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著者	SASAKAWA Tatuya, HORACEK Jiri, SAWADA Tatsuro
journal or publication title	The science reports of the Tohoku University. Ser. 8, Physics and astronomy
volume	1
number	3
page range	119-154
year	1980-12-15
URL	http://hdl.handle.net/10097/25454

Some Preliminary Studies for Treating the Three-Body Scattering Problems

Tatuya SASAKAWA, Jiří HORÁČEK* and Tatsuro SAWADA+

Department of Physics, Faculty of Science
Tohoku University, Sendai 980

+Department of Applied Mathematics, Faculty of Engineering Science
Osaka University, Toyonaka 560

(Received November 7, 1980)

As a preliminary step for handling the three-body scattering problems, here we present the following three new methods: A method of acceleration for iterative calculations, a method for treating the scattering from a non-local potential, and a method for treating the three-body breakup channel. Convergence of elastic scattering channel is studied.

Keywords: Three new methods. Method of acceleration for iterative calculations. Scattering from non-local potential. Three-body breakup.

§1. Introduction

For a long time, the study for convergence of perturbation-iteration has attracted a great deal of interest[1)-6)]. Although the development of computers has made this subject less attractive for a simple problem, such as the scattering from a potential, the study of this subject for more complicated problems is still very important. For instance, we have used a perturbation-iteration method in solving the Faddeev equation for the bound state of the triton and ^{3}He[7),8)]. In these cases, it turned out numerically that the convergence is very fast. However, we don't know at this moment whether a similar rapid convergence is expected or not when the method used for the bound state is extended to the scattering state. As a result, we feel a need for exploring a powerful method for handling the perturbation-iteration.

As a perliminary for solving a three-body scattering problem in a full scale, we study the problem of convergence in the present paper. Since we have written several papers on this subject, it might be helpful to readers, if we give a brief review of what we have done in our previous papers, as well as what we are going to do at the present time. We devote sec. 2 to this subject. In other words, sec. 2 is an introduction to the present paper in mathematical

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*Permanent address: Department of Mathematical Physics, Charles University, 18000 Prague 8, Povltavská 1, Czechoslovakia.

language. In sec. 3, a new method of acceleration for the iterative calculation of a Fredholm equation is presented. In sec. 4, some examples for this method are given for the scattering from a local potential. In sec. 5, the treatment of scattering problems for a non-local potential is studied in detail. In sec. 6, the method[6] is applied to the three-body breakup channels. In sec. 7, we show how to treat the breakup channel neglecting the elastic and closed channel.

§2. Decomposition of Faddeev Equation into Channels

A. Decomposition into channels

As stated in the Introduction, the present paper is a preliminary study for solving a three-body scattering problem. For this purpose, we decompose the three-body wave functions into the closed-, elastic- and break-up channels. Then we study the treatment of each channel separately. This was done in previous papers[9)-11)]. However, since we have got some new ideas after that, we are writing the present paper.

For simplicity, we restrict ourselves to the central potential and to s-waves. We name three particles 1, 2 and 3. We call the channel 1 such a state that the particle 1 sits as a spectator while the pair of particles 2 and 3 are interacting. We express the wave function of the channel 1 by $\Psi^{(1)}$.

Suppose that the position of the particles 2 and 3 is fixed. Then the waves $\Psi^{(3)}$ and $\Psi^{(2)}$ representing the final state interactions that took place between 2 and 1, and 3 and 1, respectively, will interfer after the final state interaction. As a result, the wave function Ψ of the total system may be expressed as

$$\Psi = \Psi^{(3)} + \Psi^{(2)} \quad . \tag{1}$$

In general, when all particles are treated symmetrically, Ψ should be represented as the sum of $\Psi^{(i)}$, i = 1, 2 and 3,

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)} \quad . \tag{2}$$

If we let $\Phi^{(1)}$ denote the initial state for the channel 1, the Faddeev equation reads

$$\Psi^{(1)} = \Phi^{(1)} + G_0 t_1 (\Psi^{(2)} + \Psi^{(3)}) \quad , \tag{3}$$

where G_0 is the three-body Green's function in free space, t_1 is the scattering matrix for the pair 23. Sometimes it is convenient to express Eq. (3) as

$$\Psi^{(1)} = \Phi^{(1)} + G_0 t_1 P \Psi^{(1)} \quad , \tag{4}$$

where the exchange operator P is defined by

$$P\Psi^{(1)} = \Psi^{(2)} + \Psi^{(3)} \tag{5}$$

We shall suppress the superfix (i) of Eq. (3) if there is no fear of confusion.

If we denote by K_1, K_{23} and v_{23} the kinetic energy operator of the particle 1 relative to c.m. of the pair 23, that of the relative motion of 2 and 3 and the potential acting between the pair 23, the following identity holds,

$$G_0 t_1 = \frac{1}{E - (K_1 + K_{23}) + i\varepsilon}\, t_1 = \frac{1}{E - (K_1 + K_{23} + V_{23}) + i\varepsilon}\, V_{23}. \qquad (6)$$

In coordinate space, $G_0 t_1$ is always calculated in the limit of $\varepsilon \to 0$. For simplicity, however, the notation of $\lim_{\varepsilon \to 0}$ is omitted throughout.

We let ϕ_{23} represent the wave function of deuteron, whose binding energy is $-|E_d|$. The Faddeev equation (3) for the two-body triplet state may be expressed as

$$\psi^{(1)} = \Phi^{(1)} + |\phi_{23}\rangle \frac{1}{E + |E_d| - K_1 + i\varepsilon} \langle \phi_{23}| V_{23} P\psi^{(1)} \qquad (7.1)$$

$$+ \frac{1}{E - (K_1 + K_{23} + V_{23}) + i\varepsilon} (1 - |\phi_{23}\rangle\langle\phi_{23}|) V_{23} P\psi^{(1)}. \qquad (7.2)$$

The term (7.1) represents the *elastic* channel. We decompose the term (7.2) by a complete set of $|\phi_p\rangle$, which is defined by a product of the normalized plane wave of the spectator 1 with energy E_p, and the spin-angular wave function of the total system. The term (7.2) is then decomposed into

$$(7.2) = \int'_{E-E_p>0} dE_p |\phi_p\rangle G^{(2)}_{0,q}\, t_{1,q} \langle \phi_p| P\psi^{(1)} \qquad (8.1)$$

$$- \int'_{E-E_p>0} dE_p |\phi_p\rangle |\phi_{23}\rangle \frac{1}{E - E_p + |E_d|} \langle \phi_p| \langle \phi_{23}| V_{23} P\psi^{(1)} \qquad (8.2)$$

$$+ \int''_{E-E_p \leq 0} dE_p |\phi_p\rangle\, G^{(2)}_{0,q}\, t_{1,q} (1 - V_{23}^{-1} |\phi_{23}\rangle\langle\phi_{23}| V_{23}) \langle \phi_p| P\psi^{(1)}. \qquad (8.3)$$

Here the suffix q represents the energy given by

$$E_q = E - E_p \quad , \qquad (9)$$

and $G^{(2)}_{0,q}$ denotes the Green's function of the two-body system

$$G^{(2)}_{0,q} = \frac{1}{E_q - K_{23} + i\varepsilon} \, . \qquad (10)$$

The term (8.1) is the *break-up* channel, because both E_p and E_q are positive. The term (8.2) is a *closed* channel, because the denominator is positive definite and there is no pole. The term (8.3) is also a *closed* channel because

$E-E_p<0$. At $E_q=-|E_d|$, this term vanishes as demonstrated in refs. 9,10 and 11.

B. Closed channel

The bound state problem consists only of the term (8.3). For treating the bound states, we have expressed $G_{0,q}^{(2)}\, t_{1,q}$ as[7,8]

$$G_{0,q}^{(2)}\, t_{1,q} = -|\Psi_{23,q}\rangle\xi_q\langle\psi_{23,q}|V_{23} + (\omega_q - 1), \tag{11}$$

with

$$\xi_q = \frac{\lambda_q}{1-\lambda_q} \quad . \tag{12}$$

Here $|\psi_{23,q}\rangle$ represents the first Sturm-Liouville function of energy $E_q(<0)$, λ_q being its eigenvalue. The function $\psi_{23,q}$ is regular at the origin and falls off as e^{-qx}/x, $q=\sqrt{(m/\hbar^2)|E_q|}$ at large distances from the origin.

The decomposition (11) has the following merits: Firstly, the large part of $G_{0,q}^{(2)}\, t_{1,q}$ is involved in the first term on the right hand side, which is separable, and the remainder $\omega_q - 1$ may be treated as the perturbation. In fact, refs. 7 and 8 have revealed that the convergence of the perturbation-iteration of $(\omega_q - 1)$ is very fast. Secondly, both the first and second terms on the right hand side are regular at the origin and falls off as e^{-qx}/x at large distances. Although there are many possible ways[6] of decomposing $G_{0,q}^{(2)}\, t_{1,q}$, these two properties should always be taken into account and restrict the manner of decomposition. Since our approach to the bound state is so successful, we presume that the same method may be utilized to the closed channel. We do not discuss the treatment of closed channels in the present paper.

C. Elastic channel

To treat the elastic channel represented by the term (7.1), we have to have a method for dealing with the scattering from a non-local potential. In refs. 6 and 12, a method for this purpose was proposed. The idea was that we express the non-local potential as the sum of a real symmetric part and an Hermitian part. Each part can be expressed as the (infinite) sum of separable terms by respective Sturm-Liouville functions. In practice, we take some Sturm-Liouville functions that give main contributions as unperturbed part and treat the remainder as the perturbation.

A numerical example for which only one term is taken as the unperturbed part seems satisfactory[10,12]. However in these references, this method was not tested for all energy. Later, it turned out that we need more terms if the energy is very low (around 1 MeV). As a result, we should try another

method that works better for all low energies, say below 25 MeV, since the merit of this method manifests itself, when the number of unperturbed terms is one or at most two. In sec. 4 and 5 of the present paper, we propose new methods for the treatment of a scattering from a non-local potential.

D. Breakup channel

The Green's function $G^{(2)}_{0,q}$ is real symmetric for a closed channel, but complex and non-Hermitian for the breakup channel. As a result, both the Sturm-Liouville function and the eigenvalue λ_q becomes complex. The norm of this Sturm-Liouville function (the Weinberg function) is not positive definite. This makes the accurate and stable calculation of $G^{(2)}_{0,q}\, t_{l,q}$ difficult, if we decompose $G^{(2)}_{0,q}\, t_{l,q}$ for positive energies as Eq. (11).

To overcome this difficulty, we have proposed in ref. 6, the following decomposition to positive energies

$$G^{(2)}_{0,q}\, t_q = (\bar{\omega}_q - 1) - |\hat{\psi}_q + i\bar{\omega}_q\, u_q\rangle \frac{1}{1 - \lambda_q + i\langle u_q|V\bar{\omega}_q|u_q\rangle} \langle u_q|V\bar{\omega}_q \quad . (13)$$

Here u_q is the spherical Bessel function times $\sqrt{q}$, and $\hat{\psi}_q$ is the first Sturm-Liouville function so normalized that at large distances $\hat{\psi}_{23,q}$ behaves as $\sqrt{q}\frac{\cos(qx_{23})}{qx_{23}}$. So far as an example shown in ref. 6 is concerned, the decomposition (13) seemed very promissing.

Concerning Eq. (13), we should remark two things.

(i) A detailed study[13] has revealed that the perturbative calculation of $\bar{\omega}_q$ is useful provided that the potential is monotonic or the repulsive part is weak. In such a case, the convergence is usually very fast. If the potential is very non-monotonic, this method is not useful[13].

However, in practice, the operator $\bar{\omega}_q - 1$ need not to be calculated in the perturbational manner. But we can calculate this quantity as a function obtained by solving a differential equation. In fact, this is the way that we have adopted in refs. 7 and 8. Therefore, at the present moment, it is too rush to say that Eq.(13) is useless. In fact, what we should examine is the convergence of the series

$$\sum_n [\int'_{E-E_p>0} dE_p |\phi_p\rangle(\bar{\omega}_q - 1)\langle\phi_p| P]^n \qquad (14)$$

as obtained from Eqs. (8.1) and (13).

(ii) In Eq. (13), the denominator involves the effect only from the first eigenvalue. It may be useful if the denominator involves the effects from all poles. This requirement is satisfied by the following decomposition for S wave

$$G^{(2)}_{0,q}\, t_q = G^{(2)}_{0,q}\{J_q - J_q|w_q\rangle \frac{1}{1 + \langle u_q|J_q|w_q\rangle} \langle u_q|J_q\}, \qquad (15)$$

where w_q is the spherical Hankel function times $\sqrt{q}$. In general, for a partial wave $\ell \geq 1$

$$G_{0,q}^{(2)}\, t_q = \sum_{n=1}^{[\frac{\ell+1}{2}]} (G_{0,q}^{(2)}\, V)^n + (G_{0,q}^{(2)}\, V)^{[\frac{\ell+1}{2}]}\, G_{0,q}^{(2)} \{ J_q$$

$$- J_q |w_q\rangle \frac{1}{1 + \langle u_q | J_q | w_q \rangle} \langle u_q | J_q \} \ . \qquad (16)$$

In Eqs. (15) and (16), J_q is the J-matrix defined in terms of the Green's function of Jost g_q by

$$J_q = V + V g_q J_q . \qquad (17)$$

In Eq. (16), $[\frac{\ell+1}{2}]$ stands for the integral part of $\frac{\ell+1}{2}$. The device of Eq. (16) for $\ell \geq 1$ is necessary to let each term on the right hand side be regular at the origin. On this account, see Appendix A. The denominator $1 + \langle u_q | J_q | w_q \rangle$ is the Jost function. All poles of t_q are given by zeros of this function.

The operator J_q multiplied by a function is obtained by solving a differential equation as in ref. 7 and 8. In the application of Eq. (15) or (16) to a three-body problem, we should examine the convergence of the series

$$\sum_n \Big(\int'_{E-E_p>0} dE_p |\phi_p\rangle\, G_{0,q}^{(2)}\, J_q \langle \phi_p | P \Big)^n \qquad (18)$$

or

$$\sum_n \Big[\int'_{E-E_p>0} dE_p |\phi_p\rangle \Big(\sum_{m=1}^{[\frac{\ell+1}{2}]} (G_{0,q}^{(2)}\, V)^m + (G_{0,q}^{(2)}\, V)^{[\frac{\ell+1}{2}]}\, G_{0,q}^{(2)}\, J_q \Big) \langle \phi_p | P \Big]^n . \qquad (19)$$

Derivation of Eq. (15) and (16):

We suppress the suffix. The two-body Green's function G_0 is decomposed as[6]

$$G_0 = - |w\rangle\langle u| + g, \qquad (20)$$

where g is defined by Eq. (20). The Green's function g is the Green's function of Jost. It is irregular at the origin and vanishes at large distances.

With the usual t-matrix

$$t = V + V\, G_0\, t = V + V(-|w\rangle\langle u| + g)\, t , \qquad (21)$$

we define the Jost matrix J by

$$J = V + Vg\,J\,. \tag{22}$$

If we subtract (22) from (21) we obtain

$$\begin{aligned} t - J &= Vg\,(t-J) - V|w\rangle\langle u|t \\ &= -\frac{1}{1-Vg}\,V|w\rangle\langle u|t \\ &= -J|w\rangle\langle u|t. \end{aligned} \tag{23}$$

Multiplying by $\langle u|$ from both sides, we get

$$\begin{aligned} \langle u|t &= \langle u|J - \langle u|J|w\rangle\langle u|t \\ &= \frac{1}{1+\langle u|J|w\rangle}\langle u|J. \end{aligned} \tag{24}$$

If we use Eq. (24) into Eq. (23), we get

$$t = J - J|w\rangle\frac{1}{1+\langle u|J|w\rangle}\langle u|J. \tag{25}$$

From Eq. (25), we get Eq. (15)

Now we should remark that

$$G_0 J w_\ell \xrightarrow[x\to 0]{} \frac{(2\ell-1)!!}{2\ell+1}\left(\frac{1}{2}+\frac{1}{2\ell-1}\right)\frac{1}{x^{\ell-1}}\Big|_{x\to 0}\,. \tag{26}$$

This is regular only for $\ell = 0$. Each term on the right hand side to be regular, we iterate t making use of Eq. (21). Eq. (16) is obtained by this procedure. Each term on the right hand side of Eq. (16) is regular at the origin.

E. The subjects that we study in the present paper

With the above background, let us describe what we study in the present paper.

(i) Before studying the series (14) or (18) and (19) that involve a rather formidable operator P, a new method of acceleration for iterative calculations of a Fredholm integral equation with a usual simple kernel is proposed in secs. 3 and 4.

(ii) For dealing with the elastic scattering of three-body problems, we study the treatment of the scattering problem for a non-local potential in sec. 5.

(iii) The method of calculating the series (14) or (18) and (19) is presented

in secs. 6. and 7.

In the present paper, the study will be made channel by channel. A combined treatment of these channels will be the subject of the future papers.

§3. A Method of Acceleration

A. Equation with square integrable kernel

In three-body scattering problems, we have to calculate the series given by Eq. (14) or Eqs. (18) and (19). These series have a rather complicated kernel and we will defer the study to secs. 6 and 7. As a preliminary, let us consider the iterative calculation of a Fredholm integral equation of the second kind

$$\phi(x) = f(x) + \lambda \int_0^\infty K(x,x')\phi(x')dx' \ . \tag{27}$$

Here λ is a strength parameter. In the potential scattering, $K(x,y)$ is given by

$$K(x,x') = G_0(x,x';E)V(x') \ , \tag{28}$$

where $G_0(x,x';E)$ is the Green's function in free space and V is the interaction potential.

The function $\phi(x)$ is not usually square integrable. However, in many cases of interest, a similarity transformation makes the function square integrable. For instance, if the kernel is given by Eq. (28), we multiply $[V(x)]^{1/2}$ from the left of Eq. (27) and obtain the equation

$$\xi(x) = \eta(x) + \lambda \int_0^\infty \hat{K}(x,x')\xi(x')dx' \tag{29}$$

where

$$\xi = V^{1/2}\phi \ , \qquad \eta = V^{1/2}f \tag{30}$$

and

$$\hat{K}(x,x') = [V(x)]^{1/2} \ G_0(x,x')[V(x')]^{1/2} \ . \tag{31}$$

Since $[V(x)]^{1/2}$ is a multiplying factor, the properties of Eqs. (27) and (29) as linear algebra should be the same, but $\xi(x)$ and $\eta(x)$ are now square integrable provided that the following equations hold.

$$\int_0^\infty |V^{1/2}\phi|^2 dx = \int_0^\infty \phi^*(x)V(x)\phi(x)dx = M \ , \tag{32.1}$$

and

$$\int_0^\infty |V^{1/2} f|^2 dx = \int_0^\infty f^*(x) V(x) f(x) dx = M' \quad . \tag{32.2}$$

Here, M and M' are finite numbers. Equations (32.1) and (32.2) are satisfied by usual potentials.

The Kernel $\hat{K}(x,x')$ is L^2 under the following condition;

$$\int_0^\infty x |V(x)| dx = M'' \quad , \tag{33}$$

where M'' is a finite number. The proof of this statement for s-wave is done as follows.

The square integral of the kernel $\hat{K}(x,x')$ is given by

$$\int\int |\hat{K}(x,x')|^2 dx\, dx' \leq \int\int G_0^*(x,x')\, G_0(x,x')\, |V(x)|\, |V(x')|\, dx\, dx' \quad . \tag{34}$$

If we make use of the expression

$$G_0(x,x') = -\frac{1}{k}[\theta(x-x') e^{ikx} \sin kx' + \theta(x'-x) \sin kx\, e^{ikx'}] \quad , \tag{35}$$

with the step function

$$\theta(x) = \begin{cases} 1, & \text{for } x>0 \ , \\ 0, & \text{for } x<0 \ , \end{cases}$$

the right hand side of Eq. (34) becomes as follows.

The RHS of Eq. (34)

$$= \frac{1}{k^2}\int\int [\theta(x-x') \sin^2 kx + \theta(x-x') \sin^2 kx'] \times |V(x)|\, |V(x')|\, dx\, dx'$$

$$= \frac{2}{k^2}\int_0^\infty \sin^2 kx\, |V(x)|\, dx \int_x^\infty |V(x')|\, dx' \leq \frac{2}{k^2}\int_0^\infty \left(\frac{1-e^{-2ikx}}{2i}\right)^2 |V(x)|\, dx \int_x^\infty |V(x')|\, dx'$$

$$= 2\int_0^\infty \left(\int_0^x e^{-2ik\xi} d\xi\right)^2 |V(x)|\, dx \int_x^\infty |V(x')|\, dx' \leq 2\int_0^\infty x^2 |V(x)|\, dx \int_x^\infty |V(x')|\, dx'$$

$$\leq 2\int_0^\infty x |V(x)|\, dx \int_x^\infty x' |V(x')|\, dx' \leq 2M''^2 \quad . \tag{36}$$

This proves that the kernel $\hat{K}(x,x')$ is L^2.

In what follows, we start from Eq. (29) under assumption that $\xi(x)$ and $\eta(x)$ are square integrable functions and define the Hilbert-Schmidt norm $||\hat{K}||$ of $\hat{K}$ by

$$||\hat{K}|| = \sqrt{\int_0^\infty \int_0^\infty |\hat{K}(x,y)|^2 dx\, dy} \quad . \tag{37}$$

B. Neumann series

If the strength parameter is sufficiently small and satisfies the condition

$$|\lambda| \cdot \| \hat{K} \| < 1 \ , \tag{38}$$

we can write the solution of Eq. (29) in the Neumann series

$$\xi(x) = \eta(x) + \lambda \int_0^\infty \hat{K}(x,x') \, \eta(x') \, dx' + \lambda^2 \int_0^\infty \hat{K}_2(x,x') \, \eta(x') \, dx' + \ldots \ , \tag{39}$$

where

$$\begin{aligned} \hat{K}_n(x,x') &= \int_0^\infty \hat{K}(x,x'') \hat{K}_{n-1}(x'',x') \, dx'' \\ \hat{K}_1(x,x') &= \hat{K}(x,x') \ . \end{aligned} \tag{40}$$

If the condition (38) is satisfied, the series (39) is uniformly and absolutely convergent to the solution ξ of Eq. (29)[14]. On the other hand, if this condition is not satisfied, the series (39) is usually divergent.

In the three-body scattering problem, we have to calculate the series such as Eq. (14) or Eq. (18) and (19). If these series turn out to be divergent, we will be in a serious trouble, unless we consider other separations of $G_{0,q}^{(2)} t_q$ than those given by Eq. (13) or by Eqs. (15) and (16), or we consider a new technique of summing up series in a manner that they converge. Even when a series is convergent, if it converges very slowly, we will be in trouble in performing numerical calculations. In such a case, we should consider some method which works well beyond the convergence radius of the Neumann series. With these background, we shall propose a method of acceleration in the next subsection.

C. A method of acceleration

First, let us suppose that for a given energy E, λ is not an eigenvelue of $\hat{K}$. In the usual case, this is always expected, since the kernel $\hat{K}$ is complex and non-Hermitian. As a result, the eigenvalue of $\hat{K}$ is complex and not real. On the other hand, we assume that the parameter λ is real.

For simplicity, hereafter we write Eq. (29) as

$$\xi = \eta + \lambda \hat{K} \xi \ . \tag{41}$$

The usual first approximation, taking first two terms on the right hand side of Eq. (39) reads

$$\xi_1 = \eta + \lambda \hat{K} \eta \ . \tag{42}$$

Instead of Eq. (42), let us introduce a parameter α and the "accelerated" first approximation ζ_1 as

$$\zeta_1 = \eta + \alpha\lambda\hat{K}\eta \tag{43}$$

The solution ξ is then given in the form

$$\xi = \zeta_1 + u_1 \ . \tag{44}$$

The function u_1 satisfies the following equation

$$u_1 = \eta_1 + \lambda\hat{K}u_1 \ , \tag{45}$$

where

$$\eta_1 = (1-\alpha)\lambda\hat{K}\eta + \alpha\lambda^2\hat{K}_2\eta \ . \tag{46}$$

Here we note that Eq. (45) takes the same form as Eq. (41). Therefore, to proceed further, we replace η of Eqs. (43) and (46) by η_1 and define equations

$$\zeta_2 = \eta_1 + \alpha\lambda\hat{K}\eta_1 \ , \tag{47}$$

and

$$u_2 = \eta_2 + \lambda\hat{K}u_2 \tag{48}$$

where

$$\eta_2 = (1-\alpha)\lambda\hat{K}\eta_1 + \alpha\lambda^2\hat{K}_2\eta_1 \ . \tag{49}$$

As the function ξ [Eq. (41)] is expressed as (44), the function u_1 [Eq. (45)] is expressed as

$$u_1 = \zeta_2 + u_2 \ . \tag{50}$$

Combined with Eq. (50), the function ξ [Eq. (44)] is given by

$$\xi = \zeta_1 + \zeta_2 + u_2 \ . \tag{51}$$

Let up proceed further, defining an operator $T(\alpha)$ by

$$T(\alpha) = \lambda(1-\alpha)\hat{K} + \alpha\lambda^2 \hat{K}_2 \ . \tag{52}$$

In terms of this operator, Eq. (49) is expressed as

$$\eta_2 = T(\alpha)\eta_1 \ , \tag{53}$$

and generally

$$\eta_n = T(\alpha)\eta_{n-1} \ , \tag{54}$$

with

$$\eta_0 = \eta \ .$$

Repeating the iteration n-times, we obtain

$$\xi = \zeta_1 + \zeta_2 + ... + \zeta_n + u_n \ , \tag{55}$$

where

$$u_n = \eta_n + \lambda\hat{K}u_n \ . \tag{56}$$

The solution u_n of Eq. (56) can be expressed in terms of the resolvent $R(\lambda)$ of Eq. (56)

$$R(\lambda) = (1 - \lambda\hat{K})^{-1}, \tag{57}$$

as

$$u_n = R(\lambda)\eta_n \ . \tag{58}$$

As a result of our postulate that λ is not an eigenvalue of $\hat{K}$ we have

$$|| R(\lambda) || = M(\lambda) < \infty \ , \tag{59}$$

where $M(\lambda)$ is some non-negative function of λ. From Eq. (58), we then have

$$|| u_n|| < M(\lambda) || \eta_n|| \ . \tag{60}$$

Using Eqs. (54) and (60), we finally obtain

$$\| u_n \| < M(\lambda) \cdot \| T(\alpha) \|^n \| \eta \| \quad . \tag{61}$$

When the norm of $T(\alpha)$ is less than unity

$$\| T(\alpha) \| < 1 \ , \tag{62}$$

it follows from Eq. (61) that

$$\lim_{n\to\infty} \| u_n \| = 0 \tag{63}$$

and $\sum_{i=1}^{n} \zeta_i$ converges strongly to the unique solution ξ.

Here, the most important fact is that the convergence of the series $\sum_{i=1}^{n} \zeta_i$ is governed by the operator $T(\alpha)$ and not by the operator $\lambda\hat{K}$ as it was in the Neumann series. From the definition (52), we make the following simple statement: The parameter α in Eq. (52) can always be chosen so that the norm $\|T(\alpha)\|$ is less than or equal to $|\lambda| \cdot \| \hat{K} \|$,

$$\| T(\alpha) \| \leqq |\lambda| \cdot \| \hat{K} \| \ . \tag{64}$$

The choice $\alpha=0$ and the continuity of T with respect to α ascertain that this statement is true. As we shall see later by examples, the parameter α can often be chosen so that

$$\| T(\alpha) \| < 1, \tag{65}$$

even if $$|\lambda| \cdot \| \hat{K} \| > 1 \ . \tag{66}$$

That is to say, by summing $\sum_{i=1}^{n} \zeta_i$, we obtain a convergent result for ξ, even if the usual Neumann series (39) is divergent.

In concluding this subsection, we should remark that we have discussed the method of acceleration for Eq. (41) [Eq. (29)], but the same method should be applied to Eq. (27), because the similarity transformation to the original equation does not change the iterative properties of the equation.

D. Various choices of α

Now the problem is how to choose the parameter α in order to obtain the optimal degree of convergence. Let us consider three methods for this purpose.

The first and the most natural way is to choose α so that the norm of $T(\alpha)$ is minimal, i.e.,

$$\frac{\partial}{\partial \alpha} \| T(\alpha) \| = 0 \ . \tag{67}$$

If we obtain α satisfying Eq. (67) and prove that $\| T(\alpha) \| < 1$, the uniform convergence is guaranteed. We cannot state, however, this choice leads to the optimal convergence. Besides, the computation of the norm of $T(\alpha)$ renders a numerical task. In fact, even in the case of the potential scattering, this is a complicated operator. We propose, therefore, two other ways of choosing α which do not require the computation of the norm of $T(\alpha)$. Both are based on the idea of choosing α optimally at each step of iterations. We note that in sofar as α is chosen to satisfy

$$\| T(\alpha_i) \| < 1 \tag{68}$$

at each step, $\sum_{i=1}^{n} \zeta_i$ converges strongly to the unique solution ξ.

The first method is to choose α_i at the i-th step of iteration so that the projection of $\eta_i(\alpha_i, x)$ on a given function $F_i(x)$ is zero.

$$\int_0^\infty \eta_i(\alpha_i, x) F_i(x) dx = 0 \quad . \tag{69}$$

This method is closely related to the variational method. In our calculation, we have chosen $F_i(x)$ as

$$F_i(x) = \eta_{i-1}(x) \tag{70}$$

As a result, the condition (69) reads

$$\int_0^\infty \eta_i(\alpha_i, x) \eta_{i-1}(x) dx = 0. \tag{71}$$

The series $\sum_{i=1}^{n} \zeta_i(x)$ will be a good approximation to ξ, if u_n is small. By Eq. (60), if the norm $\| \eta_n \|$ is small, so will be the norm $\| u_n \|$. Our last choice is based on this fact. At each step of iterations, we find α_i from the equation

$$\frac{\partial}{\partial \alpha_i} \| \eta_i(\alpha_i) \| = 0 \ , \tag{72}$$

i.e., $\| \eta_i \|$ is minimal at $\alpha = \alpha_i$.

E. Examples

Ex.1 Separable kernel

As a first example, we study the separable kernel

$$\hat{K}(x, x') = g(x) h(x') \quad . \tag{73}$$

The Neumann series (39) converges provided that

$$|\lambda|^2 \int_0^\infty |g(x)|^2 dx \int_0^\infty |h(x')|^2 dx' < 1 \tag{74}$$

However, if we chose α to be

$$\alpha = \frac{1}{1 - \lambda \int_0^\infty g(x) h(x) dx} , \tag{75}$$

the norm of $T(\alpha)$ is always zero

$$|| T(\alpha) || = 0 . \tag{76}$$

As a result, the first iteration (43) is exact. This is an example in which Eq. (64) holds. Also we note that by the choice (75), the condition (71) for i = 1

$$\int_0^\infty \eta_1(\alpha_1, x) \eta(x) = 0 , \tag{77}$$

as well as the condition (72) for i = 1

$$\frac{\partial}{\partial \alpha_1} || \eta_1(\alpha_1) || = 0 \tag{78}$$

are satified.

Ex.2

In order to have some quantitative insight into the convergence, let us consider the following simple equation

$$\xi(x) = 1 + \lambda \int_{-1}^{1} \frac{\xi(x') dx'}{\sqrt{1+x^2+x'^2}} . \tag{79}$$

The norm $|| \hat{K} ||$ in this case is

$$|| \hat{K} ||^2 = \int_{-1}^{1} \int_{-1}^{1} \frac{dx dx'}{1+x^2+x'^2} = 2.558$$

and the Neumann series converges if

$$|\lambda| < 0.6252 . \tag{80}$$

The results of iterations for various choices of α are illustrated in

Table 1 for $\lambda = 0.5$ and in Table 2 for $\lambda = 2$. The first column uses α determined by Eq. (67), the second column by Eq. (72) and the third column by Eq. (71). The last column is the usual Neumann series with $\alpha = 0$. The Neumann series is convergent for $\lambda = 0.5$, but divergent for $\lambda = 2$. From Table I and II, we reach two conclusions;

(1) The convergence of the accelerated series is much faster than that of the original one.

(2) In the acceleration method, the convergence radius is greatly enlarged. In this case, the accelerated series is convergent if

$$|\lambda| < 46 , \tag{81}$$

except around the close vicinity of the eigenvalue at $\lambda = 0.6253$.

Table I. Convergence of iterations of Eq. (79) with λ=0.5. The first column uses α determined by Eq. (67), the second by Eq. (72), and the third column by Eq. (71). The fourth column is the Neumann series (α=0). The convergent value is $\xi(x_0)$, x_0 = 0.973907. The Neumann series converges very slowly.

N \ α	Eq.(67)	Eq. (72)	Eq.(71)	α=0
1	4.31237	4.32384	4.32255	1.0
2	4.31726	4.31750	4.31674	1.66604
3	4.31757	4.31756	4.31756	2.19736
4	4.31755			2.62220
5	4.31756			2.96191
10				3.87439

Table II. Convergence of iterations of Eq. (79) with λ=2.0. For caption, see Table I. The Neumann series diverges.

N \ α	Eq.(67)	Eq.(72)	Eq.(71)	α=0
1	-0.211685	-0.211820	-0.212006	1.0
2	-0.202797	-0.202199	-0.199009	3.66417
3	-0.202242	-0.202206	-0.202347	12.1653
4	-0.202208		-0.202160	39.3550
5	-0.202206		-0.202208	126.321
6			-0.202205	404.481
7			-0.202206	1294.17

§4. Application to the Elastic Scattering

A. Removal of divergence

In this section, we apply the method proposed in §3 to the elastic scattering from various local and non-local potentials. We start from Eq. (27) with the kernel (28).

In ref. 6, it was shown that the Neumann series for nucleon-nucleon scattering diverges when the relative momentum is less than about 0.6 fm^{-1}. To get a convergence, the following method was proposed[6].

First, the Green's function is decomposed as

$$G_0 = - |\bar{w}_k><u_k| + \bar{g}_k \ . \tag{82}$$

Here u_k represents the Bessel function times $\sqrt{k}$. The function $\bar{w}_k$ represents

$$\bar{w}_k = \hat{\psi}_{k,1} + iu_k \ , \tag{83}$$

where the function $\hat{\psi}_{k,1}$ is the first Sturm-Liouville function for momentum k. See Eq. (13) and below that and also ref. 13. The Green's function $\bar{g}_k$ is defined by (82). The n-th iteration of the operator $\bar{\omega}_k$ appeared in Eq. (13) [q in Eq. (13) is being written as k] is given by this Green's function as

$$\bar{\omega}_k^{(n)} = 1 + \sum_{i=1}^{n} (\bar{g}_k V)^i \ . \tag{84}$$

With the operator (84), the n-th iteration of Eq. (27) is expressed as

$$\phi^{(n)} = \bar{\omega}^{(n)} |u> - [\hat{\psi}_1 + i\bar{\omega}^{(n)} |u>]<u|V|\phi^{(n-1)}> \ ,$$

$$T^{(n)} = - <u|V|\phi^{(n)}> \ . \tag{85}$$

Here we have suppressed the suffix k for simplicity. Since $\bar{\omega}_k^{(n)}$ does not involve contributions from the first Sturm-Lionville eigenstate, the series (84) converges, and so does the iteration (85). Numerical calculations of Eq. (85) was the subject of ref. 13. In the present paper, our concern is to study the way of improving the speed of convergence of the series (84).

B. Numerical examples

Let us denote the phase shift calculated from Eq. (85) with the series (84) by δ_n and that calculated from Eq. (85) with the series (84) but applied the method of §3 by θ_n. At each step, α_n is determined by Eq. (71). We calculate the phase shifts for 1 MeV and 10 MeV for nuclear potentials and for 0.1 eV and 1.eV for the atomic potential.

The results are given in Table III for the exponential potential[6],

$$V(r) = -\lambda e^{-\kappa r}, \qquad \lambda = 1.8927 \text{ fm}^{-2}, \qquad \kappa = 0.8615 \text{ fm}^{-1}, \tag{86}$$

in Table IV for the Yukawa potential[6]

$$V(r) = -\frac{\lambda}{\kappa} \frac{e^{-\kappa r}}{r}, \qquad \lambda/\kappa = 1.5734 \text{ fm}^{-1}, \qquad \kappa = 0.6329 \text{ fm}^{-1}, \tag{87}$$

and in Table V for the atomic potential[15]

$$V(r) = -2e^{-2r}\left(1+\frac{1}{r}\right) - \frac{\alpha}{r^4},$$

$$\alpha = \frac{9}{2} - \frac{2}{3}e^{-2r}\left(r^5 + \frac{9}{2}r^4 + 9r^3 + \frac{27}{2}r^2 + \frac{27}{2}r + \frac{27}{4}\right). \qquad (88)$$

The elastic channel in the n-d scattering is described as the scattering from a nonlocal nonsymmetric potential[12]

$$U(y_1,y_2) = \frac{8\pi}{3} \times \int_{\frac{4}{3}|y_2-y_1/2|}^{\frac{4}{3}(y_2+y_1/2)} x_1 dx_1 \phi_d(x_1) V(x_1) \times \phi_d\left(\sqrt{|x_1^2 + \tfrac{4}{3}(y_1^2 - y_2^2)|}\right) \qquad (89)$$

where $\phi_d(x)$ and $V(x)$ denote the wave function of the deuteron and the phenomenological nucleon-nucleon potential, respectively. The potential $V(x)$ is chosen so that it reproduces the $^3S_1$ low energy data[7].

$$V(x) = \hbar c(-p_1 e^{-p_2 x} + p_3 e^{-2p_2 x} + (p_1-p_3)e^{-5p_2 x})/x$$

with

$$\hbar c = 197 \text{ MeV fm},$$

$$p_1 = 3.1344, \quad p_2 = 1.5502 \text{ fm}^{-1}, \quad p_3 = 7.4616. \qquad (90)$$

Table III. Phase shifts δ_n and θ_n calculated by Eq. (85) for the exponential potential (86). In calculating θ_n, the acceleration method is applied to the series (84). At each step, the parameter α_n is determined by Eq. (71).

Energy	1 MeV		10 MeV	
n	tan δ_n	tan θ_n	tan δ_n	tan θ_n
1	-1.507	-1.150	8.831	5.537
2	-1.197	-1.117	4.659	5.350
3	-1.139		5.569	
4	-1.123		5.298	
5	-1.119		5.365	
6	-1.117		5.346	
7			5.351	

Table IV. Phase shifts δ_n and θ_n calculated by Eq. (85) for the Yukawa potential (87). For captions, see Table III.

Energy	1 MeV		10 MeV	
n	tan δ_n	tan θ_n	tan δ_n	tan θ_n
1	-1.471	-1.028	264.7	169.3
2	-1.083	-0.9641	155.3	158.6
3	-1.002		161.6	
4	-0.9766		158.8	
5	-0.9681		158.7	
6	-0.9652		158.6	
7	-0.9642			

With the potential (89), we have solved the Schrödinger equation

$$\phi = f + PG_0 U\phi . \qquad (91)$$

We calculated (91) for E=10 MeV and 26 MeV, respectively. The value of $\phi(x_0)$, x_0=2 fm is printed in Table VI. The first and fourth (second and fifth) columns show the accelerated values of $\phi(x_0)$, where α_i is chosen by Eq. (71) [Eq. (72)]. The third and sixth columns show the Neumann series (α=0).

Table V. Phase shifts δ_n and θ_n calculated by Eq. (85) for the atomic potential (88). For captions, see Table III.

Energy	0.1 eV		1 eV	
n	tan δ_n	tan θ_n	tan δ_n	tan θ_n
1	-1.271	-0.8755	-9.374	-6.381
2	-0.9416	-0.8502	-6.602	-6.195
3	-0.8761	-0.8504	-6.301	
4	-0.8582		-6.223	
5	-0.8529		-6.203	
6	-0.8512		-6.197	
7	-0.8507		-6.195	
8	-0.8504			

Table VI. The value of $\phi(x_0)$ at x_0=2 fm obtained as the solution of Eq. (91) for the non-local potential (89). The first, second (fourth and fifth) columns are calculated values for 10 MeV (26 MeV) by the acceleration method. In the first and fourth (second and fifth) columns, α_i are chosen by Eq. (71) [eq. (72)]. The third and sixth columns show the Neumann series (α=0).

Energy	10 MeV			26 MeV		
n α_i	Eq.(71)	Eq.(72)	α=0	Eq.(71)	Eq.(72)	α=0
1	0.547	0.526	0.906	0.274	0.327	9.75
2	0.707	0.768	0.306	0.368	0.360	5.01
3	0.952	0.909	1.22	0.361	0.363	4.38
4	0.996	0.991	0.444	0.363		3.51
5	1.06	1.04	1.42			2.87
6	1.08	1.07	0.484			2.34
7	1.09	1.08	1.56			1.91
8	1.10	1.09	0.469			1.55

C. Concluding discussions

Examples in the present section show that the method proposed in sec. 3 is very useful; the acceleration of convergence is remarkable and the radius of convergence is substantially enlarged. This fact allows us to treat the elastic channel of the n-d scattering by the perturbation-iteration for as small energy as 10 MeV without making reductions into separable terms[6),12)].

Below 8 MeV, the convergence for this problem becomes very slow and useless. In sec. 5, we shall present a method that works even at lower energies.

Let us describe the price that we have to pay to enjoy the fantastically fast convergence. This, of course, depends on the way of choosing the parameter α. If we calculate α from the condition (67), we have to compute $\hat{K}_2$, $\hat{K}_3$, $\hat{K}_4$ and their traces. If we calculate α from the condition (71), we have to compute two one-dimensional integrals $\langle \eta_i | \hat{K} \eta_i \rangle$ and $\langle \eta_i | \hat{K}_2 \eta_i \rangle$ at each step of iterations. If we calculate α from the condition (72), we have to compute three one-dimensional integrals $\langle \hat{K}\eta_i | \hat{K}\eta_i \rangle$, $\langle \hat{K}_2\eta_i | \hat{K}_2\eta_i \rangle$ and $\langle \hat{K}\eta_i | \hat{K}_2\eta_i \rangle$ at each step of iterations.

Of course, there might be some other possibilities of choosing the operator $T(\alpha)$ than Eq. (52), or of taking the first approximation different from Eq. (43); e.g.

$$\zeta_1 = \eta + \alpha\lambda\hat{K}\eta + \alpha_2\lambda^2\hat{K}_2\eta \ . \tag{92}$$

However, the method presented in secs. 3 and 4 certainly enjoys the rapid convergence with a simple procedure.

§5. Elastic Scattering from a Non-local Potential

As shown in refs.6,10,12,the elastic scattering of neutron from deuteron [Eq. (7.1)] is represented as the scattering from a non-local potential (89). In these references, a method for treating this problem was presented. The basic idea is first to express the non-local potential as a sum of real symmetric and hermitian parts and then to express each part as a sum of separable terms. The separable terms that give large contributions are taken as the unperturbed part and the remainder is treated as the perturbation. This method converges very quickly for 26 MeV, but slowly for 6 MeV. At low energies we need many separable terms. Thus the method is impracticable at low energy.

In §4, the method of acceleration proposed in §3 was studied for the potential (89). Table VI shows that this method works well at 26 MeV, but also impracticable at low energies, say, below 8 MeV.

In the present section, we present two methods that we have studied, following the general idea of decomposing an operator K into two terms: A large part is expressed as a separable term and the remainder is treated as the perturbation.

A. Method I: Decomposition of the non-local potential

We decompose the non-local potential U as

$$U(x,y) = U^{B}_{x_1,y_1}(x,y) + \frac{U(x,y_1)U(x_1,y)}{U(x_1,y_1)} \quad . \tag{93}$$

The separable term on the right hand side of Eq. (93) is known as the Bateman approximation[16]. It coincides with U(x,y) on two lines $x=x_1$ and $y=y_1$ (We call these lines the Bateman lines). A serious problem is how to choose the point (x_1,y_1). If the choice is wrong, the Bateman approximation might be very poor and

$$\| U^{B} \| \gg \| U \| \quad . \tag{94}$$

The set of the Bateman lines (x_1,y_1) may be found by the following manner. (1) For each point, we compute the norm $\| U^{B} \|$ and find a point for which $\| U^{B} \|$ takes a minimum value. (2) For a fixed energy and for each point, we solve the equation

$$\phi = f + G_0 U^{B} \phi \qquad (f=\sin kx) \tag{95}$$

by the perturbation iteration [Eq. (95) is a part of Eq. (91)] and look for the best choice of (x_1,y_1).

As an example, we studied the second method for E=0.01 MeV and tried to find (x_1,y_1) in the range $1 \leqq x_1 \leqq 4$ fm and $1 \leqq y_1 \leqq 6$ fm. We got the following results: (1) For this energy, this method converges only for very few choices of (x_1,y_1). Actually only two choices have been successful; (2.0, 2.25) and (2.75, 2.75). The region of convergence is thus very restricted. (2) The rate of convergence is not very high. (3) If we fix the values of (x_1,y_1) at the best choice, i.e. at (2.0, 2.25) and change the energy E, this method gives convergent results only for E<1 MeV. As a consequence, this method does not prove to be practical.

It turned out that the best choice for the Bateman lines are x_1=2 fm and y_1=2.25 fm below 1 MeV. We illustrate in Table VII. the iterative solution ϕ(x=2 fm) of Eq. (95).

As we see from this table, this method converges rather slowly.

B. Method II: Decomposition of the kernel

In this subsection, we study the decomposition of the kernel $K = G_0 U$. We decompose K similarly to the previous case as

$$K(x,y) = K^{B}_{x_1,y_1}(x,y) + \frac{K(x,y_1)K(x_1,y)}{K(x_1,y_1)} \quad . \tag{96}$$

The separable term on the right hand side of Eq. (96) is energy dependent.

Let us define the norms of the operators K and K^B by

$$N_0(E)=\sqrt{T_r(K\times K)} \qquad (97)$$

and

$$N(E,x_1,y_1)=\sqrt{T_r(K^B\times K^B)}\ , \qquad (98)$$

where T_r represents "the trace of". We calculated the minimum N_M of N for fixed energies and found the best set of Bateman lines (x_1,y_1) for each energy. In this method, the set of the Bateman lines (x_1,y_1) coincide with the Bateman lines for which the best convergence is guaranteed. This is not the case for the method I. The ratio of N_M/N_0 is given in the Table VIII.

The results of iterations of the equation

$$\phi = f + K^B\phi \qquad (99)$$

for various energies are given in Tables IX-XI. For all energies in

Table VII. Solution of Eq. (95) at the point x=2 fm. It turns out that $x_1=2$, $y_1=2.25$ are the best choice as the Bateman lines for small energies. For energies higher than 2 MeV the series diverges.

n \ E	0.01 MeV	0.1 MeV	0.5 MeV	1 MeV
1	0.03591	0.1133	0.2512	0.3515
2	0.04570	0.1249	0.1216	$-7.582\cdot10^{-3}$
3	0.05094	0.1288	0.1497	0.2512
4	0.05290	0.1278	0.1446	0.7387
5	0.05373	0.1276	0.1462	0.1945
6	0.05407	0.1274	0.1457	0.1125
7	0.05420	0.1273	0.1458	0.1682
8	0.05426			0.1304
9	0.05428			0.1561
10	0.05429			0.1386

Table VIII. The best set of Bateman lines (x_1,y_1) for a given energy. For the ratio N_M/N_0, see Eqs. (97) and (98) and below that.

E(MeV)	(x_1,y_1)	N_M/N_0
1	1.5, 3	0.018
5	1.5, 4	0.049
10	1.5, 3.5	0.075

Table IX. Eq. (99) is solved. The kernel K_B is defined by Eq. (96) with $K=G_0U$, U being given by Eq. (89). The Bateman lines for all energies in this Table are chosen as $x_1=1.5$ fm and $y_1=3.0$ fm. This set is the best choice for E=1 MeV. See Table VIII. This table illustrates the solution of Eq. (99) at x=6.25 fm.

n \ E(MeV)	0.01	1	2	4	5	6	10	26
1	0.1983	0.9871	0.4357	-1.941	divergence	3.627	0.1062	-0.5757
2	0.2080	0.9865	0.4950	-2.445		3.794	0.1102	-0.5648
3	0.2088	0.9864	0.4942	-2.084		3.749	0.1052	-0.5617
4	0.2089		0.4939	-2.070		3.754	0.1056	-0.5608
5				-2.113		3.759		-0.5606
6				-2.105		3.758		-0.5605
7				-2.103				
8				-2.104				

Table IX (X, XI), the Bateman lines (x_1,y_1) are taken for the best set of the Bateman lines for energy E=1 MeV (5 MeV, 10 MeV), namely $(x_1,y_1)=(1.5,3),[(1.5,4),(1.5,3.5)]$. The convergence in Table X is faster than in other Tables and the choice of the set of Bateman lines at (1.5,4) probably makes the perturbative series of (99) convergent for all energies.

Table X. Eq. (99) is solved. The kernel K_B is defined by Eq. (96) with $K=G_0U$, U being given by Eq. (89). The Bateman lines for all energies in this Table are chosen at $x_1=1.5$ fm and $y_1=4$ fm. This set is the best choice for E=5 MeV. See Table VIII.[1] This Table illustrates the solution of Eq. (99) at x=6.25 fm. From this Table, we see that this choice of the Bateman lines yield a good convergence for the iterative procedure of Eq. (99) for all energies.

n \ E(MeV)	0.01	1	2	3	5	10	15	26
1	-0.04109	-0.1344	3.488	-3.472	-2.476	-2.108	-1.332	-0.3409
2	-0.04676	-0.1234	2.956	-4.113	-2.533	-2.100	-1.336	-0.3679
3	-0.04686	-0.1233	2.782	-4.182		-2.099		-0.3669
4	-0.04687		2.847	-4.190				
5			2.851	-4.191				
6			2.846					

Table XI. Eq. (99) is solved. The kernel K_B is defined by Eq. (96) with $K=G_0U$, U being given by Eq. (89). The Bateman lines for all energies in this Table are chosen at $x_1=1.5$ fm, $y_1=3.5$ fm. This set is the best choice for E=10 MeV. See Table VIII. This Table illustrates the solution of Eq. (99) at x=2 fm.

n \ E(MeV)	0.01	1	2	3	5	10	14	26
1	0.02545	0.2513	0.3531	very slowly convergent	0.5065	0.6486	0.7051	0.6901
2	0.02580	0.2555	0.3577		0.5149	0.6630	0.7211	0.7242
3	0.02578	0.2553			0.5144	0.6623	0.7203	0.7215
4								0.7216
5								
6								

In this method we used a slightly different way of the iteration. At every iteration step we use for computation of $\phi^{(n)}(x_i)$ all previously calculated values of $\phi^{(n)}(x_j)$,

We have found the following aspects concerning the method II.

(1) The possible region of Bateman lines (x_1, y_1) for which the method II yields convergence is much larger than that for the method I. This region does not significantly depend on energy. See, Fig. 1 and 2.

(2) The rate of convergence is higher than that in the method I.

(3) It is possible to choose a set of Bateman lines (x_1, y_1) for which the convergence is guaranteed for all energies.

(4) The position of the best set of Bateman lines (x_1, y_1) does not significantly depend on energy.

(5) This reduction leads to convergent results even if the potential V is stronger by a factor 10.

C. Conclusion of sec. 5

We may safely use the method II for treating the elastic channel of the n-d scattering.

Fig. 1. The set of Bateman lines (x_1, y_1) and the convergence of the iterative solutions of Eq.(99) for E=1 MeV. The dark square ▨ denotes the most favorable region. The region marked by a cross x yields a very slow convergence or divergence. The scale of this Fig. is 0.25 fm.

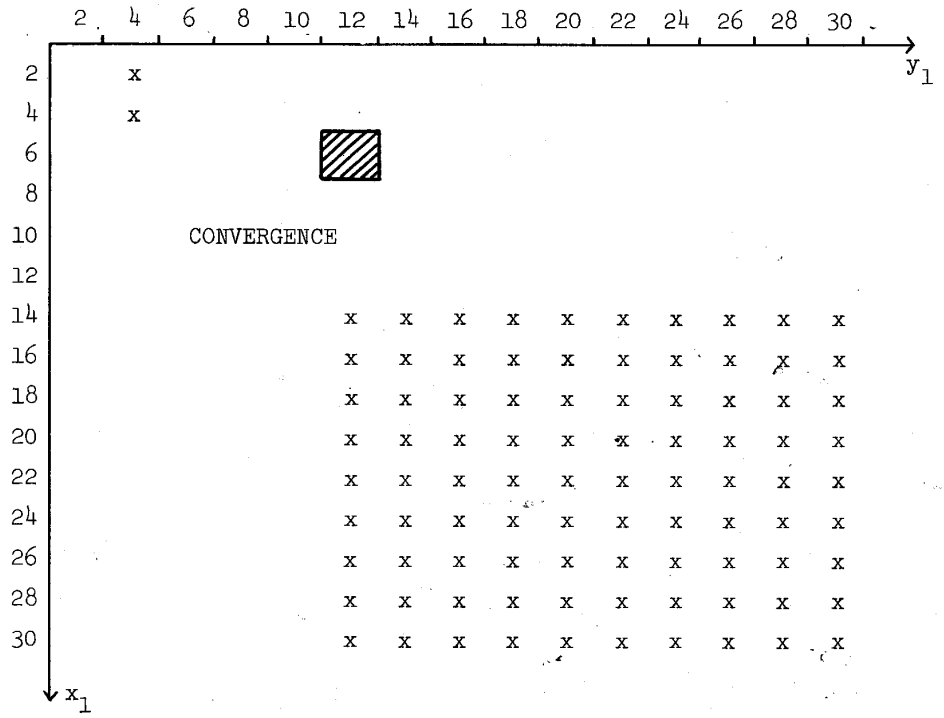
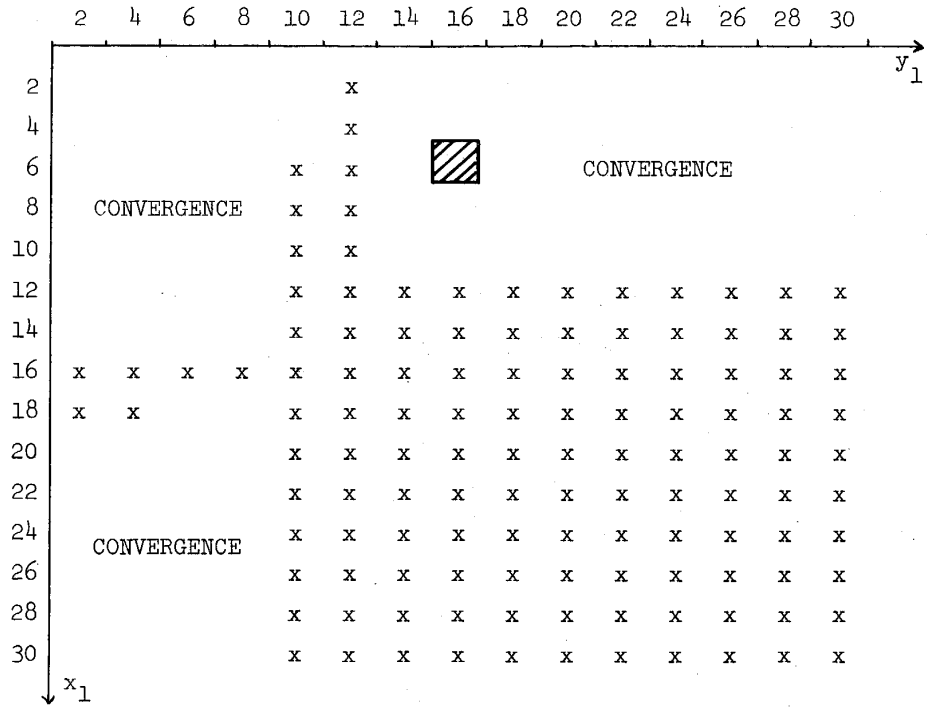


Fig. 2. The set of Bateman lines (x_1, y_1) and the convergence of the iterative solutions of Eq.(99) for E=5 MeV. See Fig. 1 for captions. The scale of this Fig. is 0.25 fm.



§6. Breakup Channel I

A. Introduction

In solving three-body problems, the most difficult part in practice is to transform the wave function from one set of coordinates to the other. This is represented by the operator P in Eq. (14) or Eqs. (18) and (19).

In the bound three-body system[7),8)], the wave function of the spectator is expressed by the plane wave, while the wave function for the interacting pair does not spread out to large distances, since the energy of the relative motion of the interacting pair is negative. Besides, the wave functions of the spectator and the interacting pair are folded to meet the binding energy of the system, which is negative. As a result, the wave function of the three-body bound state is restricted in a finite region. This fact makes the treatment of three-body bound states much simpler than that of scattering problems. In the breakup channel, both wave functions for the spectator and the interacting pair are spread out to infinitely large distances. However, the transformation of the wave functions described by one set of coordinates to another

can not be done numerically at large distances. As a result, we have to find an analytical formula for this purpose. In the sections to follow, we shall demonstrate how we can handle the breakup channel restricting ourselves to a system without Coulomb force.

We divide this section into two parts: First, we discuss the transformation of the set of coordinate for the spectator wave function. This subject was treated previously[11),7),8)].

However, we recapitulate it in sec. 6B for completeness. Next, using the formula in sec. 6B, we discuss how to treat the interacting pair. This is done in sec. 6C.

B. Coordinate transformation of spectator functions

Let $|\alpha>$ denote the spin-angular function in J-j coupling as defined by Eq. (3-9) of ref.17. We designate by $u_\ell(p)$ the spectator function [Eq. (6-13) of ref.17]

$$u_\ell(py) = \sqrt{\frac{2}{\pi}}\, p j_\ell(py) \quad . \tag{100}$$

This function is normalized so that

$$\int_0^\infty u_\ell(py)\, u_\ell(p'y)\, y^2 dy = \delta(p-p') \tag{101.1}$$

and

$$\int_0^\infty u_\ell(py)\, u_\ell(py')\, dp = \frac{1}{yy'}\, \delta(y-y') \quad . \tag{101.2}$$

We define q to denote the momentum of the interacting pair,

$$q^2 = \left(\frac{m}{\hbar^2}\right)\left(E - \frac{3\hbar^2}{4m}\, p^2\right) \qquad \text{for} \qquad E - \frac{3\hbar^2}{4m}\, p^2 > 0 \quad . \tag{102}$$

The heart of calculations in three-body bound states was to perform the quadrature of the quantity

$$A_{\alpha_1 q_1}^{\alpha q}\, F^{\alpha_1}(x) = <\alpha|<u_\ell(p)|P|u_{\ell_1}(p_1)>|\alpha_1>F^{\alpha_1}(x) \quad , \tag{103}$$

as defined by Eq. (7-14) of ref.17. We have seen in ref.17 that Eq. (103) is brought to the form

$$A_{\alpha_1 q_1}{}^{\alpha q} F^{\alpha_1}(x) = 2pp_1 \int_{-1}^{1} du j_L(\lambda x) \Lambda_{pp_1}(\alpha\alpha_1;u) \int_0^\infty x'^2 dx' j_{L_1}(\lambda_1 x') F^{\alpha_1}(x'), \tag{104}$$

as given by Eq. (8-1) of ref.17. Here x and x' represent the coordinates of a pair, say i, and that of another pair, say j. λ and λ_1 are given by

$$\vec{\lambda} = \frac{\vec{p}}{2} + \vec{p}_1 \qquad \text{and} \qquad \vec{\lambda}_1 = -(\vec{p} + \frac{\vec{p}_1}{2}) \ . \tag{105}$$

The function $\Lambda_{pp_1}(\alpha\alpha_1;u)$ is given by Eq. (8-3) of ref. 17. For bound states, the function $F^{\alpha_1}(x')$ is decreasing with increasing x'. The FORTRAN program (PERFECT I) for calculating the quantity (104) was given in ref. 18.

C. Treatment of interacting pair

Now let us assume that the function $F^{\alpha_1}(x)$ in Eq. (103) takes the following form

$$F^{\alpha_1}(x) = \frac{1}{q_1^2 + [\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{L_1(L_1+1)}{x^2}] + i0} (\frac{m}{\hbar^2}) \ \Xi^{\alpha_1}(x') \ . \tag{106}$$

In fact, when we iterate the series (14) or (18), (19), we have to deal with the expression (106). We defer the study of $\Xi^{\alpha_1}(x')$ to the end of sec. 7B. In this subsection, we discuss how we can calculate Eq. (104) if the function $F^{\alpha_1}(x)$ takes the form of (106).

If we put Eq. (106) into the last integral of Eq. (104), we get

$$\int_0^\infty x^2 dx j_{L_1}(\lambda_1 x) \frac{1}{q_1^2 + [\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{L_1(L_1+1)}{x^2}] + i0} (\frac{m}{\hbar^2}) \ \Xi^{\alpha_1}(x')$$

$$= \frac{2}{\pi} \int_0^\infty x^2 dx j_{L_1}(\lambda_1 x) \int_0^\infty q^2 dq j_{L_1}(qx) \frac{1}{q_1^2 - q^2 + i0} (\frac{m}{\hbar^2}) \int_0^\infty x'^2 dx' j_{L_1}(qx') \Xi^{\alpha_1}(x')$$

$$= \int_0^\infty dq \ \frac{\delta(\lambda_1 - q)}{q_1^2 - q^2 + i0} (\frac{m}{\hbar^2}) \int_0^\infty x'^2 dx' j_{L_1}(qx') \ \Xi^{\alpha_1}(x')$$

$$= \frac{1}{q_1^2 - \lambda_1^2 + i0} (\frac{m}{\hbar^2}) \int_0^\infty x'^2 dx' j_{L_1}(\lambda_1 x') \ \Xi^{\alpha_1}(x')$$

$$= [-i\pi\delta(q_1^2 - \lambda_1^2) + P(\frac{1}{q_1^2 - \lambda_1^2}] (\frac{m}{\hbar^2}) \int_0^\infty x'^2 dx' j_{L_1}(\lambda_1 x') \ \Xi^{\alpha_1}(x') \ . \tag{107}$$

As a result, Eq. (104) is expressed as

$$A_{\alpha_1 q_1}{}^{\alpha q_F \alpha_1}(x) = -i2\pi j_L(qx)\Lambda_{pp_1}[\alpha\alpha_1;\frac{1}{pp_1}(E-\frac{\hbar^2}{m}p^2-\frac{\hbar^2}{m}p_1^2)](\frac{m}{\hbar^2})\int_0^\infty x'^2 dx' j_{L_1}(q_1x')\Xi^{\alpha_1}(x')$$

$$+4\int_{\lambda\ \min}^{\lambda\ \max}\lambda d\lambda j_L(\lambda x)\Lambda_{pp_1}(\alpha\alpha_1;u)P(\frac{1}{q^2-\lambda^2})(\frac{m}{\hbar^2})\int_0^\infty x'^2 dx' j_{L_1}(\lambda_1 x')\Xi^{\alpha_1}(x') \ , \tag{108}$$

Here, we have changed the integral over u to that over λ as in ref. 17. Also, we made use of the identity

$$q_1^2-\lambda_1^2 = q^2-\lambda^2 \ . \tag{109}$$

For given values of p and p_1, the first term on the right hand side is separable, and hence easily calculated. The second term involves the principal value integral, which can be calculated with some care[18].

Now only remaining problem is to study the method of calculating the last integral $\int_0^\infty x^2 dx j_L(qx)\Xi^\alpha(x)$. If $\Xi(x)$ is of short range with respect to x, this integral can be calculated numerically.

§7. Breakup Channel II

A. Breakup amplitude

In this section, we derive the breakup amplitude. For the purpose of studying the manner of calculations of the breakup channel, here we neglect contributions from the elastic and closed channels. Namely, we approximate Eq. (7) by

$$\Psi^{(1)} = \Phi^{(1)} + \int'_{E-E_p>0} dE_p |\phi_p> G_{0,q}^{(2)} t_{1,q} <\phi_p| P\Psi^{(1)} \ . \tag{110}$$

For simplicity and to be explicit, we write this equation as

$$|\Psi> = |\Phi> + \sum_\alpha \int' dp |u_\ell(p)>|\alpha> G_{0,a}^{(2)}(q) t_a(q) <\alpha|<u_\ell(p)|P\Psi> \ . \tag{111}$$

Except the region of integration, the notations in this equation are in conformity with ref. 17. The operator $G_{0,a}^{(2)}(q)t_a(q)$ was given in the from of Eq. (13), (15) or (16) which take in general the form of

$$G_{0,a}^{(2)}(q)t_a(q) = A_a(q) - |B_a(q)>F_a(q)<C_a(q)| \ . \tag{112}$$

Putting Eq. (112) into Eq. (111), we get

$$|\Psi\rangle = |\Phi\rangle + \sum_{\alpha}\int' dp\,|u_{\ell}(p)\rangle|\alpha\rangle A_a(q)\langle\alpha|\langle u_{\ell}(p)|P\Psi\rangle$$

$$- \sum_{\alpha}\int' dp\,|u_{\ell}(p)\rangle|\alpha\rangle|B_a(q)\rangle V_a(q) \quad , \tag{113}$$

where

$$V_a(q) = F_a(q)\langle C_a(q)|\langle\alpha|\langle u_{\ell}(p)|P\Psi\rangle \quad . \tag{114}$$

We introduce the operator B defined by

$$B = [1 - \sum_{\alpha}\int' dpP\,|u_{\ell}(p)\rangle|\alpha\rangle A_{\alpha}(q)\langle\alpha|\langle u_{\ell}(p)|]^{-1} \quad , \tag{115}$$

and express the equations for $P|\Psi\rangle$ and $V_a(q)$ in the form

$$P|\Psi\rangle = B[P|\Phi\rangle - \sum_{\alpha}\int' dpP\,|u_{\ell}(p)\rangle|\alpha\rangle|B_a(q)\rangle V_a(q)] \quad , \tag{116}$$

$$V_a(q) + \sum_{\alpha'}\int' dp' M_{\alpha q,\alpha' q'} V_{a'}(q') = F_a(q)\langle C_a(q)|\langle\alpha|\langle u_{\ell}(p)|BP|\Phi\rangle \quad , \tag{117}$$

where

$$M_{\alpha q,\alpha' q'} = F_a(q)\langle C_a(q)|\chi_{\alpha' q'}{}^{\alpha q}(x)\rangle \quad , \tag{118}$$

$$\chi_{\alpha' q'}{}^{\alpha q}(x) = \langle\alpha|\langle u_{\ell}(p)|X_{\alpha' q'}\rangle \quad , \tag{119}$$

$$X_{\alpha' q'} = B\cdot P|u_{\ell'}(p')\rangle|\alpha'\rangle|B_{a'}(q')\rangle \quad . \tag{120}$$

If we use Eq. (120), Eq. (116) reads

$$P|\Psi\rangle = BP|\Phi\rangle - \sum_{\alpha}\int' dpX_{\alpha q}V_a(q) \quad . \tag{121}$$

Suppose that we have calculated the function $BP|\Phi\rangle$ as well as the function $X_{\alpha q}$. From the function $X_{\alpha' q'}$, we can calculate $\chi_{\alpha' q'}{}^{\alpha q}(x)$, $M_{\alpha q,\alpha' q'}$ and $V_a(q)$, successively. Hence let us assume that all quantities on the right hand side of Eq. (121) are calculated.

By definition,

$$u_q = \sqrt{q}\,j_{\ell}(qx) \quad , \qquad w_q = \sqrt{q}\,h_{\ell}^{(+)}(qx) \quad , \tag{122}$$

where $h_\ell^{(+)}(qx)$ is the spherical Bessel function defined by Messiah[19]. From Eq. (20), the Green's function $G_{0,q}^{(2)}$ behaves asymptotically as

$$G_{0,q}^{(2)} \xrightarrow[x\to\infty]{} -|w_q\rangle\langle u_q| \xrightarrow[x\to\infty]{} -\frac{e^{i(qx-\frac{1}{2}L\pi)}}{x} \langle j_L(qx')| \ . \tag{123}$$

As a result, the breakup amplitude for the channel 1 with momentum p and spin-isospin α is obtained by Eq. (111) as

$$T_{\alpha,p}^{(1,B)} = -\langle j_L(qx')|t_a(q)\langle\alpha|\langle u_\ell(p)|P|\Psi\rangle. \tag{124}$$

On the other hand, Eq. (15) and Eq. (16) read respectively,

$$G_{0,q}^{(2)}t_q = G_{0,q}^{(2)}[J_q - J_q|w_q\rangle\frac{1}{1+\langle u_q|J_q|w_q\rangle}\langle u_q|J_q] \ , \quad \text{for } L=0, \tag{125}$$

and

$$G_{0,q}^{(2)}t_q = G_{0,q}^{(2)}V[\sum_{n=0}^{[\frac{L-1}{2}]}(G_{0,q}^{(2)}V)^n + (G_{0,q}^{(2)}V)^{[\frac{L-1}{2}]}G_{0,q}^{(2)}\{J_q$$

$$- J_q|w_q\rangle\frac{1}{1+\langle u_q|J_q|w_q\rangle}\langle u_q|J_q\}], \quad \text{for } L\geq 1. \tag{126}$$

Putting Eq. (125) and Eq. (126) into Eq. (111), we express the amplitude (124) as

$$T_{\alpha,p}^{(1,B)} = -\langle j_{L=0}(qx)|[J_q - J_q|w_q\rangle\frac{1}{1+\langle u_q|J_q|w_q\rangle}\langle u_q|J_q]$$

$$\langle\alpha|\langle u_\ell(p)|P|\Psi\rangle \ , \qquad \text{for } L=0. \tag{127}$$

We can use this expression for $L\geq 1$, if the partial wave is not coupled. In general, however,

$$T_{\alpha,\beta}^{(1,B)} = -\langle j_L(qx)|V[\sum_{n=0}^{[\frac{L-1}{2}]}(G_{0,q}^{(2)}V)^n + (G_{0,q}^{(2)}V)^{[\frac{L-1}{2}]}G_{0,q}^{(2)}\{J_q$$

$$- J_q|w_q\rangle\frac{1}{1+\langle u_q|J_q|w_q\rangle}\langle u_q|J_q\}]\langle\alpha|\langle u_\ell(p)|P|\Psi\rangle \ , \quad \text{for } L\geq 1 \ . \tag{128}$$

The amplitude obtained from the decomposition (13) is not so neat as Eqs. (127) and (128). By this reason, we shall not consider Eq. (13) in the present paper any more. Also since essential aspects of our formulation becomes clear if we demonstrate it for Eq. (127), we shall not discuss Eq. (128) from now on.

We remark that the factor $F_a(q)$ of Eq. (112) is

$$F_a(q) = \frac{1}{1 + \langle u_q | J_q | w_q \rangle} \tag{129}$$

for Eq. (127). If we use Eq. (121), the first term on the right hand side of Eq. (127) reads

$$-\langle j_{L=0}(qx) | J_q \langle \alpha | \langle u_\ell(p) | P | \Psi \rangle = -\langle j_{L=0}(qx) | J_q \langle \alpha | \langle u_\ell(p) BP | \Phi \rangle$$

$$+ \sum_{\alpha'} \int' dp' \langle j_{L=0}(qx) | J_q \chi_{\alpha' q'}{}^{\alpha q}(x) \rangle V_{a'}(q'), \tag{130}$$

where we have used Eq. (119). The second term on the right hand side of Eq. (127) reads

$$\langle j_{L=0}(qx) | J_q | w_q \rangle F_a(q) \langle C_a(q) | \langle \alpha | \langle u_\ell(p) | P \Psi \rangle$$

$$= \langle j_{L=0}(q) | J_q | w_q \rangle V_a(q) \quad . \tag{131}$$

Here we have used the definition (114).

If we put Eqs. (130) and (131) into Eq. (127), we obtain

$$T^{(l,B)}_{\alpha,p} = -\langle j_{L=0}(qx) | J_q \langle \alpha | \langle u_\ell(p) | BP | \Phi \rangle + \sum_{\alpha'} \int' dp' \langle j_{L=0}(qx) | J_q \chi_{\alpha' q'}{}^{\alpha q}(x) \rangle$$

$$\times V_{a'}(q') + \langle j_{L=0}(q) | J_q | w_q \rangle V_a(q) \quad . \tag{132}$$

In conclusion, what we have to calculate are the functions $\langle \alpha | \langle u_\ell(p) | BP | \Phi \rangle$, $\chi_{\alpha' q'}{}^{\alpha q}(x)$ and $V_a(q)$. Here, the function $V_a(q)$ is calculated by Eq. (117), once the first two functions are calculated. As seen by Eqs. (119) and (120), two functions $\langle \alpha | \langle u_\ell(p) | BP | \Phi \rangle$ and $\chi_{\alpha' q'}{}^{\alpha q}(x)$ are of a similar structure. Therefore, from now on, we will concenrate ourselves to the treatment of the

function $\chi_{\alpha' q'}^{\alpha q}$.

B. Treatment of function $\chi_{\alpha' q'}^{\alpha q}$

If we make use of Eq. (115), the function $X_{\alpha' q'}$ defined by Eq. (120) is expressed as an integral equation

$$X_{\alpha q} = P|u_\ell(p)>|\alpha>|B_a(q)> + \sum_{\alpha''}\int' dp'' P|u_{\ell''}(p'')>|\alpha''>A_{\alpha''}(q'')<\alpha''|<u_{\ell''}(p'')|X_{\alpha q}> \tag{133}$$

Here, the operator $A_\alpha(q)$ and the function $|B_a(q)>$ is obtained by comparing Eqs. (112) and Eq. (125), as

$$A_a(q) = G_{0,q}^{(2)} J_q \tag{134}$$

and

$$|B_a(q)> = G_{0,q}^{(2)} J_q |w_q> \quad . \tag{135}$$

If we use the transformation coefficient $A_{\alpha_1 q_1}^{\alpha q}$ defined by Eq. (103), we obtain the integral equation for $\chi_{\alpha' q'}^{\alpha q}$ by utilizing Eqs. (119) and (133) as

$$\chi_{\alpha' q'}^{\alpha q} = A_{\alpha' p'}^{\alpha p}|B_{a'}(q')> + \sum_{\alpha''}\int' dp'' A_{\alpha'' p''}^{\alpha p} A_{\alpha''}(q'')\chi_{\alpha' q'}^{\alpha'' q''} \quad . \tag{136}$$

We solve Eq. (136) by the iterative procedure.

$$\chi_{\alpha' q'}^{\alpha q} = \sum_{m=0}^{\infty} \chi_{\alpha' q'}^{\alpha q(m)} \tag{137}$$

with

$$\chi_{\alpha' q'}^{\alpha q(0)} = A_{\alpha' p'}^{\alpha p}|B_{a'}(q')> \quad , \tag{138}$$

and

$$\chi_{\alpha' q'}^{\alpha q(m)} = \sum_{\alpha''}\int' dp'' A_{\alpha'' p''}^{\alpha p} A_{\alpha''}(q'')\chi_{\alpha' q'}^{\alpha'' q''(m-1)} \quad . \tag{139}$$

Equations (134), (135), (138) and (139) show that in the course of calculations, we have to perform integrations of the form (104). Besides, the function $F^{\alpha_1}(x)$ appeared in Eq. (104) always take the form of Eq. (106). By

comparing these equations, we see that the function $\Xi^{\alpha_1}(x')$ in Eq. (106) stands

$$\Xi^{\alpha_1}(x') = J_{q'}|w_{q'}> \qquad \text{for Eq. (138)}, \tag{140}$$

and

$$\Xi^{\alpha_1}(x') = J_{q'}\chi_{\alpha'q'}{}^{\alpha''q''(m-1)} \qquad \text{for Eq. (139)} . \tag{141}$$

Since both of these functions are short ranged, we can numerically calculate Eq. (108).

C. Calculation of $\chi_{\alpha'q'}{}^{\alpha q(m)}$

In this section, we demonstrate the method for calculations of Eq. (138) [Eq. (139)] with the function $B_{a'}(q')$ [the operator $A_{\alpha''}(q'')$] given by Eq. (134) [Eq. (135)].

(i) Calculation of Eq. (138)

The equation (138) is calculated by Eq. (108), making use of Eq. (140). From Eq. (17), we see immediately that the function $J_{q'}|w_{q'}>$ is a product of potential and the Jost solution $f_{q'}$,

$$J_{q'}|w_{q'}> = V\cdot f_{q'} \quad . \tag{142}$$

with

$$f_{q'} = w_{q'} + g_{q'}Vf_{q'} \quad . \tag{143}$$

The Jost solution is obtained by solving the Schrödinger equation for the potential V with the boundary condition that it behaves as $w_{q'}$ at a large distance from the origin. Since the Jost kernel $g_{q'}V$ is of the Volterra type, we obtain the solution numerically by simply integrating inwards.

(ii) Calculation of Eq. (139)

Eq. (139) is calculated by Eq. (108), if we make use of Eq. (141). Here, what we have to demonstrate is the method of calculating $J_{q'}\chi_{\alpha'q'}{}^{\alpha q(m-1)}$, when $\chi_{\alpha'q'}{}^{\alpha q(m-1)}$ is known. For this purpose, let us introduce a function $X_{\alpha'q'}{}^{\alpha q(m)}(x)$ defined by

$$J_{q'}\chi_{\alpha'q'}{}^{\alpha q(m-1)} = VX_{\alpha'q'}{}^{\alpha q(m)} \quad . \tag{144}$$

By Eq. (17) $X_{\alpha' q'}^{\alpha q(m)}$ satisfies the integral equation

$$X_{\alpha' q'}^{\alpha q(m)} = (1 + g_{q'} J_{q'}) \chi_{\alpha' q'}^{\alpha q(m-1)}$$

$$= \chi_{\alpha' q'}^{\alpha q(m-1)} + g_{q'} V X_{\alpha' q'}^{\alpha q(m)} \quad . \tag{145}$$

This integral equation is solved by a simple iterative procedure with the boundary condition that at a large distance from the origin the solution is given by $\chi_{\alpha' q'}^{\alpha q(m-1)}$. As in the case of the usual Jost solution the iteration converges for the usual class of potentials satisfying the condition (33).

D. Singularity of the operator $A_{\alpha'' p''}^{\alpha p}$

The operator $A_{\alpha'' p''}^{\alpha p}$ defined by Eq. (108) is singular for some values of p and p". The singularity is logarithmic and arises from the pincipal value integral in the last term of Eq. (108). It appears if

$$q^2 = \lambda_{max}^2 \qquad \text{or} \qquad q^2 = \lambda_{min}^2$$

$$\text{i.e. for} \quad p'' = p_s = \pm \left(\frac{p}{2} \pm \sqrt{\frac{mE}{h^2} - \frac{3}{4}p^2}\right) \quad . \tag{146}$$

In the neighbourhood of p_s the last term of Eq. (108) behaves like

$$\sim \ln \left| \frac{q^2 - \frac{p^2}{4} - p''^2 + pp''}{q^2 - \frac{p^2}{4} - p''^2 - pp''} \right| \tag{147}$$

For computation of $\chi_{\alpha' q'}^{\alpha q}$, from Eq. (136) we have to integrate such function over p". Integral over p" of a function which has logarithmic singularity like (147) is finite and so is the integral on the right hand side of (136).

Appendix A. Proof for regularization in Eq. (16)

In this section, we prove that Eq. (16) is regular at the origin for a partial wave ℓ. For simplicity, we suppress the suffix q and keep only the suffix ℓ.

The function w_ℓ behaves near the origin as

$$w_\ell \xrightarrow[x \to 0]{} \frac{(2\ell-1)!!}{x^{\ell+1}} \quad . \tag{A.1}$$

As a corollary, let us demonstrate that once the function w_ℓ is multiplied by G_0J, this behaves as

$$G_0Jw_\ell \xrightarrow[x\to 0]{} \frac{(2\ell-1)!!}{(2\ell+1)}\left(\frac{1}{2}+\frac{1}{2\ell-1}\right)\frac{1}{x^{\ell-1}} \quad . \tag{A.2}$$

Proof: With the explicit expression of G_0, we get

$$G_0Jw_\ell = h_\ell(x)\int_0^x j_\ell(x')Jw_\ell(x')x'^2dx' + j_\ell(x)\int_x^\infty h_\ell(x')Jw_\ell(x')x'^2dx'. \tag{A.3}$$

To see the behavior at small x, we divide the integral $\int_x^\infty$ into two parts $\int_x^{x_c}+\int_{x_c}^\infty$, where x_c is a sufficiently small fixed value. For small x, Eq. (A.3) behaves as

$$G_0Jw_\ell \xrightarrow[x\to 0]{} \frac{(2\ell-1)!!}{(2\ell+1)!!x^{\ell+1}}\int_0^x \frac{x'^\ell}{(2\ell+1)!!}\frac{(2\ell-1)!!}{x'^{\ell+1}}x'^2dx'$$

$$+\frac{x^\ell}{(2\ell+1)!!}\int_x^{x_c}\left[\frac{(2\ell-1)!!}{x'^{\ell+1}}\right]^2 x'^2dx' \tag{A.4}$$

(A.4) leads (A.2).

Now we observe that once the function w_ℓ that behaves as $x^{-(\ell+1)}$ is multiplied by the operator G_0J, the singularity is reduced to $x^{-(\ell-1)}$. Further if the function G_0Jw_ℓ is multiplied by G_0V, the singularity is further reduced to $x^{-(\ell-3)}$. In fact, we have

$$G_0VG_0Jw_\ell \xrightarrow[x\to 0]{} \frac{(2\ell-1)!!}{(2\ell+2)}^2\left(\frac{1}{2}+\frac{1}{2\ell-1}\right)\left(\frac{1}{4}+\frac{1}{2\ell-3}\right)\frac{1}{x^{\ell-3}} \tag{A.5}$$

In this manner, we see that Eq. (15)[Eq.(16)] is regular at the origin for the s-wave [the partial wave of $\ell\geqq 1$].

One of the authors (J.H.) thanks to the Ministry of Education of Japan for providing him the fellowship.

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