The Airflow over the Mountain with Constant Wind Shear,1. Two-dimensional Solution

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The Airflow over the Mountain with Constant Wind Shear
I. Two-dimensional Solution

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Abstract

This paper presents a two-dimensional small perturbation approach to the problem of waves produced in a shearing air stream flowing over a ridge. The fundamental solution for a constant wind shear is calculated by Fourier transformation. The second boundary condition for the solutions is determined in a rigorous manner.

While there are some points of correspondence between these results and the single or two-layer's solution, there is a considerable degree of difference. The amplitude of the non-wave disturbance decreases with height and decays like x^{-1}. The lee-wave is composed of an infinite number of harmonic waves and several waves predominate over others on account of the resonance between the width of the mountain and the character of the air stream. In an extreme case the lee-wave cannot be appreciated on account of the interference of these wave components. Larger barriers cause an over-development of the waves, which have closed circulation or negative horizontal velocity.

1 Introduction

From a meteorological viewpoint, considerable interest is attached to the airflow over mountains. For this reason numerous studies have been carried out on this problem. The solution of meteorological equations is particulary difficult, because the behaviour of the atmosphere can be described almost completely only by five equations with five variables.

In the first place, the difficulty arises from that the many variables have been diminished by considering the two-dimensional motion, i.e. an infinite mountain range parallel to the y-axis which is crossed at right angle by air moving in the positive x-direction. Secondly, the method of approximation with simplified models has been applied. LYRA (1943) aud QUENEY (1947) have studied from the viewpoint of the perturbation theory. The underlying assumptions then are: The motion is stationary; the wind velocity u is considered as independent of height; the static stability is constant with height, so that the incompressible stratified model atmopshere can be used; and if the mountains are low and their slopes gentle, the perturbation theory may be applied. These studies have made it possible to explain several features of the motion.

In general the flow from the surface to the stratosphere has a considerable shear of velocity with height. In the westerlies, for example, the flow is roughly in the same direction at all leveles but several times faster aloft than near the surface.

Therefore these models may not be useful even if the solution is obtained for considerable amplitude.

SCORER (1949) has proposed as an approximate atmospheric model a simple two-layer model to calculate the streamlines as an air current, whose velocity varies with height. His example is one in which the atmosphere is divided into two layers in each of which Scorer's parameter l^2 has a constant but different value. In this model almost regular stational sine waves (lee-waves) form on the lee side of mountain ranges.

Multi-layer models with several different values of l^2 could in theory be set up and examined in a similar fashion. However, the mathematical treatment would become inordinately complex and no new general principle could emerge. CORBY and WALLINGTON (1956), applying the two-layer or three-layer model illustrate the effect of mountain and stability on the lee-wave amplitude. In order to examine the effects of the upper boundary and high-level conditions, CORBY and SAWYER (1958) studied the four-layer model, where the two lower layers representing the troposphere are surmounted by two higher layers representing the stratosphere. WALLINGTON and PORTNAL (1958) have computed the wavelength and amplitude of lee-waves in a 17-layer modely by a high-speed computer and compared them with those of the observed lee-waves.

There is really no evidence to support such multi-layer assumption and it appears to be desirable, for this and other reasons, to extend the theory to the model with continuous wind velocity. It is the purpose of this paper to present the results of this investigation.

2 Fundamental equation

The frictionless stationary flow in two-dimensional motion (in the X, Z plane, X-axis being along the general direction of the current and Z-axis being directed upward) has been given by the perturbation method, and the differential equation of vertical displacement obtained for small perturbation takes the form

$$\nabla^2\bar{\zeta} + \left(\frac{2}{\bar{u}}\frac{d\bar{u}}{dZ} - \beta - \frac{g}{c^2}\right)\frac{\partial\bar{\zeta}}{\partial Z} + \frac{g\beta}{\bar{u}}\zeta = 0, \tag{1}$$

where β is the static stability in the form $\beta = 1/\theta\cdot\partial\theta/\partial z$; θ the potential temperature; c, the velocity of sound; $\bar{u}$ undisturbed horizontal wind speed; and g, the gravity. If Z_0 is the height of a streamline far upstream and Z is the height at a considering point, the vertical displacement $\bar{\zeta}$ is defined by $\bar{\zeta} = Z - Z_0$. According to LONG (1953) the the linear equation (1) is valid even when the amplitude is considerably large under a special condition: the ratio $g\beta/c^2 \ll 1$ and $\bar{u}$ is constant. If $\bar{u}$ varies continously with height, the equations for waves are no longer linear when the amplitude is large. In this paper we concern with $\bar{u}$ which is not constant, thus we are obliged to apply (1) as a first approximation.

In order to discuss a model airstream, some simplifications are assumed.

i SCORER neglected the second and third factors of $\partial\zeta/\partial Z$ in (1) on the grounds that its effect is small and quantitative only, then,

$$\left|\frac{1}{\bar{u}}\frac{d\bar{u}}{dZ}\right| \gg \left|\beta + \frac{g}{c^2}\right| \qquad \text{and} \quad \beta + \frac{g}{c^2} \approx 0 \tag{2}$$

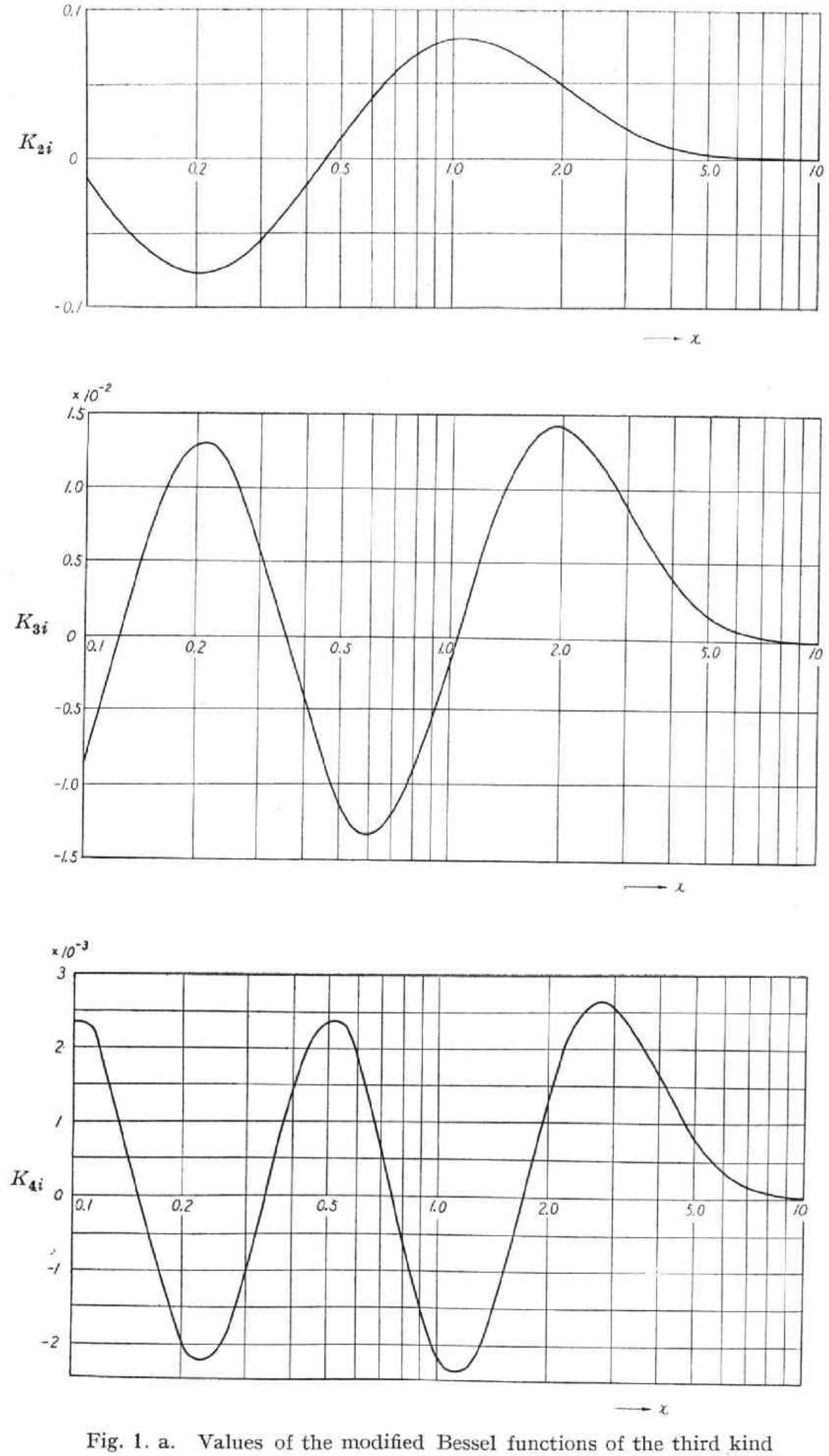

Fig. 1. a. Values of the modified Bessel functions of the third kind calculated by the author.

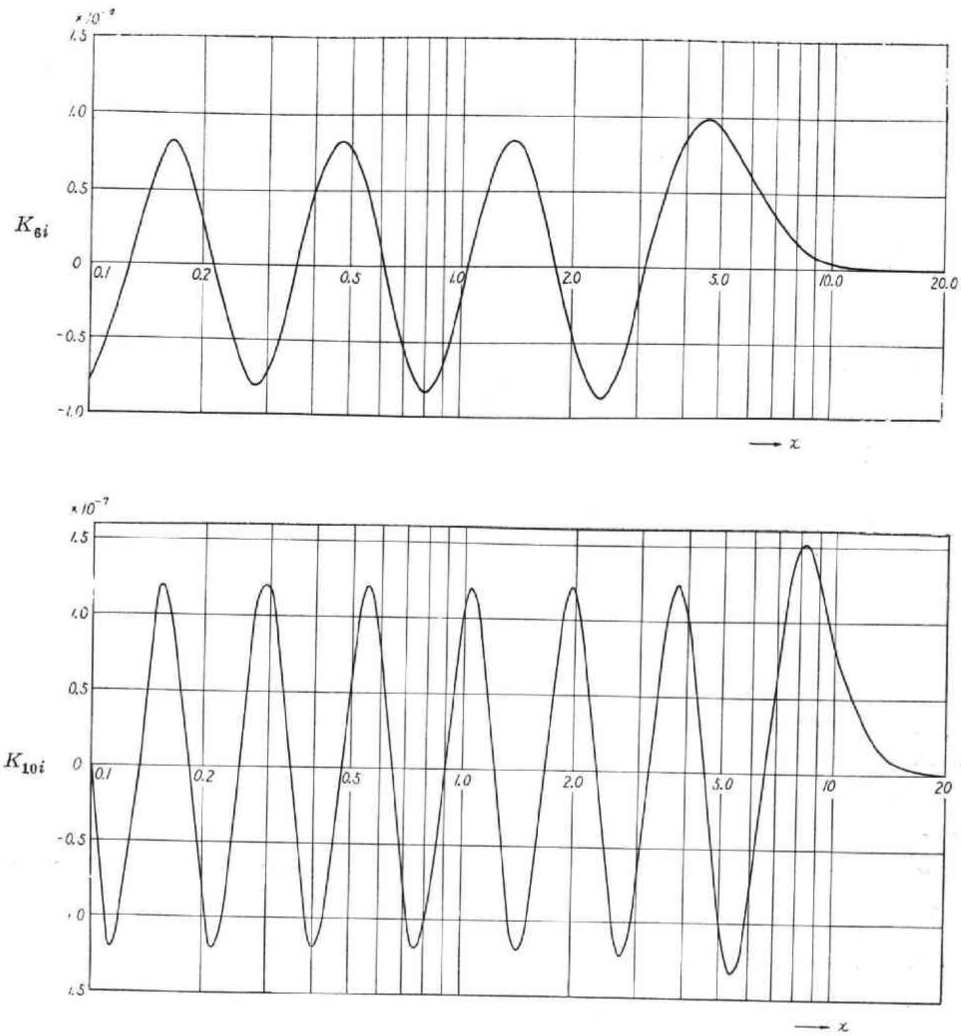

Fig. 1. *b*. Cont.

ii The static stability is consant through the whole atmosphere and the wind velocity $\bar{u}$ is a linear function of Z,

$$\bar{u} = A(L + Z) \tag{3}$$

Accordingly (1) becomes

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{1}{z}\frac{\partial \Psi}{\partial z} + \frac{l^2 - \frac{1}{4}}{z^2}\Psi = 0 \tag{4}$$

where Ψ, x, z are non-dimensional variables, given by

$$L + Z = Lz\,, \quad X = Lx\,, \quad \text{and} \quad \bar{\zeta} = L\, z^{-1/2}\,\Psi\,. \tag{5}$$

and where

$$l^2 = \left(\frac{g\,\beta}{\bar{u}^2}\right)_{z=1} = \frac{g\,\beta}{L^2 A^2}\left(=\frac{1}{R_i}\right) \tag{6}$$

R_i is the Richardson number.

If the solution of (5) is the form,

$$\Psi_k = K(kz)e^{ikx} \tag{7}$$

the simplified equation for $K(k, z)$ is

$$\frac{d^2}{dz^2}K(k,z) + \frac{1}{z}\frac{d}{dz}K(k,z) + \left(\frac{l^2-\frac{1}{4}}{z^2} - k^2\right)K(k,z) = 0\,. \tag{8}$$

The solution may be

$$K(k,z) = Z_m(ikz) \tag{9}$$

where $m^2=1/4-l^2$ and Z_m is a cylinder function.

We now consider the flow when a disturbance at infinity is diminished, the only solution of (8) may be given by

$$K(k,z) = K_m(kz) \tag{10}$$

where $K_m(x)$ is the modified Bessel function of the third kind.

The function $K_m(x)$ has the following properties: 1 $K_m(x)$ is real if x is real positive and m is real or pure imaginary, 2, $K_m(x)$ tends to zero if x increases without limit but remains real. 3, also K-function has an important property: it has zeros. If m is pure imaginary $K_m(x)$ has an infinite number of simple zeros which are real and positive, and $x=0$ is the limiting point of zeros. $K_m(x)$ has no complex zeros and has no zeros on the negative real axis.(1) K-function for pure imaginary m is shown in Fig. 1. Since in the stable atmosphere the Richardson number is positive, the value of m is pure imaginary or real and positive less than 1/4. Then $K_m(x)$ has zeros only when m is pure imaginary (or when $R_i<4$). The zero of $K_m(x)$ refers to lee-waves, and it will be seen in the following section.

3 Flow over an isolated hill

The simplified equation in non-dimensional unit used here is summarized as follows:

$$\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial z^2} + \frac{2}{z}\frac{\partial\zeta}{\partial z} + \frac{l^2-\frac{1}{4}}{z^2}\zeta = 0 \tag{12}$$

and one of its solutions is given by

$$\zeta_k = z^{-1/2}K_m(kz)\,e^{ikx}$$

By the principle of superpositions, the new solution can be constructed. If the bottom

(1) DYSON (1960) has proved that the confluent hypergeometric function $W_{k,m}(x)$ has no complex zeros when the index k is real while m is pure imaginary, and under these conditions, there is an infinite number of positive real zeros with a point of accumulation at zero. Using the fact

$$K_m(x) = \left(\frac{\pi}{2x}\right)^{1/2} W_{0,\,m}(x) \tag{11}$$

we can obtain the above properties for zero of $K_m(x)$ when m is pure imaginary.

topography is given in a non-dimensional unit,

$$H(x) = \int_{-\infty}^{\infty} (H_k \cos k x + L_k \sin k x)\, d k \tag{13}$$

the displacement of the stream lines along the ridge is given by

$$\zeta(x, z = 1+H) = \int_{-\infty}^{\infty} (H_k \cos k x + L_k \sin k x)\, d k \tag{14}$$

A simple boundary form, after QUENEY (1947), may be taken by

$$H(x) = \frac{h\, b^2}{b^2 + x^2} \tag{15}$$

where Lh is the mountain height and Lb is the half-width. Using the Fourier transformation we have

$$\zeta(x, z) = z^{-1/2} \mathcal{R} \int_0^{\infty} hb' \frac{K_{il}(\varepsilon k)}{K_{il}(k)} e^{-b'k + i x' k}\, d k \tag{16}$$

here

$$\varepsilon = \frac{z}{1+H}, \quad b' = \frac{b}{1+H} \quad \text{and} \quad x' = \frac{x}{1+H}$$

If h is infinitesimal (16) may be written

$$\zeta(x, z) = \mathcal{R} \int_0^{\infty} z^{-1/2} h\, b \frac{K_{il}(k z)}{K_{il}(k)} e^{-bk + i k x}\, dk \tag{17}$$

The simplified equation corresponding to (17) is applied by many writers.

If a ridge inclines at angle δ to the wind direction x-axis, (16) may be used with some modification, which becomes

$$\zeta(x, y, z) = z^{-1/2} \mathcal{R} \int_0^{\infty} h\, b' \frac{K_{il}(\varepsilon k)}{K_{il}(k)} e^{-k b' + i(x' \cos\delta - y' \sin\delta) k}\, d k \tag{18}$$

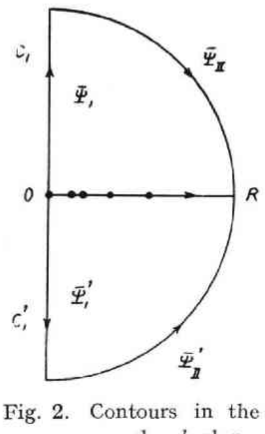

Fig. 2. Contours in the complex k-plane.

As far as the displacement of streamlines is concerned, the airflow over a long ridge presents an essentially two-dimensional problem. For this reason we shall examine only the airflow across a long ridge perpendicular to the wind.

Integration (16) can be evaluated by deforming the contour in the complex plane of k, noting that the integrand has no poles off the axes (see Fig. 2). Using instead of (16) a new integral

$$\psi = \int \frac{K_{il}(\varepsilon k)}{K_{il}(k)} e^{-k b + i k x}\, d k \tag{19}$$

we have

$$\psi = \psi_I + \psi_{II} + \pi i \quad (\Sigma \text{ residues at the poles}) \quad \text{for} \quad x > 0$$

and (20)

$$\psi = \psi_I' + \psi_{II}' - \pi i \quad (\Sigma \text{ residues at the poles}) \quad \text{for} \quad x < 0$$

We now assume from practical experience that the disturbance of the flow must decrease to zero upstream and we therefore annul the waves upstream by adding the term of residues on both sides of the mountain, according to CORBY and SAWYER (1958). Then,

$$\psi = \psi_I + \psi_{II} + 2\pi i \quad (\Sigma \text{ residues at the poles}) \quad \text{for} \quad x > 0$$

and (21)

$$\psi = \psi_I' + \psi_{II}' \qquad \text{for} \quad x < 0$$

The second term in each case (ψ_{II} or ψ_I') tends to zero as R tends to infinity on account of the exponential term.

The first term is written by

$$\psi_I = i \int_0^\infty \frac{H_{il}{}^2(\varepsilon k)}{H_{il}{}^2(k)} e^{-ikb-kx} dx \qquad (x > 0) \tag{22}$$

or

$$\psi_I' = -i \int_0^\infty \frac{H_{il}{}^1(\varepsilon k)}{H_{il}{}^1(k)} e^{ikb+kx} dx \qquad (x < 0) \tag{23}$$

The solution (16) is symmetric to x, so that ψ_I and ψ_I' are symmetric. It is sufficient to consider only ψ_I'.

In this case an asymptotic representation is obtained as follows: For a large value of l, the asymptotic expansion is applied.

$$\pi H_{il}{}^1(z) = (2x)^{1/2}(l^2+z^2)^{-1/4} \exp\left\{i\,(l^2+z^2)^{1/2} - i\,l \sinh^{-1}\frac{l}{x} + \frac{l\pi}{2} - i\frac{\pi}{l}\right\} + 0\,(x^{-2}) \tag{24}$$

then

$$\frac{H_{il}{}^1(k\varepsilon)}{H_{il}{}^1(k)} \doteqdot \left(\frac{1+\varepsilon^2k^2l^{-2}}{1+k^2l^{-2}}\right)^{-1/4} \exp\left[il\left\{\left(1+\frac{\varepsilon^2k^2}{l^2}\right)^{1/2} - \left(1+\frac{k^2}{l^2}\right)^{1/2} + \log\left(\varepsilon\frac{1+\sqrt{1+k^2/l^2}}{1+\sqrt{1+\varepsilon^2k^2/l^2}}\right)\right\}\right] \tag{25}$$

(25) is a good approximation formula even when k is small.

Let us calculate (23), by applying (25) and the principle of the stationary phase. We consider the integral

$$\int_0^\infty \left(\frac{1+\varepsilon^2k^2/l^2}{1+k^2/l^2}\right)^{-1/4} e^{ikb+kx} \exp\left[il\left\{\left(1+\frac{\varepsilon^2k^2}{l^2}\right)^{1/2} - \left(1+\frac{k^2}{l^2}\right)^{1/2} + \log\left(\varepsilon\frac{1+\sqrt{1+k^2/l^2}}{1+\sqrt{1+\varepsilon^2k^2/l^2}}\right)\right\}\right] dk$$
$$= \int_0^\infty \left(\frac{1+\varepsilon^2k^2/l^2}{1+k^2/l^2}\right)^{-1/4} e^{ikb+kx}\, e^{if(k)}\, dk \tag{26}$$

When $|ib+x|$ is small, we may expect, according to the principle of interference, that an approximate value of the integral will be determined from a consideration of the integral in the neighbourhood of zero. Zero is a value of the k which makes the phase of $e^{f(k)}$ stationary.

This relation will be obtained when $|ib+x|$ is small. When x is large, the part of the integral outside the range δ of the value of k is negligible owing to the exponential term e^{kx}, $(x>0)$. The approximation is adequate when l is large and only then it will meet the practical purposes.

On considering the principle of the stationary phase, we have

$$\begin{aligned}\psi_I' &\doteqdot -i\int_0^\infty \frac{J_{il}(\varepsilon k)}{J_{il}(k)}\, e^{ikb+kx}\, dk \\ &= -i\int_0^\infty e^{ikb+kx}\,\varepsilon^{il}\sum_{m=0}^{\infty}\left\{\frac{J_{il+m}(k)}{J_{il}(k)}\left(\frac{1-\varepsilon^2}{2}k\right)^m\right\} d\,k \end{aligned} \tag{27}$$

according to the formula of Bessel functions (ERDÉLYI 1953, p. 66). Applying the approximation

$$\sum_{m=0}^{\infty}\frac{J_{il+m}(k)}{J_{il}(k)}\left(\frac{1-\varepsilon^2}{2}k\right)^m \doteqdot \Gamma(1+il)\left(\frac{\varepsilon^2-1}{4}k^2\right)^{il/2} J_{il}\left(\sqrt{\varepsilon^2-1}\,k\right)$$

We see that

$$\psi_I' = \frac{i\,\varepsilon^{il}}{[(x+i\,b)^2+\varepsilon^2-1]^{1/2}}\, F\left(\frac{1}{2},\, il;\, 1+il;\, \frac{\varepsilon^2-1}{(x+i\,b)^2+\varepsilon^2-1}\right) \tag{28}$$

according to WATSON (1952, § 13.2, (3)), where $F\,(\alpha,\,\beta\,;\,\gamma\,;\,z)$ is the hypergeometric function.

By using the approximation of the hypergeometric function (28) can be expressed as

$$\psi_I' = \frac{i\,\varepsilon^{il}}{\left[(x+i\,b)^2+\dfrac{\varepsilon^2-1}{1+il}\right]^{1/2}} \tag{29}$$

Particularly when ε^2-1 is small, we can rewrite this in a simpler form

$$\begin{aligned}\psi_I' &= \frac{i\,\varepsilon^{il}}{x+i\,b} \\ &= \frac{1}{x^2+b^2}\left[b\cos(l\log\varepsilon) - x\sin(l\log\varepsilon) + i\left\{x\cos(l\log\varepsilon) + b\sin(l\log\varepsilon)\right\}\right]\end{aligned} \tag{30}$$

These representations are useful for $\varepsilon \leqslant 2$ or $l \gg 1$, but for $\varepsilon \geqslant 2$ and small x these formula become invalid and another approximation will be required.

When s^2 is small, the hypergeometric function may be given by

$$\begin{aligned} F\left(\frac{\alpha}{2},\,\frac{\beta}{2}\,;\,\frac{\alpha+\beta+1}{2}\,;\,1-s^2\right) &= \frac{\pi^{1/2}\Gamma\left(\dfrac{\alpha+\beta+1}{2}\right)}{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\dfrac{\beta+1}{2}\right)} F\left(\frac{\alpha}{2},\,\frac{\beta}{2}\,;\,\frac{1}{2}\,;\,s^2\right) \\ &\quad - \frac{2\,\pi^{1/2}\Gamma\left(\dfrac{\alpha+\beta+1}{2}\right)}{\Gamma\left(\dfrac{\alpha}{2}\right)\Gamma\left(\dfrac{\beta}{2}\right)}\, sF\left(\frac{\alpha+1}{2},\,\frac{\beta+1}{2}\,;\,\frac{3}{2}\,;\,s^2\right)\end{aligned}$$

By the aid of this formula, we obtain the result

$$\psi_I' = \frac{i\,\varepsilon^{il}}{\{(x+i\,b)^2+\varepsilon^2-1\}^{1/2}}\left\{(i\,l\,\pi)^{1/2} - \frac{2\,il\,(x+i\,l)}{\{(x+i\,b)^2+\varepsilon^2-1\}^{1/2}}\right\} \tag{31}$$

This is more suitable if we are concerned with the neighbourhood of the ridge and with the upper layer, $|x+ib|^2 \leqslant \varepsilon^2-1$.

As a result of the symmetric property we see that,

$$\mathcal{R}\{\psi_I(x, \varepsilon) = \mathcal{R}\{\psi_I'(-x, \varepsilon)\} \tag{32}$$

This "non-wave" term represents a local disturbance, dying out rapidly on both sides of the mountain.

Next, we shall evaluate the residues at the poles (wave term), The pole occurs at the zero of $K_{il}(k)$ as a simple pole. The number of poles is infinite, then we have

$$\Sigma \text{ residues} = \sum_{m=0}^{\infty} \frac{K_{il}(\varepsilon k_m)}{\left(\dfrac{d K_{il}(k)}{d k}\right)_{k=k_m}} e^{-k_m b+i k_m x} \tag{33}$$

where

$$k_0 > k_1 > \cdots\cdot$$

In order to examine the residue of the pole for a large value of n, $K_{il}(k)$ can be approximated as follows: The modified Bessel function $K_{il}(k)$ becomes

$$K_{il}(k) = \frac{\pi e^{-\alpha}}{\sinh l\pi} \sin\left(\gamma - l \log \frac{k}{2}\right) \qquad \text{for} \quad 0 < k \leqslant 1$$

where

$$\Gamma(1+il) = e^{\alpha+i\gamma}$$

When ε is small, the pole is given by

$$\gamma - l \log \frac{k_m}{2} = \pi, 2\pi, \cdots\cdot \tag{34}$$

This result shows that the ratio k_{m+1}/k_m is nearly equal to $e^{-l/\pi}$ and that the value of residue tends to zero in proportion to the value of the pole when n increases.

When l is small and when we stop at the n-th term in (33), the error becomes very small. When a better approximation is required or l is large, it is necessary to sum up to the considerable terms. In this case the wave term may be split into two terms,

$$\Sigma = \sum_{m=0}^{\mu-1} + \sum_{m=\mu}^{\infty} = \Sigma_1 + \Sigma_2$$

$$\Sigma_2 = \frac{K_{il}(k_\mu \varepsilon)}{\left(\dfrac{d K_{il}(k)}{d k}\right)_{k=k_\mu}} \sum_{m=0}^{\infty} \exp\left\{-\frac{m}{p}\pi + k_\mu(-b'+ix')\, e^{-\frac{m}{p}\pi}\right\} \tag{35}$$

Substituting $e^{-ml/\pi} = \theta$, Σ_2 is transformed into an integral form, and we obtain,

$$\Sigma_2 = \frac{K_{il}(\varepsilon k_\mu)}{\left(\dfrac{d K_{il}(k)}{d k}\right)_{k=k_\mu}} \frac{l}{\pi} \frac{1-e^{(-b'+ix')k_\mu}}{(b'-ix')\, k_\mu} \tag{36}$$

Especially when $|(b'-ix)\, k_\mu| \geqq 1$, (36) becomes

$$\Sigma_2 = 0 \tag{37}$$

Solution (33) represents down-stream waves, whose amplitudes do not decrease and which are called lee-waves.

By using (19), (22), (28), (29), (30), (33) and (36) the final results are shown as

$$\zeta(x,z)=\mathcal{J}\left\{-h\,b'\,\varepsilon^{-1/2+il}\left[(x'+i\,b')^2+\frac{\varepsilon^2-1}{1+i\,l}\right]^{-1/2}\right\}\qquad \text{for}\quad x<0 \tag{38}$$

$$\begin{aligned}\zeta(x,z)=\mathcal{J}\Bigg\{&-h\,b'\,\varepsilon^{-1/2+il}\left[(x'+i\,b')^2+\frac{\varepsilon^2-1}{1+i\,l}\right]^{-1/2}\\&-2\,\pi\,i\,h\,b'\sum_{m=0}^{\mu-1}e^{-k_m b'}\sin k_m x'\,\frac{K_{il}(k_m\,\varepsilon)}{\left(\frac{d\,K_{il}(k)}{d\,k}\right)_{k=k_m}}\\&+2\,h\,b'\,\varepsilon^{-1/2}\,\frac{1-e^{(-b'+ix')k_\mu}}{(b'-i\,x')\,k_\mu}\cdot\frac{K_{il}(k_\mu\,\varepsilon)}{\left(\frac{dM_{il}(k)}{d\,k}\right)_{k=k_\mu}}\Bigg\}\qquad \text{for}\quad x>0\end{aligned} \tag{39}$$

When $\varepsilon\approx 1$

$$\zeta(x,z)=\frac{h\,b}{\varepsilon^{1/2}(x^2+b^2)}\left\{b\cos(l\log\varepsilon)-x\sin(l\log\varepsilon)\right\}\qquad x<0$$

And when $|x'+i\,b'|\leqslant\varepsilon^2-1$

$$\zeta(x,z)=\mathcal{J}\left\{\frac{\varepsilon^{il-1/2}}{(x'+i\,b')^2+\varepsilon^2-1}\left[(i\,l\,\pi)^{1/2}-\frac{2\,i\,l(x'+i\,b')}{[(x'+i\,b')^2+\varepsilon^2-1]^{1/2}}\right]\right\}\qquad x<0$$

4 Indeterminacy of solution

The solution for the airflow over mountains is given by the summation of terms, ψ_I and the residues. The first term describes the flow in the neighbourhood of the mountain, the other describes the lee-waves, and is absent when $x<0$. It is remarkable that these terms are obtained from (10). In general the solution of (8) is given by

$$K(k,z)=A\,Z_m^{(1)}(i\,k\,z)+BZ_m^{(2)}(i\,k\,z) \tag{40}$$

where A and B are arbitrary functions of k and Z-function are independent Bessel functions. We can determine the constants by the two boundary conditions; the ground boundary condition that is called a second (upper) boundary condition. Assuming, as a second boundary condition, the disturbance vanishing for high altitude, (10) can be unique.

There has been some discussion (SCORER 1958, a, b, CORBY and SAWYER 1958, b, PALM 1958) on how to determine the unique solution. The outline is as follows: The simplified differential equation for (8) may be given by

$$\frac{d^2K(k,z)}{d\,z^2}+\left(\frac{g\,\beta}{\bar{u}^2}-k^2\right)K(k,z)=0 \tag{41}$$

and if $g\,\beta/\bar{u}^2=l_0^2$ is independent of z, and if $l_0^2>k^2$, (41) has a solution

$$K(k,z)=C\,e^{i\mu z}+D\,e^{-i\mu z} \tag{42}$$

where

$$\mu^2=l_0^2-k^2$$

Both constants C and D are indeterminate for $z=\infty$ and so there is some difficulty about the upper boundary condition. For a Fourier component cos k_x SCORER (1958. a, b) proposed the solution in the form

$$\zeta = \cos \mu z \cos k x \tag{43}$$

This comes from his opinion, "All waves are excluded which are not directly attributable to the mountain. Clearly any wave motion, however arbitrary in from, which is zero at the ground can be added to any wave motion for a given mountain, but in order to obtain a practical solution these waves must be ignored unless a second boundary condition such as a rigid lid requires their presence. In the absence of a special second condition waves corresponding to no disturbance at the ground should be excluded for the same reason that lee-waves on the upstream side of the mountain are excluded."

Then the flow over the ridge is

$$\zeta = \frac{h b^2}{x^2+b^2} \cos l_0 z \quad (+\text{ lee waves}) \tag{44}$$

Different writers have disagreed with him in the choice of a second boundary condition. QUENEY (1947) overcomes the difficulty by assuming a small amount of friction, originally used by RAYLEIGH (LAMB 1945), and obtains the solution of (41),

$$K(k, z) = \cos (k x + \mu z) \tag{45}$$

and the flow over the ridge is

$$\zeta = \frac{h b^2}{b^2+x^2} \left(b \cos l_0 z - x \sin l_0 z\right) \quad (+\text{lee waves}) \tag{46}$$

The principal advantage of this friction is that it does not in fluence the boundary conditions, but it is an artifice.

It has been proposed to treat this problem as an initial value problem, (see WURTELE 1953 and PALM 1953). The waves are created during some initial interval and approach a stationary form. It is found that the solutions are in agreement with (46). This indeterminacy may be eliminated by various other methods. CORBY and SAWYER (1958 a) assume that the atmosphere has a rigid lid on top, and then let the height of the lid become infinite. The radiation condition may be applied (ELIASSEN and PALM 1954). These results are identical with (46).

It is not the object of this paper to discuss this problem in detail, but let us consider this problem in our model, which has a shearing wind velocity. Make l large, (30) becomes

$$\mathcal{R}\,(\psi_1') = \mathcal{R} \int_0^\infty e^{-bk + ikx + il \log \varepsilon}\, dk \qquad (x < 0) \tag{47}$$

and (32) becomes

$$\mathcal{R}\,(\psi_1) = \mathcal{R} \int_0^\infty e^{-bk + ikx - il \log \varepsilon}\, dk \qquad (x > 0)$$

Replacing the summation by integration, (35) reduces to

$$i \pi \Sigma_2 = i \int_0^\infty e^{(-b+ix)k} \sin (l \log \varepsilon)\, dk - \frac{i\, e^{(-b+ix)k_\mu}}{b-ix} \sin (l \log \varepsilon) \tag{48}$$

Substituting in (22)

$$\zeta = \mathcal{R}\ \frac{h\,b\varepsilon^{il-1/2}}{b-i\,x} \qquad x<0$$

$$\zeta = \mathcal{R}\left\{\frac{h\,b}{b-i\,x}\,\varepsilon^{il-1/2} + h\,b\,\varepsilon^{-1/2}\left[-\frac{i\,e^{(-b+ix)k_\mu}}{b-i\,x}\sin(l\log\varepsilon) + 2\pi i\,\Sigma_1\right]\right\} \quad x>0 \tag{49}$$

(49) may be rewritter

$$\zeta = \frac{h\,b^2\,\varepsilon^{-1/2}}{b^2+x^2}\left\{b\cos(l\log\varepsilon) - x\sin(l\log\varepsilon)\right\} \quad (+\ \text{lee waves}) \tag{50}$$

Especially when b is large, the last term tends to zero.

Assuming $\varepsilon \approx 1$, this limit corresponds to (46), because $\log\varepsilon \fallingdotseq \varepsilon - 1$. In this model the solution is equivalent to (46).

The above result is based on the assumption that "all lee-waves occur only on the lee side of the mountain". Mathematically, there is no reason for having waves only on the lee side. However, from physical considerations it is easily seen that the definite lee-waves created by the mountain must always be behind the mountain. This assumption, originally used by KELVIN (LAMB 1945) in studying the surface wave, was applied by CORBY and SAWYER (1958 a). But when b is too large, as shown in (50), the wave term has no longer a wave form, even though the formal Fourier component of lee-waves can occur. In such a case the above assumption seems to be a more assumption whose plausibility cannot be confirmed either mathematically or physically.

This result shows that this is another method leading to the conclusion (46). But it does not mean that SCORER's assertion is incorrect. I suppose that if the effect of friction is discounted very long lee-waves can occur on both sides with non-wave form. Actually we substitute (48) in (21), then we have,

$$\zeta = \frac{h\,b^2\,\varepsilon^{-1/2}}{b^2+x^2}\cos(l\log\varepsilon) + \text{wave term}$$

the first term is equivalent to the non-wave solution found by SCORER. If we assume that only definite lee-waves occur on the lee side, the solution is represented by (44), which means that in the non-viscous fluid the flow is represented by a symmetric non-wave term and lee-waves on the lee side. But in the atmosphere, which is viscous fluid, the airflow is probably represented by an asymmetric function, because the air flow is not symmetric to x on account of Reynold's stress being asymmetric to x. It may be possible to decide a solution by substraction or addition of a wave term with a factor of some value. We cannot decide on a plausible solution. In this paper we obtain solutions (38) and (39) based on the assumption that an infinite train of waves which is represented by residues at the poles appears only on the lee side of the mountain.

It is noted that, if the wave term is represented by (50), the train of lee-waves does not occur on the lee side with a definite form. This means that for a certain value of b Fourier components of lee-waves interfere with each other and the resultant wave has no more a wave form. The condition of occurrence of a definite lee-wave is given by

$$-\frac{e^{(-b+ix)k_\mu}}{b-ix}\sin(l\log\varepsilon)+\Sigma_1\approx 0$$

or empirically by

$$\frac{1}{k_0}\ll b<\frac{10}{k_0}$$

The second condition $1/k_0 \gg b$ is necessary, because for a large value l, $(k_0 \gg 1)$ and for a very small b, we have

$$h\,b\,e^{-k_m b}\approx h\,b\approx 0$$

Concluding we can show that the lee-waves occur only when b has an adequate value in according to the atmospheric character as long as h is constant.

We shall compare (49) with LYRA's (1943) solution. His solution for a "point mountain" is essentially the same as this solution with $b=0$, hb=const. $(=M/\pi)$ and $\varepsilon=1$. Roughly to estimate the lee-waves, we assume that all poles are represented by (34). This assumption is valid for a small eigen value but when k_μ is large this representation becomes useless. For example $l=10$, the largest zero is 6.6, whereas the value calculated by (34) is 5.8. But (34) may be applied as far as the qualitative feature is concerned. The wave term is approximated by

$$\zeta_W=\frac{2M(\varepsilon-1)}{\pi}\,\frac{l}{x}\,(1-\cos k_0 x)$$

This is an equivalent formula to LYRA's but with a different form which is caused by a different ground shape and an unsuitable approximation method.

In addition, we shall touch the dependence of lee-wave amplitude on ridge width. CORBY and WALLINGTON (1956) have investigated the variations of the amplitude of the two-layer waves, and they show that if b is increased with h constant or if b and h are increased in the same proportion, the amplitude reaches a sharp maximum and then falls off rapidly. Thus the width is the important parameter of the ridge as far as the wave amplitude is concerned. According to CORBY and WALLINGTON $b=k^{-1}$ is the value which excites the largest amplitude lee-wave and it is often like "resonance". The occurrence of definite lee-waves is related to the resonance and these phenomena will be examined with examples in a later section.

5 Solution for finite height of the mountain

For simplicity we shall consider this for large b and l, in which case flow is given by (49). We are interested in the disturbance near the ridge, $x\ll b$, where (49) reduces to

$$\zeta=\mathcal{R}\left\{h\,z^{-1/2+il}\right\} \tag{51}$$

For the finite mountain height h, the horizontal velocity u rapidly fluctuates with height especially near the ridge. This may be shown analytically as follows:

$$u=-\frac{\partial\phi}{\partial z}=-\frac{\partial\phi}{\partial z_0}\frac{\partial z_0}{\partial z}=\bar{u}\left(1-\frac{\partial\zeta}{\partial z}\right) \tag{52}$$

where ϕ is the stream function and z_0 is the height of streamline ϕ =const far upstream. From (51) and (52) we have

$$u \doteqdot \bar{u}_0 z \left\{1 + h z^{-3/2}\left(l^2 + \frac{1}{4}\right)^{1/2} \sin(l \log \varepsilon + \Delta)\right\}$$

where

$$\tan \Delta = \frac{1}{2l}$$

The wind velocity is maximum nearly at $l \log \varepsilon + \Delta = \pi/2$ and if $l h \gg 1$, the horizontal velocity is zero at some level which is determined by

$$h z^{-3/2}\left(l^2 + \frac{1}{4}\right)^{1/2} \sin(l \log \varepsilon + \Delta) = -1 \tag{53}$$

Above this level that flow may be opposite to the main stream.

If h is finite and l is large, the flow near the surface has very high velocity and therefore very high shears. It is to be expected that this will lead to a breakdown into turbulence so that the solution, if it does exist, may be unstable. Stronger instability (overturning instability, after LONG 1955) may occur near the level where the flow has a negative velocity, and this means the existence of rotor-closed circulation. Although a rotor exists in a stable atmosphere, any disturbance will lead to some sort of mixing within the rotor until all the atmosphere has the same potential temperature. Static instability is produced in the rotor so that steady motion could not persist. In other words this theory can safely be applied only for small values of mountain height.

Criterion of overturning instability (after LONG 1955) is derived

$$\min\left[1 + h z^{-3/2}\left(l^2 + \frac{1}{4}\right)^{1/2} \sin(l \log \varepsilon - \Delta)\right] \leq 0 \tag{54}$$

Therefore we shall have more interest in only small h which is given by

$$h \leq \frac{e^{q\pi/4l}}{l} \tag{55}$$

We note that the lower the Richardson number, the smaller the mountain height must be which will avoid overturning instability.

Next we shall investigate for finite b, in which case the solution is not given by a simple form. The problem of obtaining the information similar to the above is one of laborious computation. In many calculations the closed circulation near the mountain appears for large l and h. It must be remembered that only smaller values of h will eliminate these cells. Besides the rotor near the mountain, strong rotors often appear near the ground in the remote distance of the mountain lee side.

In this section we examine well developed lee-waves or rotors. In general there are an infinite number of lee wave lengths, and the ratio of wave length between ajoint lee-waves is approximately $e^{-\pi/l}$ especially for large wave lengths. When l is large and b finite, the composed lee-waves have small amplitude of wave form on account of interference. They are apt to appear in large amplitude waves or rotors in the atmosphere with strong shear. According to CORBY and WALLINGTON (1956) the other factors controlling the lee-wave amplitude are mountain size and shape, which are discussed in the previous section. Consequently we may conclude that the large amplitude lee-wave or rotor appears under the following conditions: Large h, suitable width $b=k^{-1}$ and strong wind shear or small l^2.

According to SCORER (1949), the conditions of occurence of well developed lee-waves are that the wind direction is almost constant with height up to a considerable height and that an upper layer of low value of l^2 is above a lower layer of high value of l^2. These results are in agreement with us.

Lastly in this section we shall add a few words on the effect of the upper boundary. It is of interest to consider the effect of introducing a rigid lid roughly as a tropopause, at height z_2 the solution of (8) is shown by

$$K(k, z) = I_m(k z) - \frac{I_m(k z_2)}{I_{-m}(k z_2)} I_{-m}(k z) \tag{56}$$

where $I_m(x)$ is the modified Bessel function of the first kind. When m is pure imaginary, $I_m(x)$ is conjugate to $I_{-m}(x)$ and argument of $I_m(x)$ tends to zero as x increases

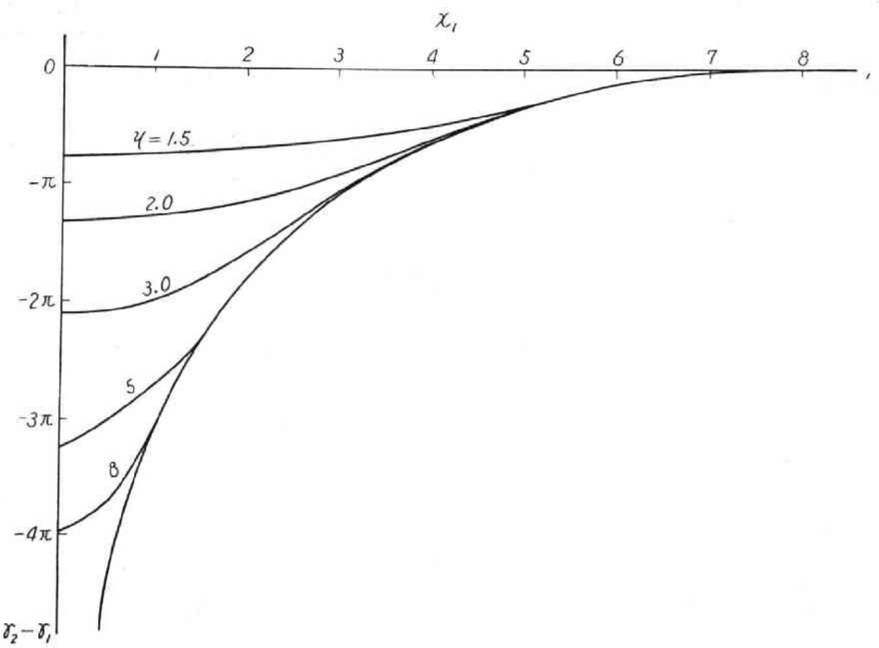

Fig. 3. The differnce of argument of modified Bessel function of the first kind. Putting

$$I_{6i}(x_1)=e^{\alpha_1+\gamma_1 i} \quad \text{and} \quad I_{8i}(x_2)=e^{\alpha_2+\gamma_2 i}$$

$\gamma_2-\gamma_1$ is shown as a function of x_1. Lines of equal $\eta=x_2/x_1$ are entered as auxiliary lines.

to infinity. For example, the difference in argument of $I_{6i}(x)$ and $I_{-6i}(x)$ is shown in Fig. 3. When this difference is integer times π, the value of x is the zero of (56). The greatest zero varies very little when $z_2/z=\eta$ changes from ∞ to 2, this value increases from 3.15 at $\eta=\infty$ to only 2.70 at $\eta=2$. But when $\eta<2$, the variation in the value of the greatest zero becomes great and there is no zero for $\eta<1.689$. For the second zero, the variation with η is greater than that of the greatest zero, and they have no zero when $\eta<2.850$. This and several computations suggest that it would give a good approximation to the flow of an unbounded atmosphere unless the lid be very low.

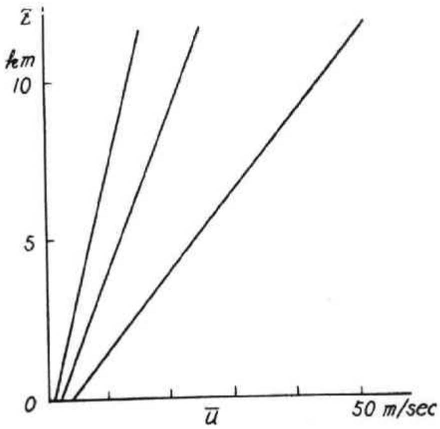

Fig. 4. Models of wind used in this paper.

6 Examples

In order to examine the airflow over the ridge with numerical examples, suitable values must be assigned to the constant parameter, L. l etc.. Fig. 4 is a graph of the wind profile used in this computation. Taking $\beta=1.4\times10^{-7}\,\text{cm}^{-1}$, the following values are obtained.

$\bar{u}_{z=0}$	4.0 m/sec	2.0	1.2
l	2.96	5.91	9.86
$\|m\|$	2.92	5.89	9.85

In this paper, for simplicity, the following values are assumed,

$$L=1\text{ km}$$
$$l(\fallingdotseq|m|)=3,\ 6\text{ and }10$$

First, we shall examine the flow for $l=3$. The largest eigen value of $K_{3i}(k)$ is 1.02 and others are 0.35, 0.12, 0.043 etc. So far as the perturbation method, is used, the amplitude of disturbances is proportional to the mountain height, and also it will vary with mountain width. According to CORBY and WALLINGTON (1956) $b=k^{-1}$ is the value which excites the largest amplitude lee-wave and it often resembles the resonance which is mentioned in §4. The variation of lee-wave amplitude with mountain width is illustrated in Fig. 5. In a discussion of the lee wave for the largest k_0, a maximum amplitude is attained in the lower layer and in the upper layer the amplitude falls off exponentially (see Fig. 6). The waves for smaller k have finite numbers of nodal surfaces, one for k_1, two for k_2, and so on. It is interesting to see here that the maximum amplitude in the upper layer is smaller than that in the lower layer. Many of these longer waves often attain a greater amplitude in the upper layer than that of short waves even when their amplitude is negligible in the lower layer.

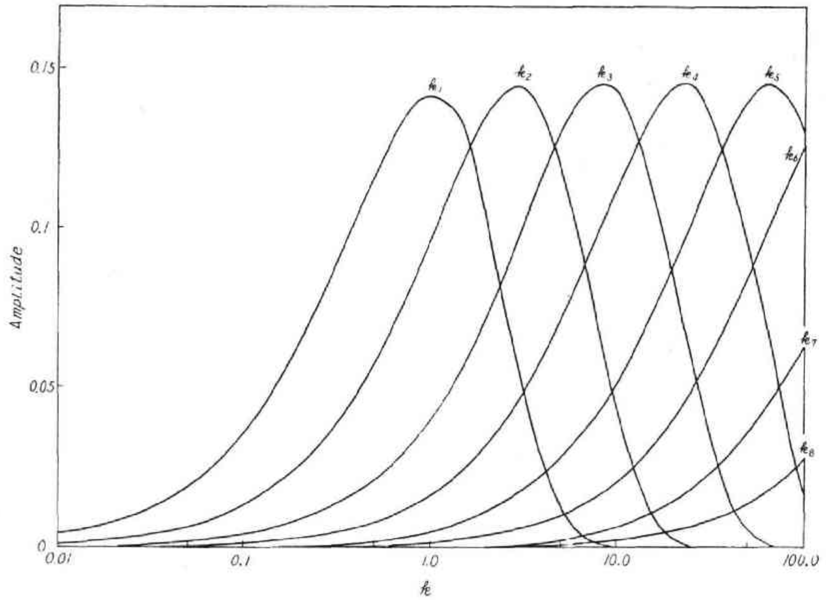

Fig. 5. Variation of lee wave amplitudes with size of mountain, when mountain height is kept constant, $h=1$. The amplitude is given by

$$k\,b\,e^{-bk}\,\frac{K_{zi}(k_n\,\varepsilon)}{\left(\frac{d\,K_{zi}(k)}{\sigma\,k}\right)_{k_2k_3}}.$$

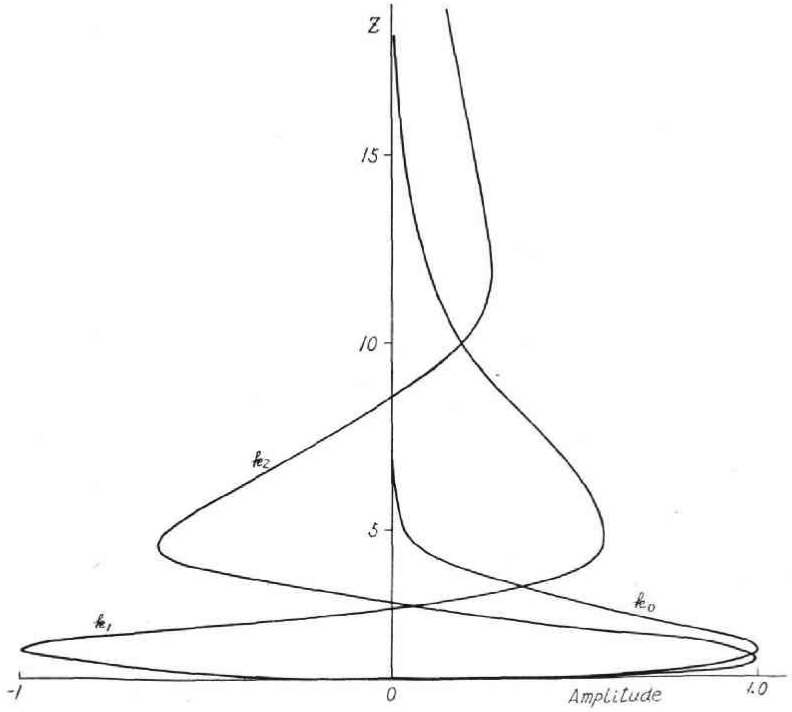

Fig. 6. Variation of amplitude with height, each lee-wave being associated with its optimum-width ridge. $l=3$. But the sign for k_1 is reverse.

The variation of airflow with mountain height is shown in Figs. 7–9, where b has been taken as unit, nearly equal to k_0^{-1}. The amplitude increases with height h. When the ridge is low, $h=0.20$, the lee-waves appear as infinite wave trains on the lee side of the mountain. When h is increased to 0.35 there is one point at the ground under the wave crest near $x=16.2$ at which the velocity is zero. With $h=0.50$, the amplitude of lee waves is so large that a well-developed rotor appears with reversed flow at the ground. The condition of the existence of the rotor has been described in the previous section. It is shown in these figures that the disturbances die away with height in the upper layer and no disturbances can be detected at levels higher than 10 km, even near the ridge. This contradicts the study of the two-layer model (SCORER 1949) in which the amplitude of disturbances increases upward for a wide range of wave length. The cause of this discrepancy is the difference of the model. The solution for $l=$ constant is given by the harmonic functions (42), whereas the solution in this model is represented by the modified Bessel function (10), which decreases with height but behaviour near the ground coincides with each other. One of the remarkable features of the airflow is the strong surface wind on the lee slope under lee-wave troughs. In Fig. 7 strong surface winds are found on the lee side $x=1.5$ and apart from the ridge, $x=7.3$, 13.5 etc. The strong wind on the lee slope is often observed and this calculation may be applied in explaining this observation.

The main features of calculations for various b are revealed in Figs. 9–14 in which $h=0.5$. Fig. 10 shows the flow over a very narrow mountain, $b=0.1$. The amplitudes of lee-waves are negligible and the motion involves mainly symmetrical elevation of the streamlines over the mountain. When b increases to 0.3 (Fig. 11) a wave with small amplitude appears in the lee side and, practically, it consists of only a single sine wave whose wave number is $2\pi k_0$. The disturbances at levels heigher than 7 km are negligible.

Fig. 9 is an example for flow in resonance. The barrier is of the optimum width for exciting the shortest lee-wave and the wave developed to a remarkable amplitude becomes a rotor. The wave for $k_1=0.35$ is found in the upper layer.

For the broad mountain (Figs. 10–12) the short wave disappears and the longer lee-waves dominate, which often develope into rotors in the remote distance from the ridge. Then the disturbances in the upper layer may be shown to be with the considerable amplitude.

Next we shall examine $l=6$. The zeros of $K_{6i}(k)$ are 3.3, 1.8, 1.05, 0.64 etc.. The optimum width for the largest k is 0.3. Fig. 13 is an example of flow for $b=0.3$. Lee-waves of small wave length appear in the lower layer and some of wave crest develop into rotors. When b increases long wave increases its amplitude and disturbances extend to the upper layer. These flows may be seen in Fig. 14.

Lastly the flows for $l=10$ are shown in Figs. 15–16. The zeros of $K_{10i}(k)$ are 6.6, 4.4, 3.2 etc. The ratio of wave length between ajoint lee-waves is about 0.73, then on account of the interference of such waves, clear lee-wave can hardly be found in these examples, except for a small b. For this example $b=1$, the width is too large to

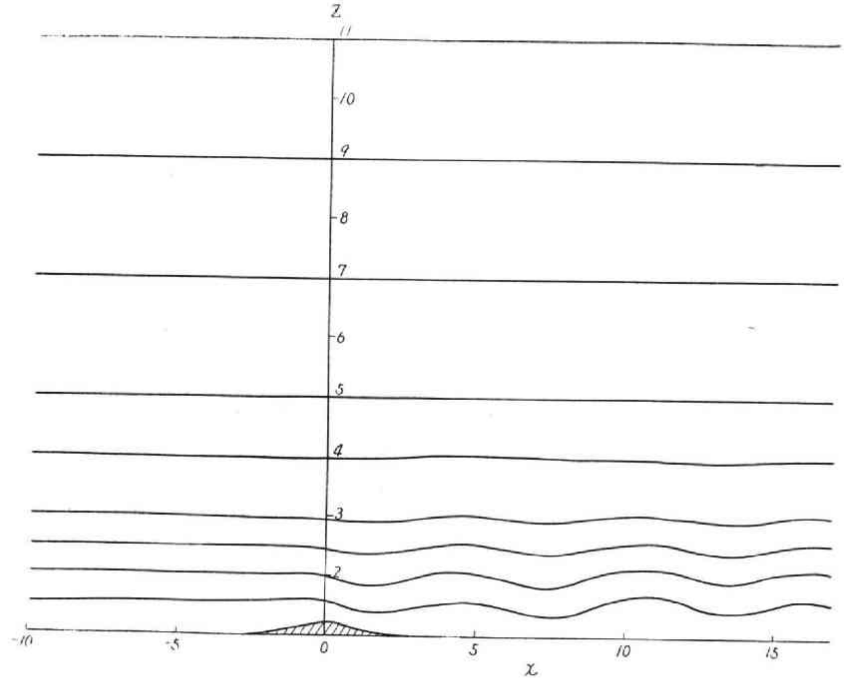

Fig. 7. Stream lines calculated for the flow over a low ridge.
$l=3,\ h=0.2,\ b=1$

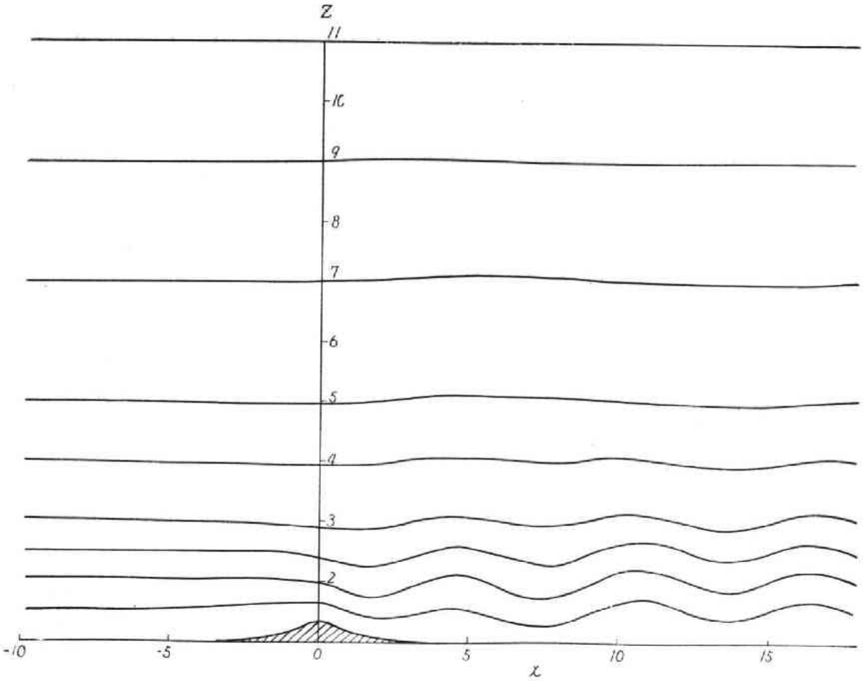

Fig. 8. Stream lines calculated for the flow over a ridge. $l=3,\ h=0.35,\ b=1$

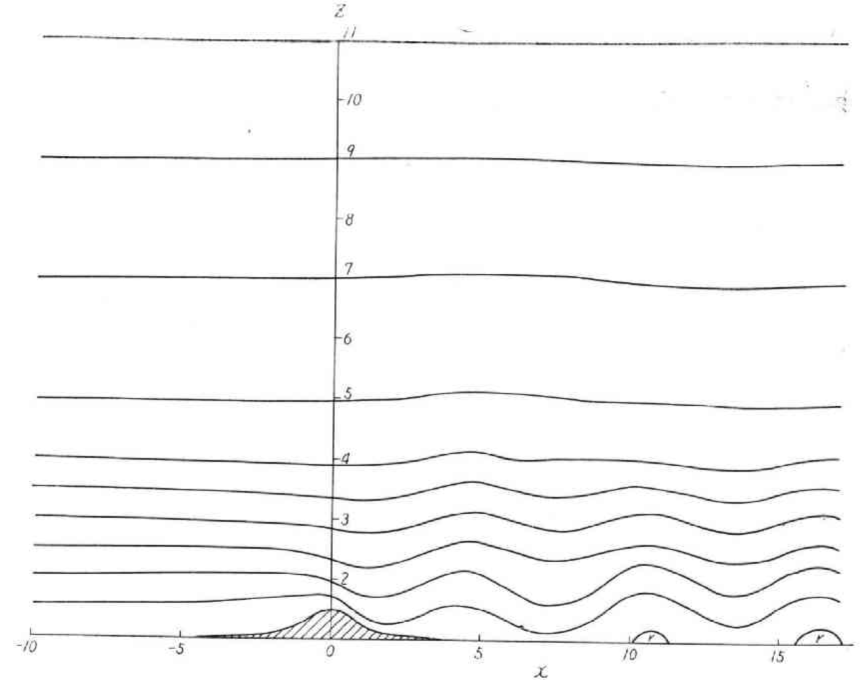

Fig. 9. Stream lines calculated for a flow over a high ridge. $l=3$, $h=0.5$, $b=1$
Rotors appear at the ground with sinusoidal wave flow at high levels.

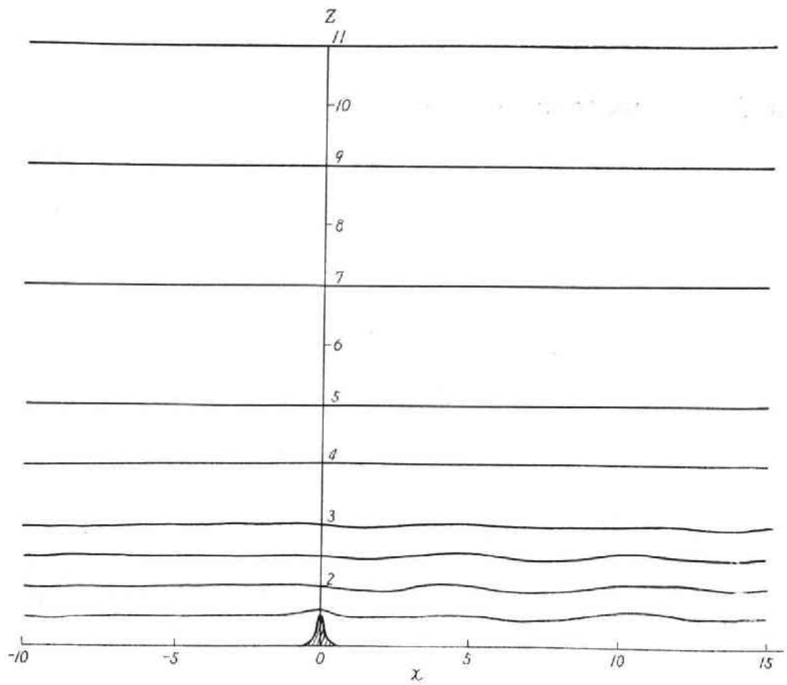

Fig. 10. Stream lines calculated for a flow over a very narrow ridge.
$l=3$ $h=0.5$ $b=0.1$

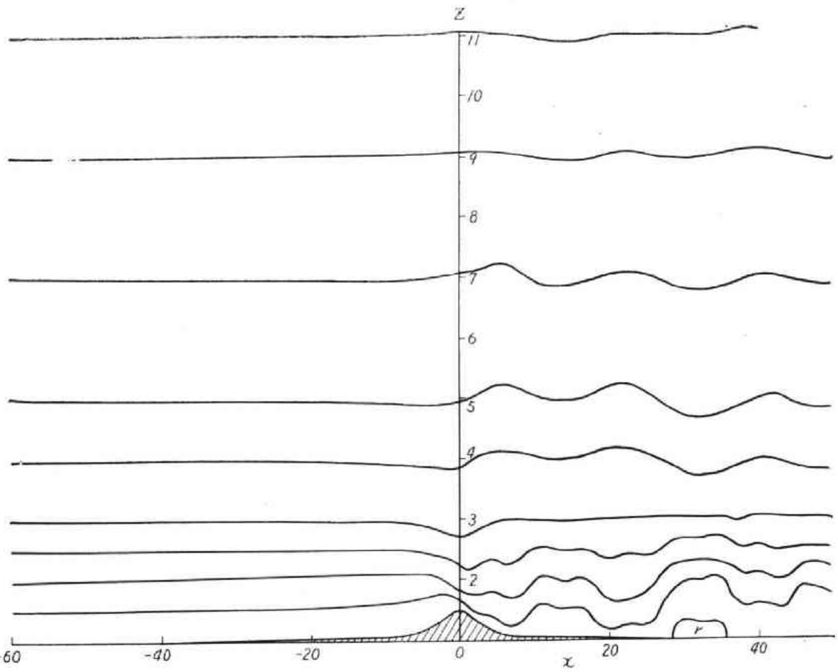

Fig. 11. Stream lines calculated for a flow over a ridge. $l=3$, $h=9.5$, $b=3$

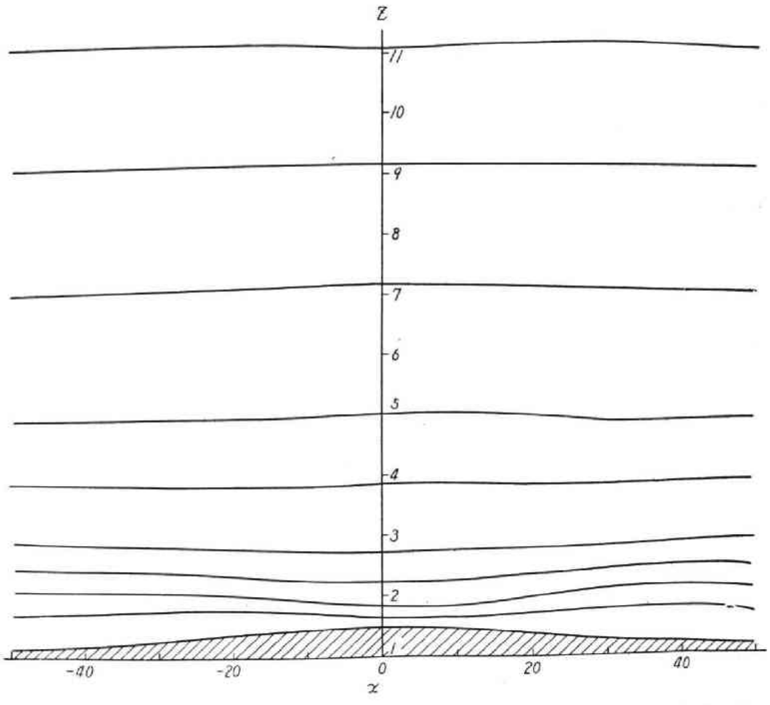

Fig. 12. Stream lines calculated for a flow over a broad ridge. $l=3$, $h=0.5$ $b=30$

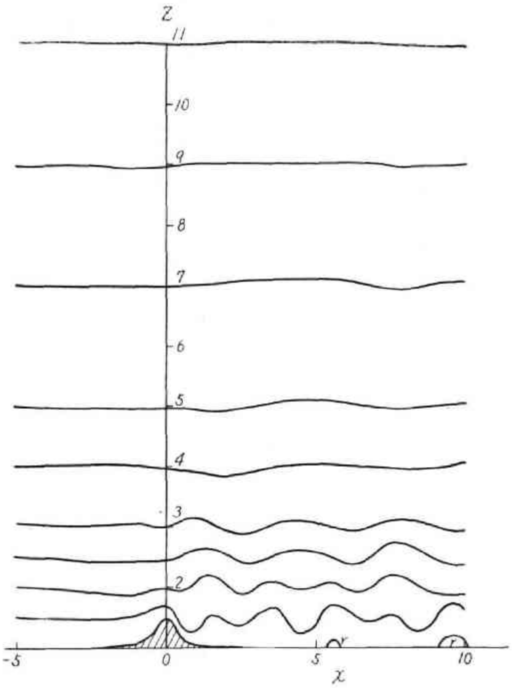

Fig. 13. Stream lines calculated for a flow over a ridge.
$l=6,\ h=0.5,\ b=0.3$

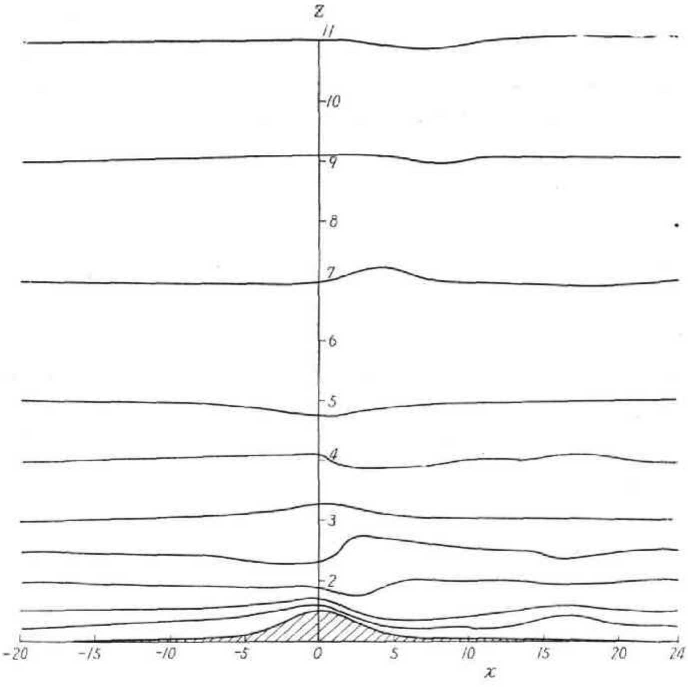

Fig. 14. Stream lines calculated for a flow over a ridge. $l=6,\ h=0.5,\ b=2$

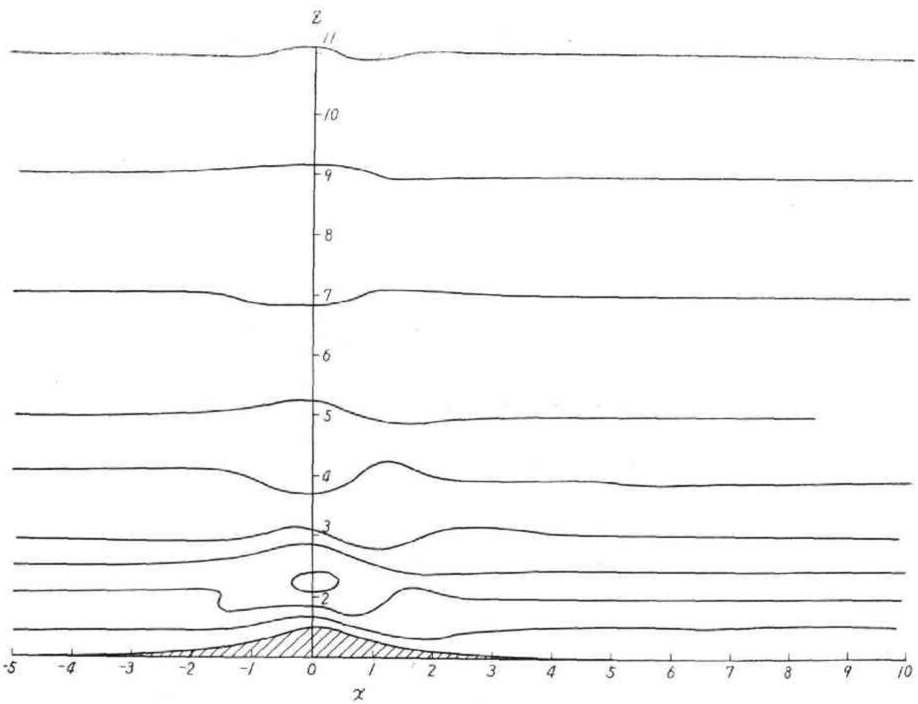

Fig. 15. Stream lines calculated for a stable atmospheric flow over a narrow ridge. $l=10$, $h=0.5$, $b=1$

excite definite lee-waves. It is interesting to note that the "S" shaped streamlines appear above the ridge. An aviator who crosses a mountian often encounters an extraordinary wind stream, especially a descending wind at the windward side of the ridge and an ascending wind at the lee side. Fig. 15 probably shows these airstreams. The flow for $b=10$ resembles the former. I suppose that in the very stable atmosphere the mountain is apt to create a turbulent flow and steady solution is hardly applicable. The investigation of this possibility must be the subject of future research.

7 Conclusions

The present paper gives a satisfactory solution of the equation for two-dimensional flow over the ridge, in which the basic wind velocity varies with height. By the assumption on choice of upper-boundary conditions we have a plausible solution which represents the occurrence of the lee-waves on the lee side of the mountain and the disturbances near the ridge. These conclusions consistent with those reached by other writers, are summarized as follows :

1. Non-wave disturbances die away over the ridge, both up- and down-wards. This is clearly seen only near the ridge.

2. On the downstream lee-waves are found, which have constant amplitudes. But it is often shown that the resultant wave is remarkable only near the ridge, on

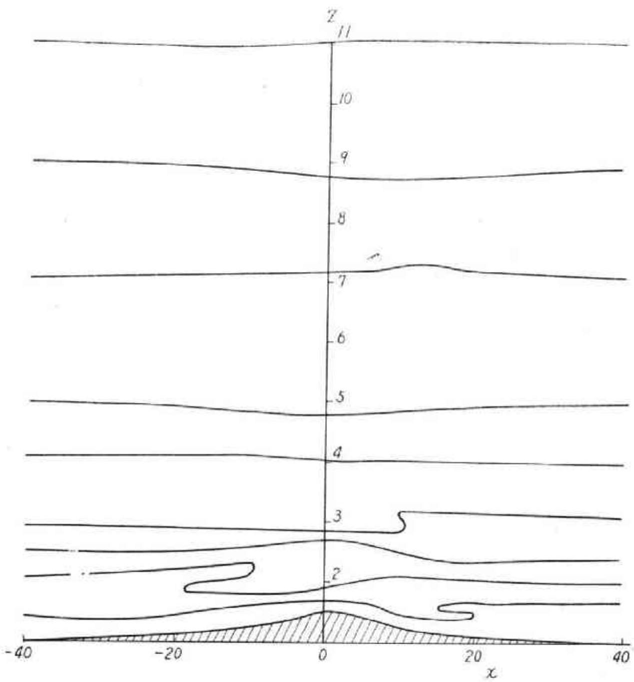

Fig. 16. Stream lines calculated for a stable atmospheric flow over a broad ridge. $l=10$, $h=0.5$, $b=10$

account of the interference of the waves.

3. Upward displacement reverses with height in some cases and often produces a closed streamline.

4. Lee-waves have their largest amplitude at some level and they die awy high up in the air. Their amplitude depends upon the character of the air stream and the shape of the ridge. Sometimes it develops into a rotor.

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