Motion of Water due to Long Waves in a Rectangular Bay of Uniform Depth

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Motion of Water due to Long Waves in a Rectangular Bay of Uniform Depth

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Abstract

The wave height at the head of a rectangular bay of uniform depth is investigated, when a packet of sinusoidal long waves is incident upon the bay mouth. It is assumed that no energy is dissipated in the form of diverging waves from the bay mouth, and the bay water possesses eddy viscosity. The bay head is assumed to be composed of either a rigid or a non-rigid wall.

The wave height at the bay head may be expressed by two different solutions, one is the mode solution and the other is the ray solution. The former may be successfully used to investigate the water motion in the later part of a marigram where few modes of free oscillation are predominant, and the latter is very convenient in constructing a theoretical marigram from which the maximum water level can be estimated.

For the case of non-viscous wave motion with a rigid wall bay head, response curves which represent the maximum wave height at the bay head as a function of T/T_0, with a parameter m, are obtained, where T and T_0 are respectively the period of incoming waves and the first mode eigen-period of the bay, and m is the number of crests and troughs contained in the incident wave packet. The response curves may be useful in estimating the wave height due to tunamis at the bay head. For the case of a certain kind of non-rigid wall bay head, it can be shown that the wave height is decreased at the bay head.

1 Introduction

Almost the whole eastern coasts of Japan were swept by the tunami which originated from the Chile Earthquake of May 24, 1960. The bays situated along the Sanriku Coast in the northeastern region of Honshu, exhibited very much different response to the invaded tunami from that at the time of near tunami originated off the Sanriku Coast in 1933. In certain bays, the wave height at the bay head is much larger than at the bay mouth, and vice versa in 1933. Also, the marigrams of the Chile tunami show much more prolonged free oscillation of bays compared with those for the Sanriku tunami in 1933. The reason for these facts may be sought in that a very distant tunami, when compared with a near one, can send into the bays waves with much longer periods and longer duration, and gives rise to turbulent motion to a less degree in the bays.

The motion of water in a bay due to incident long waves has been studied by G. Nishimura and K. Kanai (1934), G. Nishimura, T. Takayama and K. Kanai (1935), R. Takahashi (1947), T. Rikitake and S. Murauchi (1947), and S. Ogiwara (1949). All these authors have treated the case in which a shock-type wave or a sinusoidal wave train is incident upon the bay mouth. The works of NISHIMURA, TAKAYAMA and KANAI, and that of RIKITAKE and MURAUCHI are concerned with a bay of variable section, while others treat a rectangular bay. The effect of eddy viscosity is discussed by OGIWARA in an approximate manner. TAKAHASHI has showed the ray solution which will be discussed also in our paper, and has suggested a method to estimate the coefficient of reflection of long waves at an actual bay head. K. SEZAWA and K. KANAI (1936) have investigated a problem of dissipation of energy in a bay caused by the diverging waves from the bay.

It may safely be said that, for a given bay, the motion of bay water in case of an invasion of a tunami is primarily influenced by three factors ; the period and length of the incident wave packet, and the eddy viscosity relevant to the motion of bay water.

In this paper, ignoring a complication due to shape and bottom topography of the bay, the combined effect of the three factors above mentioned will be investigated by solving the motion of bay water, when a packet of long waves composed of m crests and troughs is incident upon the mouth of a bay with a rectangular shape and uniform depth. From a practical point of view, the maximum wave height at the bay head is exclusively investigated. Judging from the results of model experiment, the energy dissipation due to the diverging waves from the bay mouth will be ignored. The boundary condition at the bay head is assumed in two ways, one is the case in which the coefficient of reflection is unity, and the other is the case where it is not.

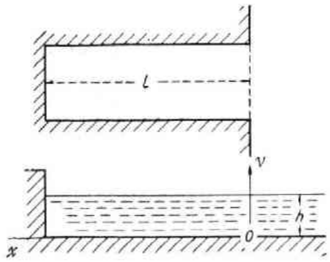


Fig. 1. Rectangular bay with uniform depth.

2 Fundamental Equations and Boundary Conditions

Consider a rectangular bay with length l and uniform depth h. The cartesian coordinates (x, y) are taken, as shown in Fig. 1, such that the origin lies at the bottom of the bay mouth, x-axis is directed to the bay head, and y-axis vertically upwards.

The equations of motion and the equation of continuity are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u, \tag{1}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v - g, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

where u and v are velocities respectively along x and y directions, ρ is the density, p the pressure, g the acceleration due to gravity, and ν may be considered to represent the coefficient of eddy viscosity. From (2), ignoring the vertical acceleration and the term $\nu\nabla^2 v$, we have

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x}. \tag{4}$$

From (3), it follows that

$$\eta = -\int_0^h \frac{\partial \xi}{\partial x} dy , \tag{5}$$

where η represents the elevation of water surface from the undisturbed state, and

$$\xi = \int u \, dt . \tag{6}$$

Substituting (4) into (1), and using (6), we have the equation for ξ,

$$\frac{\partial^2 \xi}{\partial t^2} = g \frac{\partial^2}{\partial x^2} \int_0^h \xi \, dy + \nu \frac{\partial^3 \xi}{\partial t \partial y^2} . \tag{7}$$

We take the following boundary conditions,

$$y = 0: \quad u = 0 , \tag{8}$$

$$y = h: \quad \frac{\partial u}{\partial y} = 0 , \tag{9}$$

$$x = 0: \quad \eta = f(t) , \tag{10}$$

$$x = l: \quad \frac{\partial \eta}{\partial x} = 0 , \tag{11}$$

where $f(t)$ is a given function of time. (8) indicates that there is no current at the bottom, and (9) shows that the tangential shear at the surface is zero. (10) implies no energy dissipation from the bay mouth, and (11) shows that the bay head becomes always a loop for elevation.

Initial conditions are assumed as

$$t = 0: \quad \eta = \frac{\partial \eta}{\partial t} = 0 . \tag{12}$$

Using (5), (6), and (12), we can write the Laplace transforms of (7) and (6)–(11) as follows;

$$p^2 \bar{\xi} = g \frac{\partial^2}{\partial x^2} \int_0^h \bar{\xi} \, dy + p^2 \nu \frac{\partial^2 \bar{\xi}}{\partial y^2} , \tag{7'}$$

$$y = 0: \quad \bar{\xi} = 0 , \tag{8'}$$

$$y = h: \quad \frac{\partial \bar{\xi}}{\partial y} = 0 , \tag{9'}$$

$$x = 0: \quad \eta = \bar{f}(t) , \tag{10'}$$

$$x = l: \quad \int_0^h \frac{\partial^2 \xi}{\partial x^2} dy = 0 , \tag{11'}$$

where for instance,

$$\bar{\xi} = \int_0^\infty \xi e^{-pt} dt . \tag{13}$$

3 Formal Solution

A solution of (7′) can be obtained by use of the method of separation of variables,

$$\bar{\xi} = \left[A \sin(\sqrt{K} x + B \cos(\sqrt{K} x) \right] \left[C \sinh\left(\sqrt{\frac{p}{\nu}} y\right) + D \cosh\left(\sqrt{\frac{p}{\nu}} y\right) - \frac{1}{p^2} \right] ,$$

where (14)

$$\frac{1}{gK}=\int_0^h\left[C\sinh\left(\sqrt{\frac{p}{\nu}}y\right)+D\cosh\left(\sqrt{\frac{p}{\nu}}y\right)-\frac{1}{p^2}\right]dy\,, \tag{15}$$

and A,B,C and D are constants to be determined by the boundary conditions (8′)–(11′). K is a constant of separation of variables and is ultimately dropped out from the expression for elevation.

(8′)–(11′) successively determine the constants as

$$D=\frac{1}{p^2}, \tag{16}$$

$$C=-\frac{1}{p^2}\tanh\left(\sqrt{\frac{p}{\nu}}h\right), \tag{17}$$

$$A=-g\sqrt{K}\bar{f}(t)\,, \tag{18}$$

$$B=g\sqrt{K}\bar{f}(t)\tan(\sqrt{K}l)\,. \tag{19}$$

From (14), (15), and (16)–(19), the transformed expression of (5) becomes

$$\begin{aligned}\bar{\eta}&=-\int_0^h\frac{\partial\bar{\xi}}{\partial x}dy\\&=-g\sqrt{K}\left[A\cos(\sqrt{K}x)-B\sin(\sqrt{K}x)\right]\\&=\bar{f}\,\frac{\cos\{\sqrt{K}(l-x)\}}{\cos(\sqrt{K}l)}\,.\end{aligned} \tag{20}$$

The inverse transformation gives

$$\eta=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{zt}\bar{f}\,\frac{\cos\{\sqrt{K}(l-x)\}}{\cos\{\sqrt{K}l\}}\,dz\,, \tag{21}$$

where the integration is to be carried out in the complex plane as shown in Fig. 2.

The elevation at the bay head ($x=l$) becomes

$$\eta_l=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\bar{f}\,e^{zt}}{\cos\{\sqrt{K}l\}}\,dz\,. \tag{22}$$

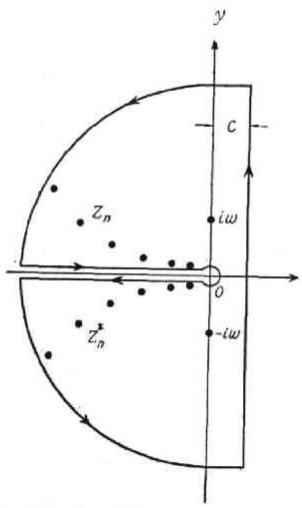


Fig. 2. Location of poles in z-plane.

We assume that a packet of sinusoidal waves, of which the number of crests and troughs is given by m, and the period by T is incident upon the bay mouth. Then we may write the function $f(t)$ as

$$f(t)=\begin{cases}0 & t<0\\ \sin\omega t & 0\leqq t\leqq\dfrac{m\pi}{\omega}\end{cases}, \tag{23}$$

where ω is the circular frequency and $T=2\pi/\omega$.

The Laplace transfom of $\sin\omega t$, $t\geqq 0$ is given by

$$\int_0^\infty e^{-pt}\sin\omega t\,dt=\frac{\omega}{p^2+\omega^2}\,, \tag{24}$$

and that of $\sin\omega t$, $t\geqq\frac{m\pi}{\omega}$, is given by

$$\int_{m\pi/\omega}^{\infty} e^{-pt} \sin \omega t \, dt = (-1)^m \frac{\omega e^{-m\pi p/\omega}}{(p^2+\omega^2)} . \tag{25}$$

Then the Laplace transform of (23) is the difference between (24) and (25). (cf. N. W. MacLachlan (1949) p. 129)

$$\bar{f} = \frac{\omega}{p^2+\omega^2} \left[1-(-1)^m e^{-m\pi p/\omega} \right], \quad t \geqq \frac{m\pi}{\omega} . \tag{26}$$

Thus we can write for the elevation at the bay head,

$$\eta_l = \begin{cases} \eta_1 & \frac{m\pi}{\omega} \geqq t \geqq 0, \\ \eta_1 - \eta_2 & t > \frac{m\pi}{\omega}, \end{cases} \tag{27}$$

where

$$\eta_1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{z^2+\omega^2} \frac{e^{zt}}{\cos(\sqrt{K}l)} dz , \tag{28}$$

$$\eta_2 = \frac{(-1)^m}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\omega}{z^2+\omega^2} \frac{e^{z[t-m\pi/\omega]}}{\cos(\sqrt{K}l)} dz . \tag{29}$$

We have two different types of solution for the elevation expressed by (27)–(29), according as the method of integration in the complex plane. One is the mode solution, the other is the ray solution.

4 Mode Solution

As the integral in (29) is readily obtained from (28) by replacing t by $t-m\pi/\omega$, we will consider only (28).

(1) Riemann surface and singular points.

From (15)–(17), it follows that

$$\frac{1}{gK} = \frac{a}{z^2} \left[\sqrt{\frac{\nu}{z}} \tanh\left(\sqrt{\frac{z}{\nu}} h\right) - h \right], \tag{30}$$

and

$$\sqrt{K} l = \frac{l z}{\sqrt{g} \left[\sqrt{\frac{\nu}{z}} \tanh\left(\sqrt{\frac{z}{\nu}} h\right) - h \right]^{1/2}} . \tag{31}$$

In the z-plane we make a branch cut from 0 to $-\infty$ along the negative real axis as shown in Fig. 2, and putting

$$\sqrt{\frac{z}{\nu}} h = w , \tag{32}$$

we assume that the real part of w is positive in the upper sheet of the Riemann surface in which the contour of integration is laid.

Since K is a single valued function of $\sqrt{z}$ as seen from (30), and $\cos(\sqrt{a/gK}\, l)$ is a single valued function of K as seen from the series expansion

$$\cos(\sqrt{K}\,l) = 1 - \frac{1}{2!}Kl^2 + \frac{1}{4!}K^2l^4 - \ldots\ldots,$$

the branch-cut-integral vanishes.

(2) Poles of the integrand.

There are poles $z=\pm i\omega$, as shown in Fig. 2, which give the forced motion due to the incident waves. Beside these poles, we have other ones determined by the relation

$$\cos(\sqrt{K}\,l) = 0. \tag{33}$$

This relation can be written as

$$\frac{z\,l}{\sqrt{gh}\left[\frac{\tanh w}{w} - 1\right]^{1/2}} = \left(n+\frac{1}{2}\right)\pi, \quad n = 0, 1, 2, \ldots\ldots \tag{34}$$

or

$$\beta_n w^4 + 1 - \frac{\tanh w}{w} = 0, \tag{35}$$

where

$$\beta_n = \frac{\nu^2}{\omega_n{}^2 h^4}, \tag{36}$$

$$\omega_n = \frac{2\pi}{T_n} = \frac{\sqrt{gh}}{l}\left(n+\frac{1}{2}\right)\pi, \quad T_n = \frac{4l}{\sqrt{gh}\,(2n+1)}. \tag{37}$$

T_n in (37) represents the period of free oscillation of n-th mode in the absence of eddy viscosity. The equation (35) was first derived by A. DEFANT (1932), in his study on the eigen-oscillation of water in a lake, and later investigated by K. HIDAKA (1935) in his problem of lake seiche generated by wind.

If we put

$$w = \xi + i\eta, \tag{38}$$

it follows from (32) that

$$z = -\frac{\nu}{h^2}\left[(\eta^2-\xi^2) - 2i\xi\eta\right]. \tag{39}$$

By separating the real and imaginary parts of (35), we obtain the simultaneous equations

$$\left.\begin{aligned}
&\left[(\xi^2-\eta^2)^2 - 4\xi^2\eta^2 - 4\eta^2(\xi^2-\eta^2)\right](\cosh 2\xi + \cos 2\eta) \\
&\qquad + \frac{1}{\beta_n}\left[\cosh 2\xi + \cos 2\eta - \frac{\sinh 2\xi}{\xi}\right] = 0, \\
&\left[(\xi^2-\eta^2)^2 - 4\xi^2\eta^2 + 4\xi^2(\xi^2-\eta^2)\right](\cosh 2\xi + \cos 2\eta) \\
&\qquad + \frac{1}{\beta_n}\left[\cosh 2\xi + \cos 2\eta - \frac{\sin 2\eta}{\eta}\right] = 0.
\end{aligned}\right\} \tag{40}$$

For a given value of β_n, ξ and η can be obtained by a trial and error method. Remembering the condition $\mathcal{R}e(w) > 0$, we can write the roots of (35) for prescribed β_n,

$$w_n = \xi_n \pm i\,\eta_n\,,$$
where $$\xi_n,\,\eta_n > 0\,. \tag{41}$$

Table 1 contains the results of our calculation of ξ and η for some values of β_n $(=\beta)$, together with those obtained by DEFANT and HIDAKA. Fig. 3 shows the behaviour of ξ and η for varied β, from which we see that η is always larger than ξ for any value of β larger than 0.5370. The positions in the z-plane of the roots of (33) may be schematically shown in Fig. 2.

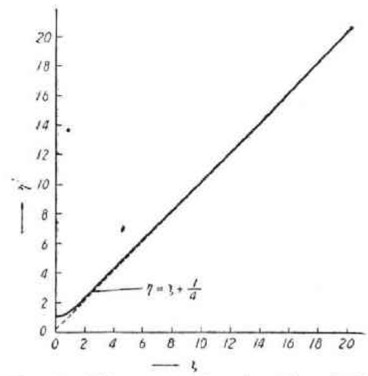


Fig. 3 The curve showing the relationship between ξ and η for varied β-values. $\xi=0$, $\eta=1.1122$ correspond to $\beta=0.5370$. The curve approches to the dotted line $\eta=\xi+1/4$ indefinitely with decreasing β-value.

(3) Residues at the poles

The sum of residues at the poles $z=\pm i\omega$ is obtained as

Table 1.

β	ξ	η	$\overline{T}/T$	ρ
* 0.5370	0.0	1.1122	∞	∞
* 0.4732	0.2	1.1307	3.217	5497.9
* 0.3376	0.4	1.1837	1.817	61.37
† 0.2500	0.53442	1.23648	1.513	19.21
* 0.2132	0.6	1.2670	1.424	13.11
* 0.1284	0.8	1.3757	1.269	5.976
† 0.0625	1.08662	1.56595	1.175	3.234
* 0.0615	1.0935	1.5708	1.174	3.200
* 0.02440	1.4	1.8916	1.128	2.086
† 0.015625	1.72059	2.08322	1.117	1.828
* 9.36×10^{-3}	2.0	2.3362	1.106	1.636
† 3.0864×10^{-3}	2.72894	3.03519	1.087	1.398
* 2.68×10^{-3}	2.8368	3.1416	1.083	1.379
† 9.7656×10^{-4}	3.73365	4.02704	1.0641	1.269
† 4.00×10^{-4}	4.73475	5.18855	1.0521	1.201
8.991×10^{-5}	7	7.264	1.0370	1.124
2.258×10^{-5}	10	10.260	1.0267	1.084
4.616×10^{-6}	15	15.256	1.0170	1.055
1.486×10^{-6}	20	20.255	1.0126	1.041
2.985×10^{-7}	30	30.253	1.0084	1.027
9.523×10^{-8}	40	40.252	1.0063	1.020
5.962×10^{-8}	45	45.252	1.0056	1.018
3.921×10^{-8}	50	50.252	1.0049	1.016
1.897×10^{-8}	60	60.252	1.0042	1.013
1.026×10^{-8}	70	70.251	1.0036	1.011

*: computed by DEFANT. †: computed by HIDAKA

$$R_1 = \frac{e^{i\omega t}}{2\,i\,[\cos(\sqrt{K}\,l)]_{z=i\omega}} + \frac{e^{-i\omega t}}{-2\,i\,[\cos\sqrt{K}\,l)]_{z=-i\omega}}$$

$$= I\,m\left[\frac{e^{i\omega t}}{\cos M}\right], \tag{42}$$

where

$$M = \frac{l\omega}{\sqrt{g\,h}}\left\{1 - \frac{\tanh w_\omega}{w_\omega}\right\}^{-1/2},$$

$$w_\omega = \sqrt{\frac{\omega}{\nu}}\, h\, e^{\pi/4 i}\ . \tag{43}$$

The sum of residues at the poles

$$\left.\begin{matrix} z_n \\ z_n{}^* \end{matrix}\right\} = -\frac{\nu}{h^2}\left\{(\eta^2{}_n - \xi^2{}_n) \mp 2\, i\, \xi_n \eta_n\right\}$$

is obtained as follows;

$$\begin{aligned} R_2 = &-\sum_{n=0}^{\infty}\left[\frac{e^{z_n t}}{\left\{\sin(\sqrt{K}\,l)\,\dfrac{d}{dz}(\sqrt{K}\,l)\right\}_{z_n}}\,\frac{\omega}{(z^2{}_n+\omega^2)}\right. \\ &\left. + \frac{e^{z_n{}^* t}}{\left\{\sin(\sqrt{K}\,l)\,\dfrac{d}{dz}(\sqrt{K}\,l)\right\}_{z_n{}^*}}\,\frac{\omega}{(z_n{}^{*2}+\omega^2)}\right] \\ = &\ 2\,\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\left[\frac{e^{z_n t}}{\left\{\dfrac{d}{dz}(\sqrt{K}\,l)\right\}_{z_n}(z_n{}^2+\omega^2)}\right], \end{aligned} \tag{44}$$

where

$$\left.\begin{aligned} &\left\{\frac{d}{dz}(\sqrt{K}l)\right\}_{z_n} = -\frac{i\,l}{4\sqrt{g\,h}}\left(1-\frac{\tanh w_n}{w_n}\right)^{-3/2}\left[5\left(1-\frac{\tanh w_n}{w_n}\right)-\tanh^2 w_n\right], \\ &\qquad w_n = \sqrt{\frac{z_n}{\nu}}\,h\ . \end{aligned}\right\} \tag{45}$$

(4) Elevation at the bay head

From (42) and (44), η_1 in (28) becomes

$$\eta_1 = I\,m\left(\frac{e^{i\omega t}}{\cos M}\right) + 2\,\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\left[\frac{e^{z_n t}}{(z_n{}^2+\omega^3)\left\{\dfrac{d}{dz}(\sqrt{K}\,l)\right\}_{z_n}}\right], \qquad 0 \leqq t \leqq \frac{m\,\pi}{\omega}\,. \tag{46}$$

From (28) and (29), we can write

$$\begin{aligned} \eta_1 - \eta_2 = &\ I\,m\left(\frac{e^{i\omega t}}{\cos M}\right) - (-1)^m\, Im\left\{\frac{e^{i\omega[t-m\pi/\omega]}}{\cos M}\right\} \\ &+ 2\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\left[\frac{e^{z_n t}}{(z_n{}^2+\omega^2)\left\{\dfrac{d}{dz}(\sqrt{K}l)\right\}_{z_n}}\left\{1-(-1)^m e^{-m\pi z_n/\omega}\right\}\right] \\ = &\ 2\,\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\left[\frac{e^{z_n t}}{(z_n{}^2+\omega^2)\left\{\dfrac{d}{dz}(\sqrt{K}\,l)\right\}_{z_n}}\left\{1-(-)^m e^{-m\pi z_n/\omega}\right\}\right], \\ &\qquad t > \frac{m\,\pi}{\omega}\,. \end{aligned} \tag{47}$$

If we put

$$\tau = \frac{t}{T_0}, \qquad u_0 = \frac{T}{T_0}, \tag{48}$$

where T_0 is the natural period of the first mode oscillation, the mode solution can be written as follows:

$$\eta_l = \begin{cases} Im\left[\dfrac{e^{i\,2\pi\tau/u_0}}{\cos M} + \sum\limits_{n=0}^{\infty} N_n\, e^{-2\pi\sqrt{\beta_0}\,[(\eta_n^2-\xi_n^2)-2i\xi_n\eta_n]\tau}\right], & 0 \leqq \tau \leqq \dfrac{m}{2}u_0, \quad (49) \\[2ex] Im \sum\limits_{n=0}^{\infty} N_n \left\{1-(-1)^m e^{-mu_0/2}\right\} e^{-2\pi\sqrt{\beta_0}\,[(\eta_n^2-\xi_n^2)-2i\xi_n\eta_n]\tau}, & \dfrac{m}{2}u_0 < \tau, \quad (50) \end{cases}$$

where

$$\left.\begin{aligned} M &= \frac{\pi}{2u_0}\left\{1-\frac{\tanh w_\omega}{w_\omega}\right\}^{-1/2}, \qquad w_\omega = u_0^{-1/2}\beta_0^{-1/4}e^{\pi/4\, i}, \\ N_n &= (-1)^n \frac{16u_0}{\pi}\left\{1-\frac{\tanh w_n}{w_n}\right\}^{-3/2}\left\{1+\beta_0 u_0^2 w_n^4\right\}^{-1}\left\{5\left(1-\frac{\tanh w_n}{w_n}\right)-\tanh^2 w_n\right\}^{-1}. \end{aligned}\right\} \tag{51}$$

When the motion of bay water is non-viscous, it can be seen that from (32), w becomes infinitely large. It follows from (32) and (35) that

$$z_n = \pm i\,\omega_n\,,$$

$$\beta_n w^4 = -1\,, \qquad 1+\beta_0 u_0^2 w_n^4 = 1+\beta_n u_n^2 w_n^4 = 1-u_n^2\,. \tag{52}$$

Thus, (49) to (51) are replaced by

$$\eta_l = \begin{cases} \dfrac{\sin\left(\dfrac{2\pi}{u_0}\tau\right)}{\cos\left(\dfrac{\pi}{2u_0}\right)} + \dfrac{4}{\pi}u_0 \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{1-u_n^2}\sin\left\{2\pi(2n+1)\tau\right\}, & 0 \leqq \tau \leqq \dfrac{m}{2}u_0, \quad (53)^* \\[2ex] \dfrac{4}{\pi}u_0\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{1-u_n^2}\left[\sin\left\{2\pi(2n+1)\tau\right\} - (-1)^m \sin\left\{2\pi(2n+1)\left(\tau-\dfrac{m}{2}u_0\right)\right\}\right], & \dfrac{m}{2}u_0 < \tau, \quad (54) \end{cases}$$

where

$$u_n = \frac{T}{T_n}\,. \tag{55}$$

(49) and (53) which are applicable until the wave packet has entered the bay, is composed of two parts, the one represents the forced oscillation due to the incident wave packet, and the other, infinite modes of the free oscillation in the bay. Whereas, (50) and (54), valid after the entrance of the wave packet, contain only the terms of free oscillation.

(5) Damping of wave height and lengthening of natural period due to eddy viscosity.

It is found from (49) and (50) that, due to eddy viscosity, the forced oscillation is decreased in amplitude, but unchanged in period, and that, the amplitude of free

* When m becomes infinitely large, (53) represents the elevation due to a periodic wave train, and corresponds to the formula (56) in the paper by NISHIMURA and KANAI (1934), however, the factor π must be multiplied to their formula.

oscillation is decreased and the period of it is lengthened.

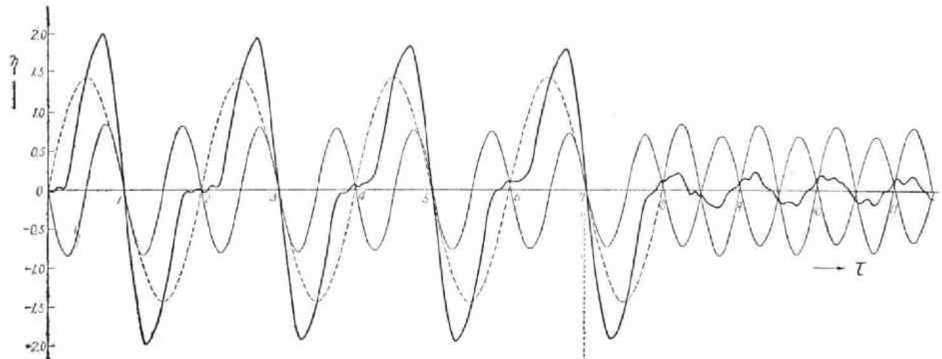
The period $\overline{T}_n$ of free oscillation of each mode, when eddy viscosity is taken into account, is given by (cf. DEFANT (1932) and HIDAKA (1935)),

$$\frac{\overline{T}_n}{T_n} = \frac{1}{2\,\xi_n\eta_n\sqrt{\beta_n}} \qquad \text{or} \qquad \frac{\overline{T}_n}{T_0} = \frac{1}{2\,\xi_n\eta_n\sqrt{\beta_0}}\,, \tag{56}$$

and the logarithmic decrement becomes

$$\rho = \log \pi\left(\frac{\eta_n{}^2-\xi_n{}^2}{2\,\xi_n\,\eta_n}\right). \tag{57}$$

For a wide range of values of β, the ratio $\overline{T}/T = 1/2\xi\eta\sqrt{\beta}$, and ρ are shown in Table 1, in which are also contained some results by DEFANT and HIDAKA. It is to be noticed that the oscillation ceases to be periodic when β is larger than 0.5370, which corresponds to $\xi=0$, $\eta=1.1122$. This indicates that, for a given values of T_n and h, there exists a certain upper limit for the coefficient of eddy viscosity in order that the free oscillation remains periodic.



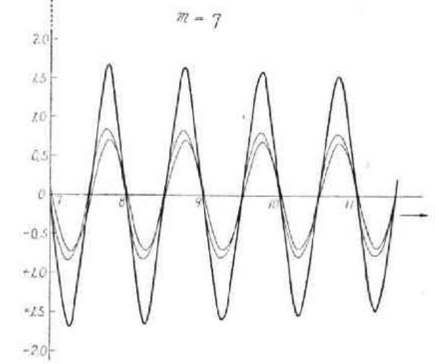


Fig. 4. The surface elevation at the bay head is indicated by thick curves which are obtained by superposing the forced oscillation and the first three modes of free oscillation. Amplitude of incident waves is unity, $u_0=2$, $\beta=9.52\times10^{-8}$. The upper figure corresponds to $m=8$, the lower to $m=7$.

—— : the first mode free oscillation
…… : forced oscillation

In constructing a theoretical marigram from the mode solution above obtained, many modes are required to obtain a correct form of it, especially when u_0 is small. When u_0 is rather large, however, the amplitudes of free oscillations of higher modes become comparatively small, so that few modes are enough to give an approximate form of a marigram. This situation is shown in Fig. 4, which indicates the marigram constructed by superposing the forced oscillation and first three modes of the free oscillation, when $u_0=2$, $\beta=9.52\times10^{-8}$, $m=7$ and 8. These constants are taken from the data of Ofunato Bay in the case of the Chile Tunami of 1960.

Although the mode solution is inadequate for constructing a precise theoretical marigram, it is very useful to investigate the nature of free oscillation at the later portion of the marigram in which a forced oscillation is no more found, and only few modes of free oscillation are predominant.

To see the aspect near the initial motion, or to know the maximum height of water, we must recourse to another kind of solution which will be obtained in the next section.

5 Ray Solution

From (28), (31) and (32), we can write

$$\eta_l=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{z^2+\omega^2}\frac{e^{zt}}{\cosh dz}dz\,,\qquad 0\leqq t\leqq\frac{m\pi}{2\omega}\,, \tag{58}$$

where

$$\alpha=\frac{l}{\sqrt{gh}}\left\{1-\frac{\tanh w}{w}\right\}^{-1/2}. \tag{59}$$

Since

$$\frac{e^{zt}}{\cosh dz}=2\sum_{n=0}^{\infty}(-1)^n e^{z[t-(2n+1)\alpha]}\,, \tag{60}$$

which is uniformly convergent, it follows that

$$\eta_l=\frac{1}{\pi i}\sum_{n=0}^{\infty}\int_{c-i\infty}^{c+i\infty}(-1)^n\frac{\omega}{z^2+\omega^2}e^{z[t-(2n+1)\alpha]}dz\,,\qquad 0\leqq t\leqq\frac{m\pi}{2\omega}\,. \tag{61}$$

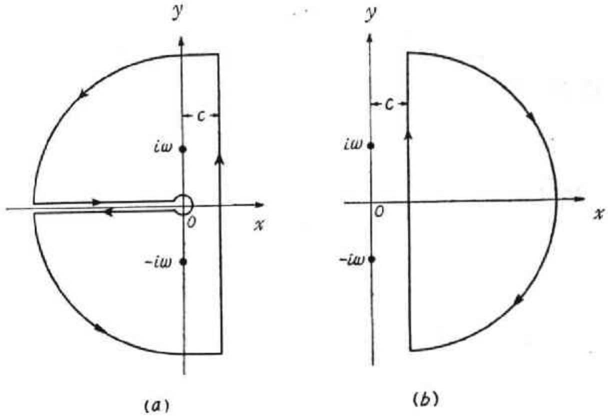


Fig. 5. Paths of integration. (a) t>0, (b) t<0

Now, the poles are $z=\pm i\omega$. When $Re\,[t-(2n+1)\alpha]<0$, the integral in (61) vanishes if we take the contour as shown in Fig. 5, b. and when $Re\,[t-(2n+1)\alpha]>0$, by taking the contour in Fig. 5, a, the integral is given by $2\pi i$ times the residues at $z=\pm i\omega$. If

$\nu=0$, the conditions $Re\,[t-(2n+1)\alpha]\lessgtr 0$ become $[t-(2n+1)l/\sqrt{gh}\,]\lessgtr 0$, but when ν is not zero, these conditions are influenced by the factor $\sqrt{\omega/\nu}\,h$.

If we consider the nature of damping in wave motion, we may put

$$[\alpha]_{\pm i\omega}=\frac{l}{\sqrt{gh}}\left\{1-\frac{\tanh\left(\sqrt{\frac{i\omega}{\nu}}h\right)}{\sqrt{\frac{i\omega}{\nu}}h}\right\}^{-1/2}=\frac{l}{\sqrt{gh}}\,(p\mp iq)\,, \tag{62}$$

where the upper sign is to be used for $z=i\omega$, and the lower sign for $z=-i\omega$. By squaring both hands of (62), we obtain

$$\left\{\begin{matrix} p \\ q \end{matrix}\right\}=\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A^2+B^2}}\sqrt{\sqrt{A^2+B^2}\ \pm A}\ , \tag{63}$$

where

$$\left.\begin{aligned} A&=2\kappa(\cosh 2\kappa+\cos 2\kappa)-(\sinh 2\kappa+\sin 2\kappa)\,,\\ B&=\sin 2\kappa-\sin 2\kappa\,,\\ C&=2\kappa(\cosh 2\kappa+\cos 2\kappa)\,, \end{aligned}\right\} \tag{64}$$

$$\kappa=\frac{h\sqrt{\omega}}{\sqrt{2\nu}}\,. \tag{65}$$

We have the relation

$$t-(2n+1)\,\alpha=\left\{t-(2n+1)\frac{l}{\sqrt{gh}}p\right\}\pm i\,(2n+1)\frac{l}{\sqrt{gh}}q\,. \tag{66}$$

By residue calculation, we obtain

$$\eta_1=\sum_{n=0}^{\infty}(-1)^n\sin\omega t\,e^{-(2n+1)lq\omega/\sqrt{gh}}\,H\left[t-\frac{(2n+1)l}{\sqrt{gh}}p\right], \tag{67}$$

where $H[t-(2n+1)\,pl/\sqrt{gh}]$ is the Heavyside's unit step function defined by

$$H\left[t-\frac{(2n+1)}{\sqrt{gh}}\,pl\right]=\left\{\begin{matrix} 1 & t>\dfrac{(2n+1)}{\sqrt{gh}}pl\,, \\ 0 & t<\dfrac{(2n+1)}{\sqrt{gh}}pl\,. \end{matrix}\right. \tag{68}$$

In a similar way, we obtain for $t>m\pi/\omega$.

$$\begin{aligned}\eta_1-\eta_2&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{z^2+\omega^2}\left[1-(-1)^m e^{-m\pi z/\omega}\right]\sum_{n=0}^{\infty}(-1)^n e^{z[t-(2n+1)\alpha]}\,dz\\ &=2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)lq\omega/\sqrt{gh}}\left[\sin\omega t\,H\left[t-\frac{(2n+1)lp}{\sqrt{gh}}\right]\right.\\ &\qquad\left.-(-1)^m\sin\omega t\,H\left[t-\frac{m\pi}{\omega}-\frac{(2n+1)lp}{\sqrt{gh}}\right]\right].\end{aligned}$$

Using τ and u_0 defined by (48), it follows that

$$\eta_l=\begin{cases} 2\sum_{n=0}^{\infty}(-1)^n e^{-\pi(2n+1)q/2u_0}\sin\left(\frac{2\pi}{u_0}\tau\right)H\left[\tau-\frac{(2n+1)}{4}p\right], & 0\leqq\tau\leqq\frac{m}{2}u_0, \\ 2\sum_{n=0}^{\infty}(-1)^n e^{-\pi(2n+1)q/2u_0}\left[\sin\left(\frac{2\pi}{u_0}\tau\right)H\left[\tau-\frac{(2n+1)p}{4}\right]\right. & \\ \qquad\left.-(-1)^m\sin\left(\frac{2\pi}{u_0}\tau\right)H\left[\tau-\frac{m}{2}u_0-\frac{(2n+1)p}{4}\right]\right], & \frac{m}{2}u_0<\tau. \end{cases} \tag{69}$$

Especially when $\nu=0$, we obtain from (59) and (62),

$$\alpha=\frac{l}{\sqrt{gh}}, \quad p=1, \quad q=0.$$

Thus we obtain

$$\eta_l=\begin{cases} 2\sum_{n=0}^{\infty}(-1)^n\sin\left(\frac{2\pi}{u_0}\tau\right)H\left[\tau-\frac{(2n+1)}{4}\right], & 0\leqq\tau\leqq\frac{m}{2}u_0, \\ 2\sum_{n=0}^{\infty}(-1)^n\left[\sin\left(\frac{2\pi}{u_0}\tau\right)H\left[\tau-\frac{(2n+1)}{4}\right]\right. & \\ \qquad\left.-(-1)^m\sin\left(\frac{2\pi}{u_0}\tau\right)H\left[\tau-\frac{m}{2}u_0-\frac{(2n+1)}{4}\right]\right], & \frac{m}{2}u_0<\tau. \end{cases} \tag{70}$$

(69) and (70) are the ray solutions which consist of the terms representing the incident waves and waves reflected n times $(n=1, 2, 3, \ldots.)$, at the bay mouth. The ray solution is very convenient to construct a theoretical marigram, since it suffices to draw successively the curves which represent the incident waves and reflected waves both at the head and mouth of the bay once, twice, and so on. This method has been studied by R. TAKAHASHI (1947), for the case of non-viscous wave motion. Although the method is rather tedious to know the nature of free oscillation at later stages of the record, it is very useful to estimate the maximum wave height which occurs at a very earlier stage.

When the observed wave height at the bay head is less than that at the bay mouth, it is possible by use of (69) to estimate the coefficient of eddy viscosity in the transient motion of the bay water. This will be described in another paper.

6 Response Curves of Bay to Tunamis

When $\nu=0$, some marigrams constructed by the use of (70) are shown in Fig. 6. By reading the maximum water level of these kinds of marigrams drawn for various u_0-and m-values, we can construct the response curves of the bay to invading tunamis, as shown in Fig. 7, where the abscissa indicates $u_0=T/T_0$, the ordinate represents the maximum wave height η_m at the bay head, and the parameter m means the number of crests and troughs of the incident wave packet.

From Fig. 7, it can be seen that the wave height becomes larger with increasing value of m, and tends to infinity at $u_0=1$ when the wave packet tends to infinitely long wave train. There exist peaks of η_m at $u_0=1/(2n+1)$, which correspond to

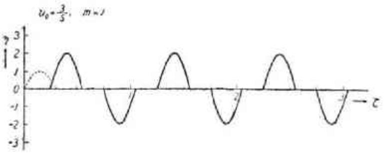


Fig. 6. a-1

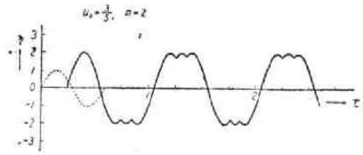


Fig. 6. a-2

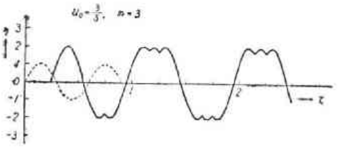


Fig. 6. a-3

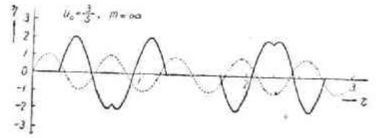


Fig. 6. a-4

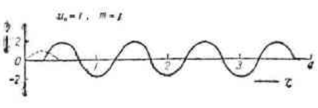


Fig. 6. b-1

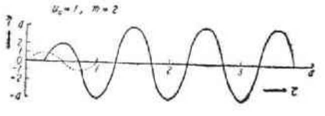


Fig. 6. b-2

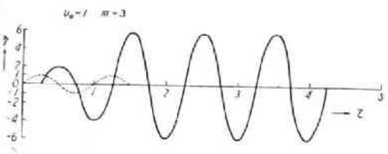


Fig. 6. b-3

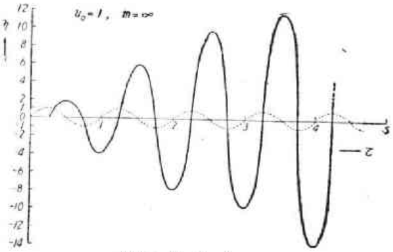


Fig. 6. b-4

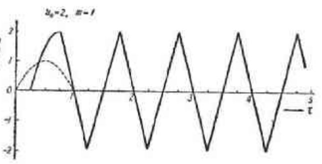


Fig. 6. c-1

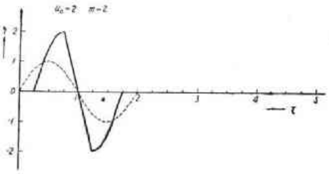


Fig. 6. c-2

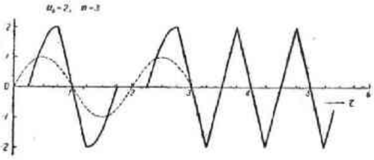


Fig. 6. c-3

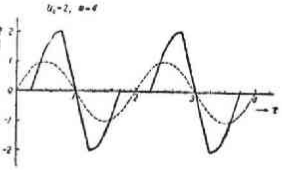


Fig. 6. c-4

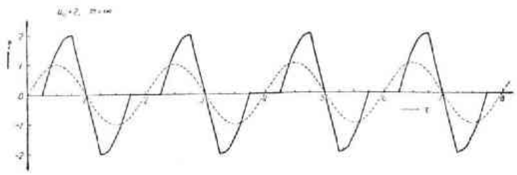


Fig. 6. c-5

Fig. 6. Marigrams at bay head obtained by ray solution. Incident waves are shown by dotted curves. $u_0=T/T_0$, $\tau=t/T_0$. T_0: seiche period, T: period of incident waves, m: number of crests and troughs.

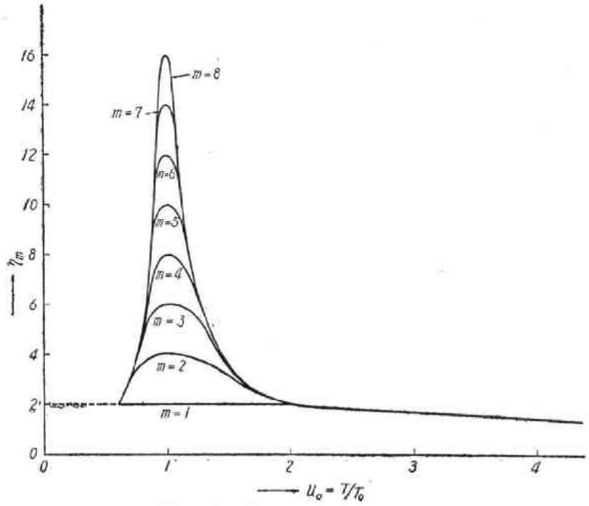


Fig. 7. Response curves.

T : period of incident waves.
T_0: seiche period.
m : number of crests and troughs contained in the incident wave packet.

the resonance period of respective modes, but the first mode shows the most conspicuous peak for a prescribed value of m and, actually, the oscillation of the first mode is often most predominant. Fig. 7 shows the response of only the first mode. It is to be noticed that η_m becomes smaller, the larger the value of u_0.

7 The Case of Non-Rigid Wall Bay Head

In the preceding analysis, we assumed the bay head to be a rigid wall. In this section, we assume instead of (11) that

$$x = l: \qquad \frac{\partial \eta}{\partial x} = k\,\eta\,, \tag{71}$$

where k is a constnat.
Then (11′) is to be replaced by

$$\int_0^h \frac{\partial^2 \bar{\xi}}{\partial x^2} dy = k \int_0^h \frac{\partial \bar{\xi}}{\partial x} dy\,. \tag{72}$$

In the sequel, the formula with dashed number will be used as the corresponding one in section 4.

We can write

$$B = \sqrt{K}\,\bar{j} \left[\frac{\sqrt{K}\sin(\sqrt{K}\,l) + k\cos(\sqrt{K}\,l)}{\sqrt{K}\cos(\sqrt{K}\,l) - k\sin(\sqrt{K}\,l)} \right] \tag{19'}$$

$$\bar{\eta} = -\int_0^h \frac{\partial \bar{\xi}}{\partial x} dy = \bar{j} \left[\frac{\sqrt{K}\cos\{\sqrt{K}\,(l-x)\} - k\sin\{\sqrt{K}\,(l-x)\}}{\sqrt{K}\cos(\sqrt{K}\,l) - k\sin(\sqrt{K}\,l)} \right] \tag{20'}$$

$$\eta_l = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i8} e^{zt}\bar{f}\left[\frac{\sqrt{K}\cos\{\sqrt{K}(l-x)\}-k\sin\{\sqrt{K}(l-x)\}}{\sqrt{K}\cos(\sqrt{K}l)-k\sin(\sqrt{K}l)}\right]dz \qquad (21')$$

(A) Mode Solution.

The elevation at the bay head becomes

$$\eta = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}\bar{f}\,\frac{\cos\gamma}{\cos(M+\gamma)}dz\,, \qquad (22')$$

where

$$\gamma = \tan^{-1}\left(\frac{k}{M}\right). \qquad (73)$$

If we again use the input function $f(t)$ expressed by (23) and (27), the relation (27) remains unchanged, and we obtain

$$\eta_1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{z^2+\omega^2}\,\frac{e^{zt}\cos\gamma}{\cos(M+\gamma)}dz\,, \qquad (28')$$

$$\eta_2 = \frac{(-1)^m}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\omega}{z^2+\omega^2}\,\frac{e^{z(t-m\pi/\omega)}\cos\gamma}{\cos(M+\gamma)}dz\,. \qquad (29')$$

Since we can verify that the integrand of (28′) is a single-valued function of $\sqrt{K}$, the branch cut integral in the z-plane becomes again zero.

As

$$\cos\gamma = \frac{M}{\sqrt{k^2+M^2}}\,,$$

the denominator would vanish if

$$M = \pm ik \qquad \text{or} \qquad \frac{zl}{\sqrt{gh}}\left[\frac{\tanh w}{w}-1\right]^{-1/2} = \pm ik\,.$$

Comparing this with (34), we obtain from (37)

$$\omega_n{}^2 = -\frac{\sqrt{gh}}{l}k^2\,,$$

from which we have the relation

$$\beta_n = \frac{\nu^2}{\omega_n{}^2 h^4} = -\frac{l\nu^2}{\sqrt{gh}\,k^2h^4} < 0\,.$$

Since (40) has no root if $\beta_n<0$, it has been ascertained that k^2+M^2 does not vanish.

Positions of the poles $\pm i\omega$ are not altered, but the poles associated with

$$\cos(M+\gamma) = 0 \qquad (33')$$

must be newly examined.

Putting $\sqrt{z/\nu}\,h = w$, we denote by z_n' and $z_n{}^*$, where the symbol * indicates the complex conjugate quantity, the roots of the equation

$$M+\gamma=\left(n+\frac{1}{2}\right)\pi\ ,$$

or

$$-\frac{ilz}{\sqrt{gh}}\left\{1-\frac{\tanh w}{w}\right\}^{-1/2}+\tan^{-1}\left[\frac{ik\sqrt{gh}}{lz}\left\{1-\frac{\tanh w}{w}\right\}^{1/2}\right]=\left(n+\frac{1}{2}\right)\pi\ . \tag{34'}$$

The sum of the residues at the poles $z=\pm i\omega$ can be written as

$$R_1=Im\left[\frac{\cos\gamma_{i\omega}}{\cos(M+\gamma)_{i\omega}}e^{i\omega t}\right].$$

The sum of the residues at the poles z_n', $z_n{}^*$ becomes,

$$R_2=2\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\{G\}\ ,$$

where

$$G=\frac{\cos\gamma_{z_n'}}{z_n'^2+\omega^2}\frac{e^{z_n't}}{\left[\frac{d}{dz}(M+\gamma)\right]_{z_n'}}\ ,$$

$$\cos\gamma_{z_n'}=-\frac{ilz_n'}{\sqrt{gh}}\left\{1-\frac{\tanh w_n'}{w_n'}\right\}\left[k^2+\left\{-\frac{ilz_n'}{\sqrt{gh}}\left(1-\frac{\tanh w_n'}{w_n'}\right)\right\}\right]^{-1/2}$$

$$\left[\frac{d}{dz}(M+\gamma)\right]_{z_n'}=\left[1-\frac{k}{k^2+\left\{-\frac{ilz_n'}{\sqrt{gh}}\left(1-\frac{\tanh w_n'}{w_n'}\right)\right\}^2}\right]\left(-\frac{il}{4\sqrt{gh}}\right)$$

$$\times\left\{1-\frac{\tanh w_n'}{w_n'}\right\}^{-3/2}\left\{5\left(1-\frac{\tanh w_n'}{w'_n}\right)-\tanh^2 w_n'\right\}.$$

The elevation at the bay head is expressed by (27), as in section 4. So that we obtain

$$\eta_l=\begin{cases} Im\left\{\frac{\cos\gamma_{i\omega}}{\cos(M+\gamma)_{i\omega}}e^{i\omega t}\right\}+2\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\left[\frac{\cos\gamma_{z_n'}}{z_n'^2+\omega^2}\frac{e^{z_n't}}{\left\{\frac{d}{dz}(M+\gamma)\right\}_{z_n'}}\right], \\ \qquad 0\leqq t\leqq\frac{m\pi}{\omega}. \qquad (46') \\ 2\omega\sum_{n=0}^{\infty}(-1)^{n+1}Re\left[\frac{\cos\gamma_{z_n'}e^{z_n't}}{(z_n'^2+\omega^2)\left\{\frac{d}{dz}(M+\gamma)\right\}_{z_n'}}\right]\left\{1-(-1)^m e^{-m\pi z_n'/\omega}\right\}, \\ \qquad \frac{m\pi}{\omega}<t. \qquad (47') \end{cases}$$

The solutions above obtained are too complicated to write the factors of amplitude decrease and phase change in the free and forced oscillations. Also, the solutions are formal ones, and, for a prescribed value of ν/h^2, there must be a certain restriction for an allowable value of k, if (71) is to represent a physically possible condition at the bay head. This circumstances will be clearly shown in the next section which treats the ray solution.

When $\nu=0$, we obtain, by increasing w indefinitely,

$$M=-\frac{ilz}{\sqrt{gh}},$$

and (34′) becomes

$$-\frac{ilz}{\sqrt{gh}}+\tan^{-1}\left\{\frac{ik\sqrt{gh}}{lz}\right\}=\left(n+\frac{1}{2}\right)\pi . \tag{34″}$$

The roots of (34″) can be written

$$z_n=\pm i\,\omega_n' ,$$

where ω_n' is determined from

$$\frac{l\,\omega_n'}{\sqrt{gh}}+\tan^{-1}\left\{\frac{k\sqrt{gh}}{l\,\omega'}\right\}=\left(n+\frac{1}{2}\right)\pi .$$

As

$$\cos\gamma_{z_n'}=\frac{l\,\omega_n'}{\sqrt{gh}}\left\{k^2+\left(\frac{l\,\omega_n'}{\sqrt{gh}}\right)^2\right\}^{-1/2}, \qquad \frac{1}{z_n'^2+\omega^2}=\frac{1}{\omega^2-\omega_n'^2},$$

$$\left\{\frac{d}{dz}(M+\gamma)\right\}_{z_n'}=-\frac{il}{\sqrt{gh}}\left[1-k\left\{k^2+\left(\frac{l\,\omega_n'}{\sqrt{gh}}\right)^2\right\}^{-1}\right],$$

it follows that

$$\eta_l=\frac{l\,\omega}{\sqrt{gh}}\frac{\sin\omega t}{\sqrt{k^2+\left(\frac{l\,\omega}{\sqrt{hg}}\right)^2}\cos\left[\frac{l\,\omega}{\sqrt{gh}}+\tan^{-1}\left\{\frac{k\sqrt{gh}}{l\,\omega}\right\}\right]}$$

$$+2\,\omega\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{\omega_n'^2-\omega^2}\frac{\frac{l\,\omega_n'}{\sqrt{gh}}}{\sqrt{k^2+\left(\frac{l\,\omega_n'}{\sqrt{gh}}\right)^2}}\left[\frac{l}{\sqrt{gh}}\left\{1-\frac{k}{\left\{k^2+\left(\frac{l\,\omega_n'}{\sqrt{gh}}\right)^2\right\}}\right\}\right]^{-1}\sin\omega_n' t ,$$

$$0\leq t\leq\frac{m\pi}{\omega}, \tag{46″}$$

$$\eta_l=2\,\omega\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{\omega_n'^2-\omega^2}\frac{\frac{l\,\omega_n'}{\sqrt{gh}}}{\sqrt{k^2+\left(\frac{l\,\omega_n'}{\sqrt{gh}}\right)^2}}\left[\frac{l}{\sqrt{gh}}\left\{1-\frac{k}{\left[k^2+\left(\frac{l\,\omega_n'^2}{\sqrt{gh}}\right)^2\right]}\right\}\right]^{-1}$$

$$\times\left[\sin\omega_n t(-1)^m\sin\omega_n\left(t-\frac{m\pi}{\omega}\right)\right], \qquad t>\frac{m\pi}{\omega}. \tag{47″}$$

If $\nu=0$ and $k=0$, ω_n' is to be replaced by $\omega_n=(\sqrt{gh/l})(n+1/2)\pi$, and (46″) and (47″) take simpler forms

$$\eta_l=\begin{cases}\dfrac{\sin\omega t}{\cos\left(\dfrac{l\,\omega}{\sqrt{gh}}\right)}+\dfrac{2\sqrt{gh}\,\omega}{l}\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\sin\omega_n t}{\omega_n^2-\omega^2}, & 0\leq t\leq\dfrac{m\pi}{\omega}, \quad (46''')\\[2ex] \dfrac{2\sqrt{gh}\,\omega}{l}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\omega_n^2-\omega^2}\left[\sin\omega_n t-(-1)^m\sin\omega_n\left(t-\frac{m\pi}{\omega}\right)\right], & \dfrac{m\pi}{\omega}>t. \quad (47''')\end{cases}$$

If we use τ and u_0, (46‴) and (47‴) become exactly the same as (53) and (54).

(B) Ray Solution.

The expression which corresponds to (58) is obtained from (21′)

$$\eta_l = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\omega}{z^2+\omega^2}\,\frac{e^{zt}}{[\cosh \alpha z - \lambda \sinh \alpha z]}\,dz\,, \tag{58′}$$

where

$$\lambda = k\Big/\Big(\frac{\alpha z}{l}\Big), \tag{59}$$

and

$$\alpha = \frac{l}{\sqrt{gh}}\Big\{1-\frac{\tanh w}{w}\Big\}^{-1/2}.$$

By expanding the hyperbolic functions in power series, we obtain

$$\frac{1}{\cosh \alpha z - \lambda \sinh \alpha z} = \frac{2}{(1-\lambda)}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{1+\lambda}{1-\lambda}\Big)^n e^{-(2n+1)\alpha z},$$

so that (58) becomes

$$\eta_l = \frac{1}{\pi i}\,\frac{1}{(1-\lambda)}\sum_{n=0}^{\infty}\int_{c-i\infty}^{c+i\infty}(-1)^n\frac{\omega}{(z^2+\omega^2)}\Big(\frac{1+\lambda}{1-\lambda}\Big)^n e^{z[t-(2n+1)\alpha]}\,dz\,. \tag{61′}$$

In a similar way as in section 5, (61′) can be evaluated by the residue calculation at the poles $z=\pm i\omega$.

We obtain

$$\eta_l = 2\,Im\sum_{n=0}^{\infty}(-1)^n F e^{-(2n+1)lq\omega/\sqrt{gh}}\,H\Big[t-\frac{(2n+1)lp}{\sqrt{gh}}\Big], \tag{67′}$$

where

$$F=\frac{1}{1-\lambda_{i\omega}}\Big\{\frac{1+\lambda_{i\omega}}{1-\lambda_{i\omega}}\Big\}^n e^{i\omega t}, \qquad \lambda_{i\omega} = k\Big/\Big(\frac{\alpha i\omega}{l}\Big),$$

and

$$\frac{\alpha i\omega}{l} = X+iY\,, \qquad \begin{cases} X=\dfrac{\omega}{\sqrt{gh}}\,q\,, \\ Y=\dfrac{\omega}{\sqrt{gh}}\,p\,. \end{cases} \tag{74}$$

If we put

$$X+iY = re^{i\theta}, \quad r=\frac{\omega}{\sqrt{gh}}(p^2+q^2)^{1/2}, \quad \theta=\tan^{-1}\frac{p}{q}, \quad 0\leqq\theta\leqq\frac{\pi}{2}, \tag{75}$$

it follows that

$$\lambda_{i\omega} = Ze^{-i\theta}, \qquad Z = kr^{-1} = \frac{k\sqrt{gh}\;T}{2\pi\sqrt{p^2+q^2}}\,, \tag{76}$$

$$\frac{1}{1-\lambda_{i\omega}} = \frac{1}{1-Ze^{-i\theta}} = Se^{i\chi}\,, \tag{77}$$

where

$$\begin{cases} S=(1+Z^2-2Z\cos\theta)^{-1/2}, \\ \chi = \tan^{-1}\Big(\dfrac{-Z\sin\theta}{1-Z\cos\theta}\Big), \end{cases} \tag{78}$$

and

$$\frac{1+\lambda_{i\omega}}{1-\lambda_{i\omega}} = \frac{1+Ze^{-i\theta}}{1-Ze^{-i\theta}} = Re^{i\mu}, \tag{79}$$

where

$$\begin{cases} R = \left\{ \dfrac{1+Z^2+2Z\cos\theta}{1+Z^2-2Z\cos\theta} \right\}^{1/2}, \\ \mu = \tan^{-1}\left(\dfrac{-Z\sin\theta}{1-Z^2} \right). \end{cases} \tag{80}$$

Since

$$F = S\,R^n\,e^{i[\omega t+\chi+n\mu]},$$

we obtian

$$\eta_l = 2\sum_{n=0}^{\infty} (-)^n SR^n e^{-(2n+1)lq\omega/\sqrt{gh}} \sin(\omega t+\chi+n\mu)\, H\left[t-\frac{(2n+1)\,l}{\sqrt{gh}}p\right], \quad 0 \leqq t \leqq \frac{m\pi}{\omega}. \tag{81}$$

In a similar way, we have

$$\eta_l = 2\sum_{n=0}^{\infty} (-1)^n SR^n e^{-(2n+1)lq\omega/\sqrt{gh}} \left[\sin(\omega t+\chi+n\mu)\, H\left[t-\frac{(2n+1)lp}{\sqrt{gh}}\right] - (-1)^m \sin(\omega t+\chi+n\mu)\, H\left[t-\frac{m\pi}{\omega}-\frac{(2n+1)lp}{\sqrt{gh}}\right]\right], \quad \frac{m\pi}{\omega} \leqq t. \tag{82}$$

By use of τ and u_0, it follows that

$$\eta_l = \begin{cases} 2\sum_{n=0}^{\infty} (-1)^n e^{-\pi(2n+1)q/2u_0} SR^n \sin\left\{\dfrac{2\pi}{u_0}\left[\tau+\dfrac{u_0}{2\pi}(\chi+n\mu)\right]\right\} H\left[\tau-\dfrac{(2n+1)}{4}p\right], \\ \qquad 0 \leqq \tau \leqq \dfrac{m}{2}u_0, \qquad (83) \\ 2\sum_{n=0}^{\infty} (-1)^n e^{-\pi(2n+1)q/2u_0} SR^n \left[\sin\left\{\dfrac{2\pi}{u_0}\left[\tau+\dfrac{u_0}{2\pi}(\chi+n\mu)\right]\right\} H\left[\tau-\dfrac{(2n+1)}{4}p\right] \right. \\ \qquad \left. -(-1)^m \sin\left\{\dfrac{2\pi}{u_0}\left[\tau+\dfrac{u_0}{2\pi}(\chi+n\mu)\right]\right\} H\left[\tau-\dfrac{m}{2}u_0-\dfrac{(2n+1)}{4}p\right]\right], \\ \qquad \dfrac{m}{2}u_0 < \tau. \qquad (84) \end{cases}$$

We have assumed that k is written formally as (75). However, as will be made clear in the following, there is a certain restriction for the value of k in order that the boundary condition (71) represents an allowable physical condition at an actual bay head.

When $k=0$, we have seen that the amplitude of reflected waves is unity both at the head and mouth of the bay, whearcas, the phase change in reflection are 0 and π at the head and mouth, respectively. But, in the case of $k \neq 0$, as the bay head is no longer a loop of vertical motion of water particle, we may suppose generally that there occur changes in amplitude and phase. Also, the waves to be measured at the bay head must be affected by the given boundary condition.

As seen from (81) and (82), R and μ indicate respectively the ratio of the amplitude of the reflected waves to that of the incident one, and the change of phase in the reflection at the bay head. S and χ represents respectively the coefficient of reduction of amplitude and the phase change, of receiving waves at the bay head. To give these quantities physical significance, we must impose the condition

$$R < 1\,. \tag{85}$$

Since $\cos\theta > 0$, $r > 0$, we obtain from the former of (85), the relation

$$k < 0\,. \tag{86}$$

It is possible to show that the relation $S<1$ is automatically satisfied by (86).

Especially when $\nu=0$, it folllows from (62) that

$$q = 0\,, \qquad p = 1\,,$$

and from (76) that

$$\theta = \frac{\pi}{2}\,.$$

Correspondingly, (83) and (84) are to be replaced by

$$\eta_l = \begin{cases} 2\sum_{n=0}^{\infty}(-1)^n S_0 R^n{}_0 \sin\left[\frac{2\pi}{u_0}\left\{\tau - \frac{u_0}{2\pi}(\chi_0 + n\mu_0)\right\}\right] H\left[\tau - \frac{(2n+1)\,p}{4}\right], \\ \qquad\qquad 0 \leq \tau \leq \frac{m}{2}u_0\,, & (88) \\ 2\sum_{n=0}^{\infty}(-1)^n S_0 R^n{}_0 \left[\sin\left[\frac{2\pi}{u_0}\left\{\tau - \frac{u_0}{2\pi}(\chi_0 + n\mu_0)\right\}\right] H\left[\tau - \frac{(2n+1)}{4}p\right]\right. \\ \qquad \left. -(-1)^m \sin\left[\frac{2\pi}{u_0}\left\{\tau - \frac{u_0}{2\pi}(\chi_0 + n\mu_0)\right\}\right] H\left[\tau - \frac{m}{2}u_0 - \frac{(2u+1)\,p}{4}\right]\right], \\ \qquad\qquad \frac{m}{2}u_0 \leq \tau\,. & (89) \end{cases}$$

where

$$R_0 = 1\,, \qquad \mu_0 = \tan^{-1}\left(\frac{2Z}{Z^2-1}\right), \tag{89}$$

$$S_0 = (1+Z^2)^{-1/2}\,, \qquad \chi_0 = \tan^{-1}(-Z)\,. \tag{90}$$

8 Some Remarks

We have assumed that the energy of wave motion in the bay is dissipated only by the turbulent motion of the bay water. The energy dissipation due to the diverging waves from the bay mouth would be a possible mechanism ofr the decay of wave height. But, the model experiments such as by OGIWARA, S. T. NAKAMURA and OGIWARA show that the bay mouth becomes approximately a node for vertical motion and a loop for horizontal motion, of a water particle. From this, it may probably be that the transfer of water particles is quite free at the bay mouth, and for a considerable time interval from the begining of the record, the energy loss due to diverging waves may be negligibly small.

The coefficient of reflection at an actual bay head is not determined from a model experiment, since, in a usual model, the bay head is formed of an almost vertical cliff as a result of the difference in horizontal and vertical scales, which is required by the law of similitude. However, because an observed eigenperiod of a bay coincides fairly well with the calculated one basing on the assumption of vertical cliff, it seems that the reflection coefficient is almost unity at least for waves with such a long period as tunami. The reflection coefficient of waves with short period would be smaller than unity.

The coefficient of eddy viscosity varies with the scale of motion of the bay water. In case of invasion of tunamis into a bay, it will differ according as the motion is in a transient or a stationary stage. The value for ν in a stationary state can be obtained, by use of the mode solution, from the curve of damped oscillation traced in a marigram. While the value of ν in a transient state, especially in the initial stage, can be estimated from a decrease in wave height of the incident wave over the distance between the mouth and head.

As stated in section 4, there is a critical value for ν, if the motion of bay water is to be periodic, but, in case of intrusion of huge waves into a bay with a very compl icated shape and bottom topography, it would occur that the height of the first wave is much diminished owing to a highly turbulent motion, the value for ν being often larger than the critical one.

When β is negligibly small, the maximum water level at the bay head is determined from Fig. 7 as a function of u_0, with m a parameter. Some examples for this at the time of the Chilean tunami of 1960, and attempts to estimate the value of ν in certain cases will be described in another paper.

9 Summary

The motion of water in a rectangular bay with uniform depth, especially, the water level at the bay head is investigated theoretically, when a packet of sinusoidal long waves with period T is incident upon the bay mouth.

A brief introduction is given in section 1, and fundamental equations and formal solution are obtained respectively in sections 2 and 3, basing on the assumption that the bay head is composed of a rigid wall, and eddy viscosity is present in the water motion.

The water level at the bay head is obtained in two different ways, one leads to the mode solution, the other to the ray solution. These solutions are derived in sections 4 and 5. The mode solution which is composed of forced oscillation and free oscillations of infinite numbers of mode is adequate to investigate the later part of a marigram. The ray solution which is expressed by direct waves and an infinite numbers of ray reflected both at the head and mouth of bay, is convenient to study the earlier stages of the record, especially, to construct a theoretical marigram and to estimate the maximum water level which occurs in the initial part of the record.

In section 6, the maximum water level at the bay head is obtained as a function of $u_0=T/T_0$, taking as a parameter m, where T_0 is the eigen-period of the first mode free oscillation, and m is the number of crests and troughs contained in the incident wave

packet. At the resonant peroid $T=T_0$, the maximum water level is $2m$ times the amplitude of the incident waves, and m times the water level at the coast near the bay mouth, if the coast is assumed to be composed of a rigid wall. The results will be useful for a project of preventing damage.

When the boundary condition at the bay head is assumed as $\partial\eta/\partial x=k\eta$, the wave height is obtained in section **7**. The mode solution is very complicated, but the ray solution reveals clearly the effect of the assumed boundary condition. It is found that k must be negative if the coefficient of reflection at the bay head is to be less than unity. If $k<0$, the boundary condition assumed gives rise to a phase change and a decrease in reflection coefficient, and a reduction of wave height at the bay head.

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