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著者	Nakamura Kohei, Suzuki Masayuki
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On the Character of Tsunami Waves Generated in a Sloped Region of the Ocean

By KÔHEI NAKAMURA AND MASAYUKI SUZUKI

Geophysical Institute, Tôhoku University, Sendai, Japan

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Abstract

A Cauchy-Poisson problem in the case where a rectangular shaped heap of water is put on the surface of a sea which has a linearly sloped bottom and connects shallow and deep seas of uniform depth, is solved to investigate size and shape of Tsunami waves generated at an oceanic sloping bottom.

The problem is restricted to a two-dimensional case and the linearized shallow water theory is used. The surface of a main heap of water that moves toward the shore tilts forward, while another heap of water which has a lesser height is propagated in the deep sea, with its surface tilting backward. In addition to the main heaps of water, multiply reflected waves from the margins of the sloped region can also be propagated away from the source area, making a kind of coda-waves of Tsunami.

Energy which is transported into the shallow sea is always less than that is transported into the deep sea. Ratio of the energies is computed for various slopes of the bottom.

From the computed amplitude spectrums it is seen that the waves which proceed toward the shore contain more short period components than the waves which proceed in the deep sea.

1. Introduction

From results of the refraction diagram method we know that many Tsunamis have their origins at bottoms of oceanic slopes. We also know from tide gauge records that a Tsunami often begins with a flood wave on the coasts near the source of the Tsunami, and that periods of nearby Tsunamis in Japan are comparatively shorter than those observed on the opposite side of the Pacific Ocean. These facts may partly be attributed to a mechanism of generation of a Tsunami and dispersive nature of the waves which becomes important when the Tsunami travels over a long distance.

Taking out of account of these effects, we intend to make clear some characters near the source of the Tsunami formed on the sea surface of an oceanic slope that separates shallow and deep seas of uniform depth. If we confine ourselves to this purpose alone, we have only to use the linearized shallow water theory and it is quite sufficient to start with a given heap of water on the sea surface over a sloped bottom.

In connection to our problem, K. YOSHIDA [1] and R. TAKAHASHI [2] have computed the coefficients of transmission and reflection of long waves when they are incident on an oceanic slope. T. RIKITAKE [3] has investigated on the change of a wave form when a rectangular shaped heap of water is propagated in a sea with sloped bottom.

2. Formal Solution

The assumed bottom topography is shown in Fig. 1, where the domain II corresponds

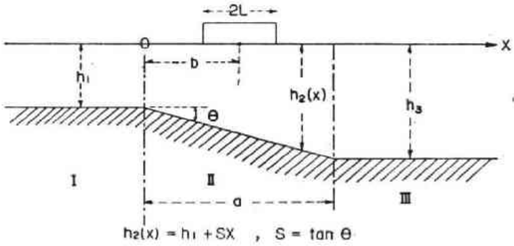

Fig. 1 Bottom topography.

to an oceanic slope with linearly changing depth, and I and III represent the domains of uniform depth. The x-axis is taken on the water surface in the direction of steepest drop with the origin on the marginal line of the shallow sea. The z-axis is taken vertically upward. Depths of domains I and III are denoted by h_1 and h_3, respectively, and depth in domain II is designated by $h_2(x)=h_2+sx$, $s=\tan\theta$, where θ is the angle of inclination and a represents the horizontal extent of the slope. The motion is supposed unchanged in the direction of y.

Let us suppose that at origin of the time $t=0$, a rectangular shaped elevation of water with unit height and length $2L$ is put without an initial velocity on the sea surface of domain II. Our problem is to solve the wave motion in domain I or III.

Let ζ and v signify elevation and velocity of long waves, respectively. The equation of motion in the sea of uniform depth is

$$\frac{\partial^2 \zeta_i}{\partial t^2} = v_i^2 \frac{\partial^2 \zeta_i}{\partial x^2}, \qquad i = 1, 3, \tag{1}$$

where the suffices 1 and 3 represent the domains I and III respectively.

Assuming that initial elevation of water particles are both zero in each domain, we obtain Laplace-transformed equation of (1)

$$\frac{\partial^2 \bar{\zeta}_i}{\partial x^2} = \frac{p^2}{v_i^2} \bar{\zeta}_i, \qquad i = 1, 3, \tag{1'}$$

where the superimposed bar represents the Laplace-transformed quantity defined by

$$\bar{\zeta}_i = \int_0^\infty \zeta_i e^{-pt} dt, \qquad p > 0.$$

We take the solutions of (1′)

$$\bar{\zeta}_1 = A_1 e^{-q_1 x}, \qquad \bar{\zeta}_3 = A_3 e^{-q_3 (x-a)}, \tag{2}$$

where

$$q_1 = \frac{p}{v_1}, \qquad q_3 = \frac{p}{v_3}. \tag{3}$$

In the domain II, the equation of motion can be written

$$\frac{\partial^2 \zeta_2}{\partial t^2} = g \frac{\partial}{\partial x} \left[h_2(x) \frac{\partial \zeta_2}{\partial x} \right]. \tag{4}$$

Putting

$$x + \frac{h_1}{s} = z, \tag{5}$$

we have

$$\frac{\partial^2 \zeta_2}{\partial t^2} = g\, s \left[z \frac{\partial^2 \zeta_2}{\partial z^2} + \frac{\partial \zeta_2}{\partial z} \right]. \tag{6}$$

When we assume that the initial elevation of water surface is expressed by $f(x)$ and the initial velocity of a water particle is everywhere zero, we can write

$$\left.\begin{aligned} \dot{\zeta}_{2,0} &= \left(\frac{\partial \zeta_2}{\partial t} \right)_{t=0} = 0, \\ \zeta_{2,0} &= (\zeta_2)_{t=0} = f(x), \end{aligned}\right\} \tag{7}$$

and the transformed equation becomes

$$\frac{\partial^2 \bar{\zeta}_2}{\partial z^2} + \frac{1}{z} \frac{\partial \bar{\zeta}_2}{\partial z} - \frac{p^2}{g\, s\, z} \bar{\zeta}_2 = - \frac{p}{g\, s\, z} f(x). \tag{6'}$$

We take the solution of (6')

$$\bar{\zeta}_2 = A_2 J_0(k\sqrt{z}) + B_2 Y_0(k\sqrt{z}), \tag{8}$$

where

$$k = \frac{2\, p\, i}{\sqrt{g\, s}}, \tag{9}$$

and J_0 and Y_0 are Bessel and Neumann functions of zeroth order.

$A_2(x)$ and $B_2(x)$ are functions which we want to determine by the method of variation of constants.

Using the relation

$$A_2' J_0(k\sqrt{z}) + B_2' Y_0(k\sqrt{z}) = 0, \tag{10}$$

where the prime means differentiation with respect to x, (7) can be written

$$A_2' J_1(k\sqrt{z}) + B_2' Y_1(k\sqrt{z}) = \frac{2p}{g\, s\, k\sqrt{z}} f(x). \tag{11}$$

From (10) and (11) we obtain

$$\left.\begin{aligned} A_2' &= \frac{\pi p}{g\, s} f(x)\, Y_0(k\sqrt{z}), \\ B_2' &= -\frac{\pi p}{g\, s} f(x)\, J_0(k\sqrt{z}). \end{aligned}\right\} \tag{12}$$

At the margins of the slope, conservations of momentum and mass require continuities in ζ and $h(\partial\zeta/\partial x)$, respectively. Thus we have four boundary conditions

$$\left.\begin{aligned}
&1)\quad \bar{\zeta}_1 = \bar{\zeta}_2\,, && \text{at}\quad x = 0\,,\\
&2)\quad \frac{\partial \bar{\zeta}_1}{\partial x} = \frac{\partial \bar{\zeta}_2}{\partial x}\,, && \text{at}\quad x = 0\,,\\
&3)\quad \bar{\zeta}_2 = \bar{\zeta}_3\,, && \text{at}\quad x = a\,,\\
&4)\quad \frac{\partial \bar{\zeta}_2}{\partial x} = \frac{\partial \bar{\zeta}_3}{\partial x}\,, && \text{at}\quad x = a\,.
\end{aligned}\right\} \tag{13}$$

These conditions can be written using (2) and (8)

$$\left.\begin{aligned}
&A_2(0)\, J_0(i\,m_1) + B_2(0)\, Y_0(i\,m_1) = A_1\,,\\
&A_2(0)\, J_1(i\,m_1) + B_2(0)\, Y_1(i\,m_1) = i\,A_1\,,\\
&A_2(a)\, J_0(i\,m_3) + B_2(a)\, Y_0(i\,m_3) = A_3\,,\\
&A_2(a)\, J_1(i\,m_3) + B_2(a)\, Y_1(i\,m_3) = -\,i\,A_3\,,
\end{aligned}\right\} \tag{14}$$

where

$$m_1 = \frac{2\,p}{g\,s}\, v_1\,, \qquad m_3 = \frac{2\,p}{g\,s}\, v_3\,. \tag{15}$$

Solving the simultaneous equations (14) for $A_2(0)$, $B_2(0)$, $A_2(a)$ and $B_2(a)$, we have

$$\left.\begin{aligned}
&A_2(0) = -\frac{\pi\, i\, m_1}{2}\, A_1 \left\{ Y_1(i\,m_1) - i\, Y_0(i\,m_1) \right\},\\
&B_2(0) = \frac{\pi\, i\, m_1}{2}\, A_1 \left\{ J_1(i\,m_1) - i\, J_0(i\,m_1) \right\},\\
&A_2(a) = -\frac{\pi\, i\, m_3}{2}\, A_3 \left\{ Y_1(i\,m_3) + i\, Y_0(i\,m_3) \right\},\\
&B_2(a) = \frac{\pi\, i\, m_3}{2}\, A_3 \left\{ J_1(i\,m_3) + i\, J_0(i\,m_3) \right\}.
\end{aligned}\right\} \tag{16}$$

On the other hand by integrating (12) from $x=0$ to $x=a$,

$$\left.\begin{aligned}
&A_2(a) = \frac{\pi p}{g\,s} \int_0^a f(\xi)\, Y_0(k\sqrt{\xi + h_1/s})\, d\xi + A_2(0)\,,\\
&B_2(a) = -\frac{\pi p}{g\,s} \int_0^a f(\xi)\, J_0(k\sqrt{\xi + h_1/s})\, d\xi + B_2(0)\,,
\end{aligned}\right\} \tag{17}$$

which become by using the relation (16)

$$\left.\begin{aligned}
&A_3 v_3 \left\{ Y_1(i\,m_3) + i\, Y_0(i\,m_3) \right\} - A_1 v_1 \left\{ Y_1(i\,m_1) - i\, Y_0(i\,m_1) \right\}\\
&\qquad\qquad = i \int_0^a f(\xi)\, Y_0(k\sqrt{\xi + h_1/s})\, d\xi\,,\\
&-A_3 v_3 \left\{ J_1(i\,m_3) + i\, J_0(i\,m_3) \right\} + A_1 v_1 \left\{ J_1(i\,m_1) - i\, J_0(i\,m_1) \right\}\\
&\qquad\qquad = -\,i \int_0^a f(\xi)\, J_0(k\sqrt{\xi + h_1/s})\, d\xi\,.
\end{aligned}\right\} \tag{18}$$

From (18), A_1 and A_3 are obtained

$$\left.\begin{aligned} A_1 &= -\frac{i}{v_1}\,[M\,\{Y_1(i\,m_3)+i\,Y_0(i\,m_3)\}-N\,\{J_1(i\,m_3)+i\,J_0(i\,m_3)\}]/\Delta\,,\\ A_3 &= \frac{i}{v_3}\,[N\,\{J_1(i\,m_1)-i\,J_0(i\,m_1)\}-M\,\{Y_1(i\,m_1)-i\,Y_0(i\,m_1)\}]/\Delta\,, \end{aligned}\right\} \tag{19}$$

where

$$\begin{aligned} \Delta = \{Y_1(i\,m_3)+i\,Y_0(i\,m_3)\}\,\{J_1(i\,m_1)-i\,J_0(i\,m_1)\}\\ -\{Y_1(i\,m_1)-i\,Y_0(i\,m_1)\}\,\{J_1(i\,m_3)+i\,J_0(i\,m_3)\}, \end{aligned} \tag{20}$$

$$\left.\begin{aligned} M &= \int_0^a f(\xi)\,J_0(k\sqrt{\xi+h_1/s})\,d\,\xi\,,\\ N &= \int_0^a f(\xi)\,Y_0(k\sqrt{\xi+h_1/s})\,d\,\xi\,. \end{aligned}\right\} \tag{21}$$

3. Rectangular Shaped Elevation of Water

In the case of a rectangular shaped elevation which is centered at $x=b$, and has a unit height and length $2L$, we can write

$$\left.\begin{aligned} f(x) &= 1\,, & |x-b| &< L\,,\\ &\;\;0\,, & |x-b| &> L\,, \end{aligned}\right\} \tag{22}$$

so that (21) becomes

$$\left.\begin{aligned} M &= \int_{b-L}^{b+L} J_0(k\sqrt{\xi+h_1/s})\,d\,\xi = \frac{2}{k}\,\{\kappa_3\,J_1(i\,\theta_3)-\kappa_1\,J_1(i\,\theta_1)\}\,,\\ N &= \int_{b-L}^{b+L} Y_0(k\sqrt{\xi+h_1/s})\,d\,\xi = \frac{2}{k}\,\{\kappa_3\,Y_1(i\,\theta_3)-\kappa_1\,Y_1(i\,\theta_1)\}\,, \end{aligned}\right\} \tag{23}$$

where

$$\kappa_3 = \sqrt{b+L+h_1/s}\,, \qquad \kappa_1 = \sqrt{b-L+h_1/s}\,, \tag{24}$$

$$\theta_3 = \frac{2\,p}{\sqrt{g\,s}}\,\kappa_3\,, \qquad \theta_1 = \frac{2\,p}{\sqrt{g\,s}}\,\kappa_1\,. \tag{25}$$

Substituting (23) into (19) gives

$$A_1 = -\frac{i}{v_1}\,\frac{2}{k}\,\frac{X_1(p)}{\Delta(p)}\,, \qquad A_3 = \frac{i}{v_3}\,\frac{2}{k}\,\frac{X_3(p)}{\Delta(p)}\,, \tag{26}$$

where

$$\left.\begin{aligned} X_1(p) &= \{\kappa_3\,J_1(i\,\theta_3)-\kappa_1\,J_1(i\,\theta_1)\}\,\{Y_1(i\,m_3)+i\,Y_0(i\,m_3)\}\\ &\quad -\{\kappa_3\,Y_1(i\,\theta_3)-\kappa_1\,Y_1(i\,\theta_1)\}\,\{J_1(i\,m_3)+i\,J_0(i\,m_3)\}\,,\\ X_3(p) &= \{\kappa_3\,Y_1(i\,\theta_3)-K_1\,Y_1(i\,\theta_1)\}\,\{J_1(i\,m_1)-i\,J_0(i\,m_1)\}\\ &\quad -\{\kappa_3\,J_1(i\,\theta_3)-\kappa_1\,J_1(i\,\theta_1)\}\,\{Y_1(i\,m_1)-i\,Y_0(i\,m_1)\}\,. \end{aligned}\right\} \tag{27}$$

On substituting (26) into (2), we have the inverse transformations

$$\left.\begin{aligned} \zeta_1 &= \frac{1}{2\pi i}\left(-\frac{\sqrt{gs}}{v_1}\right)\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda\{t+(x/v_1)\}}\frac{1}{\lambda}\frac{X_1(\lambda)}{\Delta(\lambda)}\,d\lambda\,, \\ \zeta_3 &= \frac{1}{2\pi i}\left(\frac{\sqrt{gs}}{v_3}\right)\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda\{t-(x-a/v_3)\}}\frac{1}{\lambda}\frac{X_3(\lambda)}{\Delta(\lambda)}\,d\lambda\,. \end{aligned}\right\} \tag{28}$$

Using the asymptotic expansions

$$\left.\begin{aligned} J_0(iz) &\doteqdot \frac{1}{\sqrt{2\pi z}}\left[e^z A(z) - i e^{-z} B(z)\right], \\ J_1(iz) &\doteqdot \frac{1}{\sqrt{2\pi z}}\left[i e^z C(z) - e^{-z} D(z)\right], \\ Y_0(iz) &\doteqdot \frac{1}{\sqrt{2\pi z}}\left[i e^z A(z) - e^{-z} B(z)\right], \\ Y_1(iz) &\doteqdot \frac{1}{\sqrt{2\pi z}}\left[- e^z C(z) + i e^{-z} D(z)\right], \end{aligned}\right\} \tag{29}$$

where

$$\left.\begin{aligned} A(z) &= 1 + \frac{1}{8z} + \frac{9}{128z^2} + \cdots\cdots, \\ B(z) &= 1 - \frac{1}{8z} + \frac{9}{128z^2} + \cdots\cdots, \\ C(z) &= 1 - \frac{3}{8z} - \frac{15}{128z^2} + \cdots\cdots, \\ D(z) &= 1 + \frac{3}{8z} - \frac{15}{128z^2} + \cdots\cdots, \end{aligned}\right\} \tag{30}$$

the terms $\frac{X_1(p)}{\Delta(p)}$ and $\frac{X_3(p)}{\Delta(p)}$ are expressed as follows;

$$\begin{aligned} \frac{X_1}{\Delta} = \sqrt{m_1}\Big[&\frac{1}{D(m_1)+B(m_1)}\left\{\left(\frac{\kappa_3 D(\theta_3)}{\sqrt{\theta_3}}\right)e^{m_1-\theta_3} - \left(\frac{\kappa_1 D(\theta_1)}{\sqrt{\theta_1}}\right)e^{m_1-\theta_1}\right\} \\ &+ Q_1\left\{\left(-\frac{\kappa_3 C(\theta_3)}{\sqrt{\theta_3}}\right)e^{\theta_3+m_1-2m_3} + \left(\frac{\kappa_1 C(\theta_1)}{\sqrt{\theta_1}}\right)e^{\theta_1+m_1-2m_3}\right\}\Big] \\ &\times\left[1 - Z e^{2(m_1-m_3)}\right]^{-1}, \end{aligned} \tag{31}$$

$$\frac{X_3}{\Delta} = \sqrt{m_3}\Big[Q_3\left\{\left(-\frac{\kappa_3 D(\theta_3)}{\sqrt{\theta_3}}\right)e^{2m_1-\theta_3-m_3} + \left(\frac{\kappa_1 D(\theta_1)}{\sqrt{\theta_1}}\right)e^{2m_1-\theta_1-m_3}\right\}$$

$$+ \frac{1}{A\,(m_3)+C\,(m_3)} \left\{ \left(\frac{\kappa_3\, C\,(\theta_3)}{\sqrt{\theta_3}} \right) e^{\theta_3 - m_3} + \left(- \frac{\kappa_1\, C\,(\theta_1)}{\sqrt{\theta_1}} \right) e^{\theta_1 - m_3} \right\} \Bigg]$$

$$\times \left[1 - Z e^{2(m_1 - m_3)}\right]^{-1}, \tag{32}$$

where

$$Z = \frac{\{A\,(m_1) - C\,(m_1)\}\,\{B\,(m_3) - D\,(m_3)\}}{\{A\,(m_3) + C\,(m_3)\}\,\{B\,(m_1) + D\,(m_1)\}}\,, \tag{33}$$

$$\left.\begin{aligned} Q_1 &= \frac{D\,(m_3) - B\,(m_3)}{\{A\,(m_3) + C\,(m_3)\}\,\{B\,(m_1) + D\,(m_1)\}}\,, \\ Q_3 &= \frac{C\,(m_1) - A\,(m_1)}{\{A\,(m_3) + C\,(m_3)\}\,\{B\,(m_1) + D\,(m_1)\}}\,. \end{aligned}\right\} \tag{34}$$

(31) and (32) can be rewritten as follows;

$$\frac{X_1\,(p)}{\varDelta\,(p)} \doteqdot \sqrt{\frac{h_1}{s}} \left[\left(\frac{H_3}{h_1}\right)^{1/4} D_1\, e^{2p(t_1 - T_3)} - \left(\frac{H_1}{h_1}\right)^{1/4} E_1\, e^{2p(t_1 - T_1)} \right.$$

$$\left. - \left(\frac{H_3}{h_1}\right)^{1/4} F_1\, e^{2p(t_1 + T_3 - 2t_3)} + \left(\frac{H_1}{h_1}\right)^{1/4} G_1\, e^{2p(t_1 + T_1 - 2t_3)} \right]$$

$$\times \left[1 - Z\, e^{4p(t_1 - t_3)}\right]^{-1}, \tag{35}$$

where

$$\left.\begin{aligned} D_1 &= \frac{1}{2}\left[1 + \frac{1}{16}\left(\frac{3}{T_3} - \frac{1}{t_1}\right)\frac{1}{p} + \cdots\cdots\right], \\ E_1 &= \frac{1}{2}\left[1 + \frac{1}{16}\left(\frac{3}{T_1} - \frac{1}{t_1}\right)\frac{1}{p} + \cdots\cdots\right], \\ F_1 &= \frac{1}{16\, t_3\, p}\left[1 - \left(\frac{1}{8\, t_3} + \frac{1}{16\, t_1} + \frac{3}{16\, T_3}\right)\frac{1}{p} + \cdots\cdots\right], \\ G_1 &= \frac{1}{16\, t_3\, p}\left[1 - \left(\frac{1}{8\, t_3} + \frac{1}{16\, t_1} + \frac{3}{16\, T_1}\right)\frac{1}{p} + \cdots\cdots\right], \end{aligned}\right\} \tag{36}$$

and

$$\frac{X_3\,(p)}{\varDelta\,(p)} \doteqdot \sqrt{\frac{h_3}{s}} \left[\left(\frac{H_3}{h_3}\right)^{1/4} D_3\, e^{2p(T_3 - t_3)} - \left(\frac{H_1}{h_3}\right)^{1/4} E_3\, e^{2p(T_1 - t_3)} \right.$$

$$\left. - \left(\frac{H_3}{h_3}\right)^{1/4} F_3\, e^{2p(2t_1 - T_3 - t_3)} + \left(\frac{H_1}{h_3}\right)^{1/4} G_3\, e^{2p(2t_1 - T_1 - t_3)} \right]$$

$$\times \left[1 - Z\, e^{4p(t_1 - t_3)}\right]^{-1}, \tag{37}$$

$$\left.\begin{aligned}
D_3 &= \frac{1}{2}\left[1+\frac{1}{16}\left(\frac{1}{t_3}-\frac{3}{T_3}\right)\frac{1}{p}+\cdots\cdots\right],\\
E_3 &= \frac{1}{2}\left[1+\frac{1}{16}\left(\frac{1}{t_3}-\frac{3}{T_1}\right)\frac{1}{p}+\cdots\cdots\right],\\
F_3 &= -\frac{1}{16t_1p}\left[1+\left(\frac{1}{8t_1}+\frac{1}{16t_3}+\frac{3}{16T_3}\right)\frac{1}{p}+\cdots\cdots\right],\\
G_3 &= -\frac{1}{16t_1p}\left[1+\left(\frac{1}{8t_1}+\frac{1}{16t_3}+\frac{3}{16T_1}\right)\frac{1}{p}+\cdots\cdots\right].
\end{aligned}\right\} \tag{38}$$

To derive (35)–(38), we have made use of the following relations

$$\left.\begin{aligned}
m_1 &= \frac{2p}{gs}v_1 = 2p\left(\frac{l_1}{v_1}\right) = 2pt_1,\\
m_3 &= \frac{2p}{gs}v_3 = 2p\left(\frac{l_3}{v_3}\right) = 2pt_3,\\
\theta_1 &= \frac{2p}{\sqrt{gs}}\sqrt{b-L+h_1/s} = 2p\left(\frac{L_1}{\sqrt{gH_1}}\right) = 2pT_1,\\
\theta_3 &= \frac{2p}{\sqrt{gs}}\sqrt{b+L+h_1/s} = 2p\left(\frac{L_3}{\sqrt{gH_3}}\right) = 2pT_3.
\end{aligned}\right\} \tag{39}$$

In (39), l_1, l_3, L_1 and L_3 are distances indicated in Fig. 2. In this figure, S is the point of intersection of the water surface with a line indicating the sloped bottom, and H_1 and H_3 represent the depths at end points of the initial elevation.

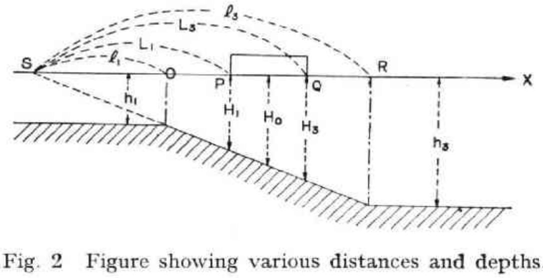

Fig. 2 Figure showing various distances and depths.

Substituting (35)–(38) into (28), and then performing the integrations, we obtain the expressions for elevation

$$\zeta_1 = \frac{1}{2}\left[\left(\frac{H_1}{h_1}\right)^{1/4} H\left\{t-\left(-\frac{x}{v_1}+2(T_1-t_1)\right)\right\}\left\{1+\frac{1}{16}\left(\frac{3}{T_1}-\frac{1}{t_1}\right)\right.\right.$$
$$\left.\times\left(t+\frac{x}{v_1}+2(t_1-T_1)\right)+\cdots\cdots\right\}$$

$$-\left(\frac{H_3}{h_1}\right)^{1/4} H\left\{t-\left(-\frac{x}{v_1}+2\left(T_3-t_1\right)\right)\right\}\left\{1+\frac{1}{16}\left(\frac{3}{T_3}-\frac{1}{t_1}\right)\right.$$

$$\left.\times\left(t+\frac{x}{v_1}+2\left(t_1-T_3\right)\right)+\cdots\cdots\right\}$$

$$+\left(\frac{H_3}{h_1}\right)^{1/4} H\left\{t-\left(-\frac{x}{v_1}+2\left(2t_3-t_1-T_3\right)\right)\right\}$$

$$\times\left\{\frac{1}{8t_3}\left(t+\frac{x}{v_1}+2\left(t_1+T_3-2t_3\right)\right)+\cdots\cdots\right\}$$

$$-\left(\frac{H_1}{h_1}\right)^{1/4} H\left\{t-\left(-\frac{x}{v_1}+2\left(2t_3-t_1-T_1\right)\right)\right\}$$

$$\left.\times\left\{\frac{1}{8t_3}\left(t+\frac{x}{v_1}+2\left(t_1+T_1-2t_3\right)\right)+\cdots\cdots\right\}\right] \tag{40}$$

+ multiply reflected waves,

$$\zeta_3=\frac{1}{2}\left[\left(\frac{H_3}{h_3}\right)^{1/4} H\left\{t-\left(\frac{x-a}{v_3}+2\left(t_3-T_3\right)\right)\right\}\left\{1+\frac{1}{16}\left(\frac{1}{t_3}-\frac{3}{T_3}\right)\right.\right.$$

$$\left.\times\left(t-\frac{x-a}{v_3}+2\left(T_3-t_3\right)\right)+\cdots\cdots\right\}$$

$$-\left(\frac{H_1}{h_3}\right)^{1/4} H\left\{t-\left(\frac{x-a}{v_3}+2\left(t_3-T_1\right)\right)\right\}\left\{1+\frac{1}{16}\left(\frac{1}{t_3}-\frac{3}{T_1}\right)\right.$$

$$\left.\times\left(t-\frac{x-a}{v_3}+2\left(T_1-t_3\right)\right)+\cdots\cdots\right\}$$

$$+\left(\frac{H_3}{h_3}\right)^{1/4} H\left\{t-\left(\frac{x-a}{v_3}+2\left(t_3+T_3-2t_1\right)\right)\right\}$$

$$\times\left\{\frac{1}{8t_1}\left(t-\frac{x-a}{v_3}+2\left(2t_1-T_3-t_3\right)\right)+\cdots\cdots\right\}$$

$$-\left(\frac{H_1}{h_3}\right)^{1/4} H\left\{t-\left(\frac{x-a}{v_3}+2\left(t_3+T_1-2t_1\right)\right)\right\}$$

$$\left.\times\left\{\frac{1}{8t_1}\left(t-\frac{x-a}{v_3}+2\left(2t_1-T_1-t_3\right)\right)+\cdots\cdots\right\}\right] \tag{41}$$

+ multiply reflected waves.

In (40) and (41), $H\{t\}$ means the Heaviside's unit-step function defined by

$$\left.\begin{aligned} H\{t\} &= 1\,, \qquad t \geqq 0\,, \\ &= 0\,, \qquad t < 0\,. \end{aligned}\right\} \tag{42}$$

We can see that respective members of (40) and (41) correspond to waves that have different paths. For instance, the factor $2(T_1-t_1)$ in the first member of (40) represents the travel time τ for the distance PO (Fig. 2). This can be shown as follows;

$$\tau = -\int_{L_1-l_1}^{0} \frac{dx}{\sqrt{g h_2(x)}} = \int_{0}^{L_1-l_1} \frac{dx}{\sqrt{g(h_1+sx)}} = \left[\frac{2\sqrt{g(h_1+sx)}}{gs}\right]_{0}^{L_1-l_1}$$

$$= 2\left[\frac{L_1}{\sqrt{gH_1}} - \frac{l_1}{\sqrt{gh_1}}\right] = 2(T_1 - t_1)\,.$$

Thus we can identify the first member of (40) as the wave which proceeds from P in the negative direction of x-axis. Similarly the second member represents the wave which is propagated in the same direction starting from Q. The third and fourth members correspond to the waves which proceed from Q and P respectively to the deeper sea and are reflected back to the shallow sea from R. The multiply reflected waves mean the waves which are propagated in the negative direction of x-axis after being reflected more than once respectively at O and R.

Quite similarly, respective members in (41) represent waves which proceed in the direction of x-axis after travelling respective paths that are different from each other. The first two members represent the direct waves from Q and P respectively, and the next two members correspond to the reflected waves from O. The multiply reflected waves are suffered reflections more than once respectively at O and R.

The factors such as $(H_1/h_1)^{1/4}$ contained in each member of (40) and (41)show change in amplitude according to the Green's law.

It should be noted that the third and fourth members in (40) and (41) are approximate solutions which hold only in the early stage of each arrival.

4. Numerical Examples

(1) Wave forms

For prescribed distances $x=-100$km and $x-a=100$km, the wave forms in the shallow and deep seas are calculated taking the following values

$$\left.\begin{array}{l} h_1 = 1\text{ km}\,, \quad h_3 = 6\text{ km}\,, \quad a = l_3 - l_1 = 50\text{ km} \\ L_1 - l_1 = 10\text{ km}\,, \quad 2L = 30\text{ km}\,, \quad (s = 0.1) \end{array}\right\} \qquad (43)$$

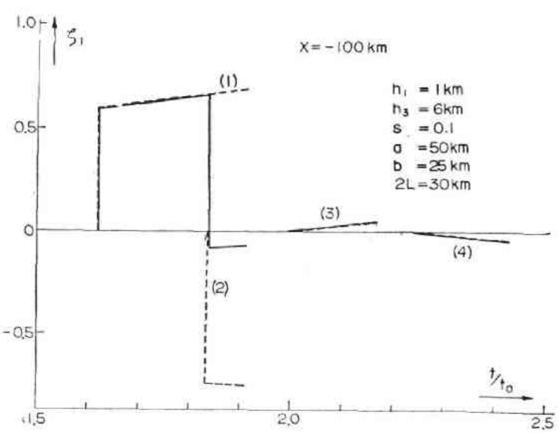

Fig. 3 Wave form in the shallow sea. Solid lines indicate sum of component waves.

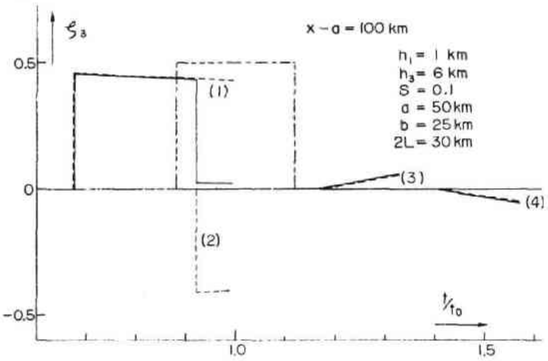

Fig. 4 Wave form in the deep sea. Solid lines indicate sum of component waves.

and they are shown in Figs. 3 and 4.

In these figures, the abscissa represents the ratio t/t_0, where t_0 is the time required for the long wave in the sea of uniform depth H_0 (Fig. 2) to travel from the center of the initial heap of water to the observation point. We note that, in case of the constants given in (43), the initial heap of water is centered at the middle point of the sloped region. The numerals attached to the broken lines in Figs. 3 and 4 indicate the contributions of the respective members. The thick solid curves show the sum of first four members. The multiply reflected waves are omitted owing to their small amplitude.

As we have mentioned earlier, the curves 3 and 4 are approximate representation of the initial stage of the waves, so that we can infer from these curves only an arrival of blunt and small amplitude waves.

In contrast to these reflected waves, main heap of water formed by the curves 1 and 2 is very sharp in form and has a much greater height. For the sake of comparison, a wave form which should be observed at $x=-100$km or $x=150$km, when the initial heap of water with the same size and shape as given in (22) is put on the surface of the sea of uniform depth $H_0(=3.5\text{ km})$, is shown by the chained lines in Fig. 4.

In the shallow sea (Fig. 3), the width of the main heap becomes narrower and its height becomes greater as compared with the case of uniform depth H_0. The surface of the heap tilts forward. On the contrary, in the deep sea (Fig. 4), the width broadens, the wave height becomes smaller and the surface of the heap tilts backward.

(2) Energy partition

Total energy of the main heap of water contained in a unit length in the y-direction is expressed by

$$E_i = \rho g \int \zeta_i^2 \, dx, \qquad i = 1, 3, \tag{44}$$

where the integral is extended over the width of the water heap. Fig. 5 shows variation of the ratio E_1/E_3 when the slope is changed, other constants remaining the same as

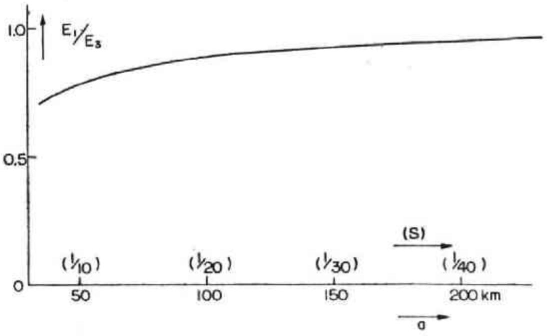

Fig. 5 Ratio of the energy E_1 contained in a wave in the shallow sea to the energy E_3 contained in a wave in the deep sea.

those given in (43). It is evident from Fig. 5 that energy partition is controlled by the magnitude of the slope.

(3) Amplitude spectrum

The Fourier integral representation of the initial elevation expressed by (22) is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\frac{2}{k} \sin (k L) \right] d k \,, \tag{45}$$

where

$$k = \frac{\omega}{\sqrt{g H_0}} \,. \tag{46}$$

Since we know that, in case of a sea of uniform depth, heaps of water with similar form but half the height of the initial elevation are propagated in both directions, the amplitude spectrum F_0 of the wave form (22) in a sea of uniform depth H_0 (Fig. 2) can be expressed by

$$F_0 = \frac{1}{2} \frac{\sin (k L)}{k L} \,. \tag{47}$$

To obtain the amplitude spectrums in the sea of uniform depth h_1 and h_3, we first replace λ by $-iw$ in (28) and using the relations

$$\left.\begin{aligned} k_1 &= \frac{\omega}{v_1} = \frac{v_0}{v_1} k \,, \\ k_3 &= \frac{\omega}{v_3} = \frac{v_0}{v_3} k \,, \qquad v_0 = \sqrt{g H_0} \,, \end{aligned}\right\} \tag{48}$$

rewrite the expressions in (28) as follows,

$$\left.\begin{aligned} \zeta_1 &= \frac{1}{2 \pi i} \left(- \frac{\sqrt{g s}}{v_1} \right) \int_{-\infty - i\delta}^{\infty - i\delta} e^{-i\omega(t+(x/v_1))} \frac{1}{\omega} \frac{X_1(\omega)}{\Delta(\omega)} d\omega \,, \\ \zeta_3 &= \frac{1}{2 \pi i} \left(\frac{\sqrt{g s}}{v_3} \right) \int_{-\infty - i\delta}^{\infty - i\delta} e^{-i\omega(t-(x-a/v_3))} \frac{1}{\omega} \frac{X_3(\omega)}{\Delta(\omega)} d\omega \,. \end{aligned}\right\} \tag{49}$$

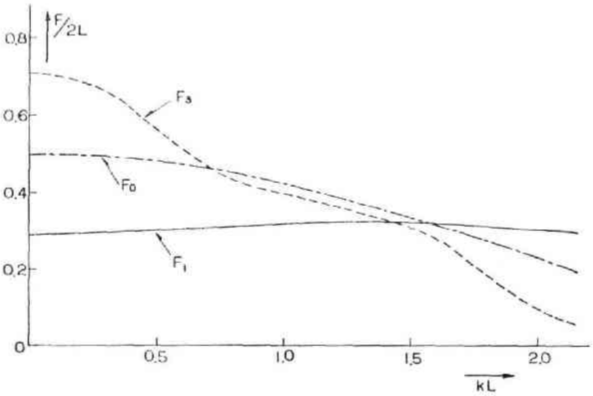

Fig. 6 Amplitude spectrums of original wave (F_0), of a wave in the shallow sea (F_1), and of a wave in the deep sea (F_3).

Thus we have the amplitude spectrums of waves in the domains I and III

$$F_1 = \sqrt{gs}\left(\frac{L}{v_1}\right)\frac{1}{k_1 L}\frac{X_1\{k_1 L(v_1/L)\}}{\Delta\{k_1 L(v_1/L)\}}, \tag{50}$$

$$F_3 = \sqrt{gs}\left(\frac{L}{v_3}\right)\frac{1}{k_3 L}\frac{X_3\{k_3 L(v_3/L)\}}{\Delta\{k_3 L(v_3/L)\}}. \tag{51}$$

Fig. 6 shows F_1, F_3 and F_0 as functions of kL. In this figure we have taken a limited range of kL in which the long wave approximation holds.

It is made clear that the waves in the shallow sea contain more short period components than the waves in the deep sea.

5. Concluding Remarks

As shown in Fig. 3, the height of the main heap of water in the shallow sea becomes greater as compared with the case of uniform depth H_0 which indicates the depth beneath the middle point of the source, and the surface of the main heap tilts forward though it is flat in the sea of uniform depth H_0. From these results it may be inferred that a Tsunami generated on a sloped bottom would have a larger height and a forward steeper front than the case of a horizontal sea bed.

It can be easily seen that, if a rectangular shaped depression of water is given initially, size and shape of main heaps of water are the same with the case of a given initial elevation except for the sign. Thus we can see that the original form of a water heap is deformed by the existence of a slope such that the features of flood and ebb of the initial wave become more pronounced. Of course, it must be noticed that degree of the slope of the surface of water heaps and growth or decay of the wave heights depend on the magnitude of a slope of the sea bed.

Partition of energy represented in Fig. 5 shows that more energy is transported into the deep sea than into the shallow sea. It may be likely that coasts not near to a tsunami-source suffer severe damage if a tsunami proceeding to deeper sea is refracted back to the shore owing to adequate topography of sea bed. It may also be possible that coasts situated at the opposite side of the ocean encounter a considerable height of tsunami waves.

Comparison of the spectral components shown in Fig. 6 will make it clear that the waves which proceed toward the shore contain more short period components than the waves that are propagated in the deep sea. This result is in accord with the fact that the period of the Tsunami of March 3, 1933 observed along the Sanriku Coast, Japan was about 15 minutes, while on the Pacific Coast of U.S.A. it was about 30 minutes, and also the fact that, in case of Chile Tsunami of May 24, 1960, the period was 30–40 minutes on the Chilean coast, and about 80 minutes on the Sanriku Coast. Of course, as mentioned in the Introduction, those facts must partly be ascribed to dispersive nature of waves over the Pacific Ocean.

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