscillation. If it is not so great, we can find only small bump. And it is evident that large α_2 yeilds weak bump.

In actual classical cepheids, the bump is not so strong, therefore rather strong stability or instability (positive or negative μ_2) and strong non-linear damping (positive or negative α_2^2 with large absolute value) are expected to fit the present model of oscillating system. We solved the resonance equation with parameters as follows:

$$\mu_2/\sigma_2 = -0.00004, \quad \text{and} \quad \alpha_2^2 = -250000,$$

using tabulated coupling coefficients. Amplitudes a_0 and b_2 in linear approximation are illustrated in Fig. 2 by the dotted lines with σ_2/σ_0. ω_0/σ_0 is also shown in the figure. Non-linear results calculated by equations in previous section are shown in Fig. 3. The non-linear result is well understood on the diagram of b_2/a_0 with σ_2/ω_0. This relation expresses the enhancement of b_2 by the external oscillation a_0 of frequency ω_0. The diagram is just like the usual relationship in non-linear forced oscillation.

Because $|\mu_0\omega_0/\sigma_0^2| \ll 1$, ψ is close to $-\pi$. Then the variation of q with the time t is expressed by the following relation.

$$a_0\cos\omega_0 t - a_{02}\cos(2\omega_0 t) + b_2\cos(2\omega_0 t + \phi_2). \tag{75}$$

The bump caused from resonant oscillation may appear at the descending slope of dq/dt. Another bump may be situated at the rising branch. The strong enhancement of harmonic oscillation may cause double bumps both at the rising and descending branch. For models with the period shorter than that for the resonance centre, the term $(-4\omega_i^2 + \sigma_j^2)/\sigma_j^2$ is negative so that $-\pi < \phi_2 < -\pi/2$, and for those with longer period $-\pi/2 < \phi_2 < 0$. In the former case a bump appears at the descending slope of dq/dt and another one melt into the main maximum, so we find usually only one bump at the rising slope of radial velocity curve. On the contrary, we see a bump at the descending slope of radial velocity curve in the latter. Thus Hertzsprung relation between the phase of

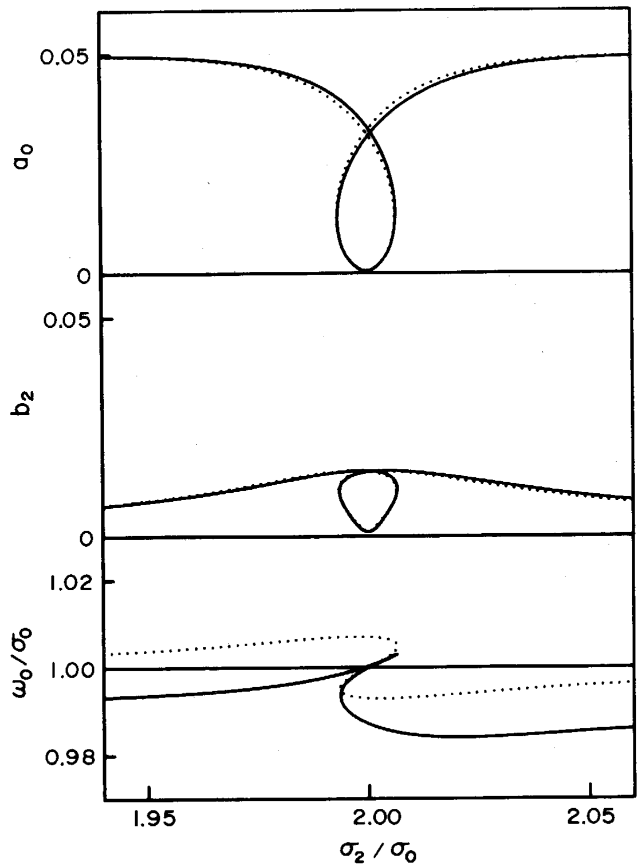

Fig. 2. Amplitudes and frequency in superharmonic resonance. The decrease of the amplitude of principal mode a_0 (top) and the increase of the amplitude of resonant mode b_2 (middle) are illustrated. Frequency of principal mode ω_0 is also shown (below). Dotted lines demonstrate the case without harmonic terms (section 2-2) and solid lines with harmonic terms (section 3-2). Coupling coefficients for model 3a are used.

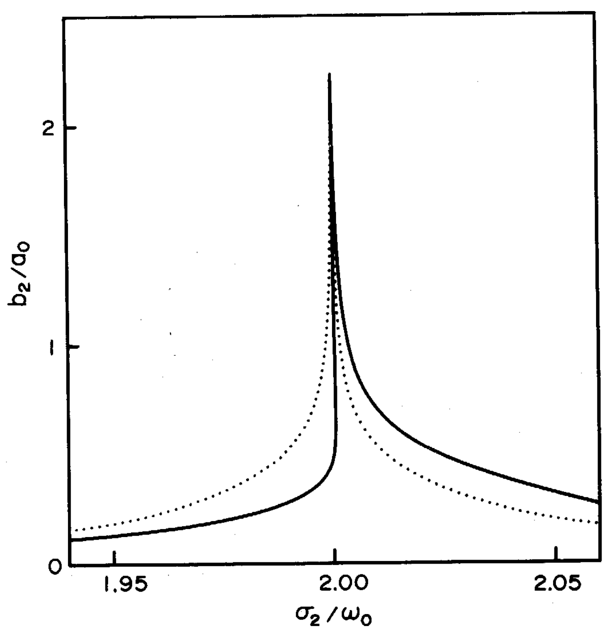

Fig. 3. Relative enhancement of the resonant amplitude b_2 to the principal amplitude a_0.

bump and the period seems to be presented by the resonance. Fig. 4 shows q(t) and -dq/dt, the radial velocity, for several models.

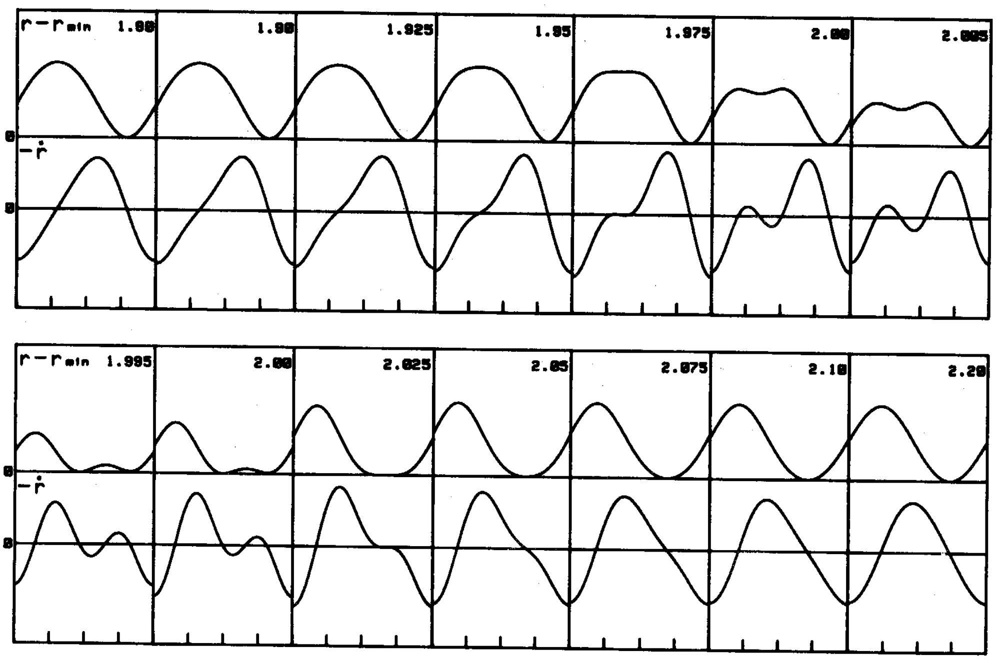

Fig. 4. Variations of radius and radial velocity. Radius $(r - r_{min})$ and radial velocity $(-\dot{r})$ are illustrated for various period ratio, which is labelled on each curve. Model 3a is also used here.

The enhancement of resonant harmonic is usually combined with the decrease of the amplitude of main mode in the self-exciting system. This decrease likely corresponds to the depression of the amplitude of classical cepheids appearing for the period of 10 days. This seems also to be evidences for the resonance theory. It should be mentioned that we are not able to state definitely that the 10-day depression is originated from the resonance phenomenon, however. In fact, the depression appears only with the sufficiently large value of α_2, although it is not denied in the present paper.

4-3. Double-mode cepheids

In the present subsection, we shall try the double-mode cepheid, which seems quite puzzling for the pulsation theory. The coexistence of single-mode and double-mode cepheids on the H-R diagram[15] leads us to the assumption that the amplitude of the first overtone and the third overtone are rather small compared with that of the fundamental mode off the resonance. So the free oscillation of the third overtone should be suppressed even if the mode is unstable. If the mode is stable, only the forced oscillation driven by the fundamental and first overtone modes is enhanced. Then we may treat again in this case the resonance among the fundamental mode, the first overtone and the third overtone assuming the free oscillation of the third overtone oscillator to be suppressed. In both cases we can use results in sections 2-3 and 3-3.

The pulsational instability has been extensively investigated on the models of classical cepheids, so it is certainly that the fundamental mode and the first overtone are pulsationally unstable for short-period classical cepheids, i.e. μ_0 and μ_1 are positive. In the case of

$$4\mu_0\omega_0/(\sigma_0^2\alpha_0^2 C(0;1,3)) - 4\mu_1\omega_1/(\sigma_1^2\alpha_1^2 C(1;0,3)) > 0, \tag{76}$$

from equation (39a), a_0^2 is restricted in two separate domains, i.e.

$$a_0^2 \geq 2/\alpha_0^2 [1 + \{1 - \frac{\alpha_0^2 \mu_1\omega_1\sigma_0^2 C(0;1,3)}{\alpha_1^2 \mu_0\omega_0\sigma_1^2 C(1;0,3)}\}^{1/2}], \tag{77a}$$

and

$$a_0^2 \leq 2/\alpha_0^2 [1 - \{1 - \frac{\alpha_0^2 \mu_1\omega_1\sigma_0^2 C(0;1,3)}{\alpha_1^2 \mu_0\omega_0\sigma_1^2 C(1;0,3)}\}^{1/2}]. \tag{77b}$$

As $a_1^2 = 2/\alpha_1^2$, a_0^2 has its extrema and b_3^2 reaches also its maximum. By using the relation between a_1^2 and b_3^2, the maximum of b_3^2 is, therefore, as follows:

$$2/\alpha_3^2 [1 - \{1 + \frac{\alpha_3^2 \mu_1\omega_1\sigma_3^2 C(3;0,1)}{\alpha_1^2 \mu_3(\omega_0+\omega_1)\sigma_1^2 C(1;0,3)}\}^{1/2}]. \tag{78}$$

So if $(\sigma_3/\mu_3)C(3;0,1)/C(1;0,3)$ is sufficiently large, we shall have strongly enhanced b_3 although a_1 remains in rather small amplitude unless $|\alpha_3^2|$ is not so large.

The result which we obtained indicates that the possibility of double-periodicity depends on the characteristics of the third overtone. This reduces the conclusion that non-linear simulation of double-mode phenomenon must be examined by using the programming code which has the accuracy enough to express the third overtone. Although we find the enhancement of three different modes showing the double-periodicity, the mode mostly enhanced by the resonance is the third overtone with the frequency $(\omega_0+\omega_1)$. If we suppose a weakly stable or weakly unstable first overtone and some unstable third overtone, the amplitude of the first overtone a_1 must be enhanced with frequency ω_1. Because the investigation on the stability shows that the first overtone of short-period classical cepheids is usually unstable, the former seems probable. It is reported that the variation of double-mode cepheid is analysed to the fundamental oscillation and the first overtone oscillation. Although the recent period determination reaches rather high quality, Henden's report[16] permits to us some possibility of the enhancement of the third overtone. It remains the problem which scheme is realized in actual double-mode cepheids.

§5. Discussions and Conclusion

We have described the non-linear oscillation of stars by using coupled self-exciting oscillator model in the present paper. The meaning of analysis presented depends on the adequacy of this model for stellar oscillation. The non-linear non-adiabatic nature of outer envelopes of cepheids is complicated. Although it is required to describe the damping term much more precisely, some conceptions we used as the synchronization seem to be essential. Another essential assumption in the present study is equation (5) that non-linear non-adiabatic displacement is expressed in sufficient precision by linear adiabatic one. This assumption is rather crude especially in outer envelope layers. The coupling constant which is affected strongly by the nature of oscillation in the hydrogen and helium ionization zones is probably subject to non-linear non-adiabatic studies by using the dynamical code. If the coupling coefficient is much greater than that estimated in the present paper, the resonance occurs at the period ratio different from that calculated by linear periods. So we have to wait for the result of non-adiabatic analysis to judge whether or not the mass reduction derived in the present work is real.

We must note here the problem about the coupling coefficient. It is difficult to enhance strongly b_3 with small a_1, if $C(3;0,1)$ is not so great enough in equation (78). It is the reason why we did'nt show the example of three-wave resonance numerically in previous section. The uncertainty comes mainly from that in coupling coefficients. If we choose those of the polytropic gas sphere (Table A1), the strongly enhanced b_3 is found against very small a_1 and nearly unchanged a_0. It is important to develope the resonance theory in the non-adiabatic case like as Auvergne et al.[17] do. Simon's generalized discussion[18] is also interesting.

Summarizing the present paper, we could state that the coupled self-exciting oscillator model seems working to study the resonance phenomenon in classical cepheids. Even if the second and third overtone are vibrationally stable, the resonance of fundamental mode and first overtone with them is important. A strongly damped overtone which couples with the fundamental mode can quench efficiently the pulsation, and a weakly damped overtone may enhance the harmonics or faint oscillation. As Takeuti[19] has pointed out, the strongly damped overtone may be a cause of non-variable stars in the cepheid instability strip.

The mass problem of bump and double-mode cepheids is not removed in the present analysis, but the disagreement appeared in the resonance distance of chemically inhomogeneous model envelopes[6,20,21] is subject to the non-linear effect. The helium-enriched envelope hypothesis is still interesting in the viewpoint of our study. The application of resonance theory to other group of variable stars seems also interesting. Simon[4] has argued the applicability on double-mode δ Scuti stars like as AI Velorum. These stars have been also studied by Petersen.[13] Takeuti and Petersen[22] have discussed the RV Tauri stars concerning with the resonance of (0+0,1) and (0+1,2). Takeuti has found that the coupling coefficient for these stars is rather great because of their low surface gravity.[23] Therefore, the complicated behavior of light variation in giant or supergiant stars is likely caused from the strong coupling of several modes. Petersen[24] has discussed the BL Herculis stars by using Simon-Schmidt's semi-empirical relation between the phase of bump and the period ratio. The superharmonic resonance studied in the present paper may permit to us another phase difference for bump than that given by the locked-in consideration. It seems still important to study these variables by the resonance theory.

Appendix Coupling Coefficients for Polytropic Gas Spheres

The coupling coefficient of polytropic gas spheres is calculated for several modes (Table Al). $\gamma = 5/3$ is chosen through the calculation. C(i;0,i), the coefficient dominating the asymmetry of each mode, is positive, so that asymmetry agrees with that derived by other method.

Table Al. Coupling coefficients for polytropic gas sphere ($\gamma = 5/3$).

n	1.5	2.0	2.5	3.0	3.5	4.0
C(0;0,0)	4.7122	3.7269	2.6113	1.9802	2.1006	2.0358
C(0;0,2)	-0.0138	-0.0092	0.0119	0.0784	0.1615	0.1276
C(2;0,0)	-0.0732	-0.1523	0.4035	2.7266	4.4376	5.3088
C(0;1,3)	-0.0016	-0.0013	0.0030	0.0162	0.0449	0.0345
C(1;0,3)	-0.0046	-0.0079	0.0271	0.1275	0.2972	0.3025
C(3;0,1)	-0.0132	-0.0444	0.2578	1.7425	4.0544	4.9409
C(0;2,2)	0.8338	0.3005	0.1590	0.1566	0.1839	0.1296
C(2;0,2)	4.4349	5.0096	5.3909	5.4433	5.0504	5.3915
C(0;1,1)	1.6386	0.7956	0.5117	0.5208	0.5707	0.4636
C(0;3,3)	0.5135	0.1465	0.0646	0.0571	0.0670	0.0436
C(1;0,1)	4.6125	4.8027	4.6865	4.0912	3.7764	4.0566
C(3;0,3)	8.2012	7.4922	6.5002	5.5445	4.5992	4.2074

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