Normal Mode Waves in an Elastic Plate (1)

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Normal Mode Waves in an Elastic Plate (1)

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Abstract

Normal mode waves formed by SH waves generated from a line source in an elastic plate are investigated, including both steady state and transient cases. A remark is made on the relationship between the ray solution and normal mode solution, the former being derived by the saddle point method, the latter by residue calculation. The results of steady state excitation show that there occurs resonance at cut-off frequencies. In the case of error-functional pulse, the variations of relative amplitude and frequency with the time elapsed from the beginning of the record are shown. The results show that the first antisymmetric mode has an extremely large amplitude compared with other modes. For prescribed horizontal distances, the motions due to the first two modes are calculated, when both the source and the receiver lie in the median plane. From these results we see that a superposition of the first few modes will give rather correct aspect of the record except for the beginning of it.

Introduction

Since the first discovery of Rayleigh waves over the surface of a semi-infinite elastic solid, and Love waves transmitted in a two-layered half space, a number of investigators have studied the theory of free surface waves for various cases of horizontally layered media. In such boundary value problems, when a simple harmonic motion is assumed, the eigenvalues are given by component wave numbers parallel to the boundary surfaces, and are denoted by ω/c_n where ω is the circular frequency, c_n the phase velocity. Although for a single prescribed frequency there is a set of infinite numbers of phase velocity, only those with real values give propagation modes which are usually of seismological importance. The dispersive property is usually displayed in the form of dispersion curves in $c_n-\omega$ or c_n-T diagrams, where T is the period.

In so far as free waves are considered, it is impossible to discuss the absolute amplitudes of eigenmotions. Actually, it may be supposed that in each normal mode, the displacement amplitude excited by the initial disturbances not only varies with the frequency, but also differs according as the types and location of source. Thus, it is of importance to investigate the excitation function of eigenmotion which will be determined by various factors.

In seismology, to identify the types and modes of recorded surface waves, we must have exact knowledge about the excitation function above mentioned and also of particular eigenmotions of a particle. In the case of shallow water explosion, C. L.

PEKERIS (1948) first gave a very thorough theory of normal mode propagation in two and three layered liquids.

In this series of papers, an appropriate method for layered solids will be presented by taking up the simplest model, an infinite elastic plate bounded by two parallel planes. The problem of a typical crustal structure characterized by two-layered half space may be treated in a similar manner, although the characteristic equation becomes much more complicated.

In the present paper, normal mode waves formed by SH waves from a line source are investigated, those which are formed by P and SV waves generated from certain types of source will be treated in the next papers. Although the first problem is particularly simple, the analysis given is very useful, for it gives us a clear insight into the eigenmotions in stratified media.

The dispersion curves of SH normal mode waves in a plate have been studied by Y. SATÔ (1951).

1 Formal Solution

Consider a homogeneous isotropic plate with uniform width $2H$ and infinite area. We choose the rectangular coordinates (x, y, z), so that z-axis is perpendicular to the plane surfaces of the plate, and x, y-axes lie in the median plane.

The motion is supposed to be due to SH waves emanated cylindrically from the line source at $x=0$, $z=d$, as shown in Fig. 1. Of course the motion is independent of y, and the displacement has only y component.

Let v be the displacement, μ the rigidity, and ρ the density. The equation of motion is

$$\nabla^2 v = \frac{1}{v_s^2} \frac{\partial^2 v}{\partial t^2}$$

where $v_s = \sqrt{\mu/\rho}$ is the velocity of S waves.
If a simple harmonic motion with the time factor $e^{i\omega t}$ is assumed, we have

$$(\nabla^2 + k^2)\, v = 0\,,$$

where

$$k = \frac{\omega}{v_s}$$

Throughout this work, the plane wave solution will be used as a particular solution of the wave equations. Generally, if the elementary solution which satisfies the boundary conditions is found, the solution for the case of a point source can be derived by integrating the elementary one with respect to two spherical angles, and the solution for a line source, with respect to a cylindrical angle. Therefore, if we can derive the elementary solution, the generalized solution will be obtained automatically by performing to the elementary one the integral operator which is adequate to the type of source.

First, we express the initial displacement v_0 due to the source by plane waves of

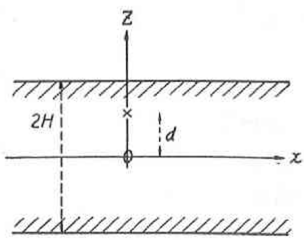

Fig. 1.

the form

$$v_0 = e^{-ik[x \sin w + |z-d| \cos w]}, \tag{1}$$

where w indicates, if it is real, the angle made by the z-axis and the wave-normal of SH waves, and may take complex values in the course of generalization to the case of a line source. Another displacement waves representing a perturbation caused by the presence of plane boundaries may be expressed by

$$v_1 = \left[A e^{-ikz\cos w} + B\, e^{ikz\cos w}\right] e^{-ikx\sin w}, \tag{2}$$

where A and B are constants to be determined by the boundary conditions. When w takes a real value, (2) may be considered to represent a pair of upgoing and downgoing waves.

The boundary conditions that the boundary planes are free from stress are written as

$$P_{zy} = \mu \frac{\partial v_2}{\partial z} = 0 \quad \text{at} \quad z = \pm H\,, \tag{3}$$

where

$$v_2 = v_0 + v_1\,. \tag{4}$$

Substituting (1) and (2) into (3), we have the simultaneous equations for A and B.

$$\left.\begin{aligned} A - Be^{-2ikH\cos w} &= e^{-ikd\cos w} \\ A - Be^{2ikH\cos w} &= e^{-ik(2H+d)\cos w} \end{aligned}\right\}$$

Thus,

$$\left.\begin{aligned} A &= \frac{-i\left\{e^{-ikd\cos w} + e^{-ik(2H-d)\cos w}\right\}}{2\sin(2\,kH\cos w)} \\ B &= \frac{-i\left\{e^{ikd\cos w} + e^{-ik(2H+d)\cos w}\right\}}{2\sin(2\,kH\cos w)} \end{aligned}\right\} \tag{5}$$

Substituting (5) into (2), using (4) and simplifying the results, we obtain the elementary solution

$$v_2 = \frac{-i\left\{\cos[k(z+d)\cos w] + \cos[k(|z-d|-2H)\cos w]\right\}}{2\sin(2\,k\,H\cos w)}\, e^{-ikx\sin w}. \tag{6}$$

To generalize the problem to the case of a line source, we must perform to (1) and (6) the operation (cf. Appendix 1)

$$\frac{\sqrt{k\,H}}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty} d\,w \tag{7}$$

From (1), the displacement v_0 due to the initial disturbances can be written

$$v_0 = \frac{\sqrt{kH}}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty} e^{-ik[x\sin w+|z-d|\cos w]}\,dw = \sqrt{kH}\,H_0^{(2)}(kr) \tag{8}$$

where

$$r=\sqrt{x^2+(z-d)^2} \tag{9}$$

When kr is large, (8) may be approximated by the asymptotic formula

$$v_0=\sqrt{\frac{2}{\pi(r/H)}}\;e^{-ikr+\pi/4\,i}, \tag{10}$$

which represents the outgoing waves from a line source, the amplitude of which is independent of the frequency.

The solution v corresponding to the line source (8) can be obtained from (6) as follows:

$$v=-\frac{i\sqrt{kH}}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty}\frac{[\cos\{k(z+d)\cos w\}+\cos\{k(|z-d|-2H)\cos w\}]}{\sin(2kH\cos w)}\,e^{-ikx\sin w}\,dw \tag{11}$$

$$=\frac{\sqrt{kH}}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty}\left[\mp\sin(\alpha'-\alpha'')-\frac{\cos\alpha\cos\alpha'\cos\alpha''}{\sin\alpha}+\frac{\sin\alpha\sin\alpha'\sin\alpha''}{\cos\alpha}\right]e^{-ikx\sin w}\,dw \tag{11$'$}$$

where $\alpha=kH\cos w$, $\alpha'=kz\cos w$, $\alpha''=kd\cos w$ and the upper sign is to be used if $z>d$, and the lower if $z<d$.

In Fig. 2 is shown the path of integration L, in which a small indentation is made to the left of the origin for convenience sake.

The integrand in (11) has poles determined by the relation

$$\sin(2kH\cos w)=0\,. \tag{12}$$

It is to be noticed that no other singular points are present in this problem, but generally, in layered media in which elastic bodily waves with various velocities can be transmitted, there are usually branch points requiring a consideration of Riemann surface and branch line integrals.

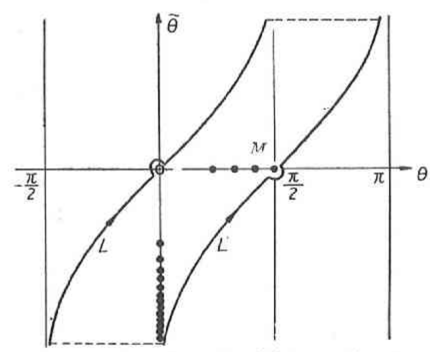

Fig. 2. Original path of integration L, a particular path L', and location of poles in the case $\bar{\gamma}=6$.

There seem to be two approaches to the evaluation of the integral in (11) or (11)$'$. One method lies in deforming the contour so that the resultant integration consists of a residue integration around the poles. The other approach is the so-called saddle point method, which consists of deforming the original path of integration in such a way that the resultant contour contains the path of steepest descent, and the main part of the integral can be obtained from the immediate neighbourhood of the saddle point. This method is, of course, applicable only when certain parameters in the exponent are relatively large.

The first method gives the normal mode solution (cf. PEKERIS (1948)), the second the ray solution, each solution being available according as the nature of the problem. The two types of solution will be discussed separately, then the relation between them will be considered. But, beforehand, the location of the poles in the w-plane, where we put $w=\theta+i\,\tilde{\theta}$, must be investigated.

2 Poles, Phase Velocity, and Group Velocity

Since w represents the angle made by the wave-normal and the vertical if it is real, we may put

$$\sin w = \frac{v_s}{c} = \bar{v}\,, \tag{13}$$

where c is the phase velocity. It may be considered that, for a given value of frequency, the poles specify the eigenvalues $\sin w_n$, which are related to the phase velocity by the equation (13).

A concept of slight absorption of a medium teaches us that we may always write

$$\sin w = a - i\,b\,, \tag{14}$$

where a and b are positive real, if the time factor be given by $e^{i\omega t}$. Paying attention to (14), the required roots of the equation (12) are obtained as follows;

$$\left.\begin{aligned} & w = \theta\,, \quad 0 \leqq \theta \leqq \frac{\pi}{2}\,, \\ & \bar{v} = \sin\theta\,, \\ & \bar{\gamma} = \frac{n\,\pi}{2}\sec\theta\,. \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} & w = -i\,\tilde{\theta}\,, \quad \tilde{\theta} > 0 \\ & \bar{v} = -i\sinh\tilde{\theta}\,, \\ & \bar{\gamma} = \frac{n\,\pi}{2}\,\mathrm{sech}\,\tilde{\theta}\,, \end{aligned}\right\} \tag{16}$$

where

$$\bar{\gamma} = k\,H\,, \quad n = 0, 1, 2, 3, \ldots\ldots \tag{17}$$

The poles represented by (15) are situated on the part of the real axis $0 \leqq \theta \leqq \pi/2$, and correspond to the propagation modes. On the other hand, the poles expressed by (16) lie on the negative imaginary axis, and give the attenuation modes, the phase velocities being positive imaginary. We have no complex poles in this problem. In both relations (15) and (16), the set of even values of n is derived from the equation $\sin(k\,H\cos w)=0$, while that of odd values of n, from the equation $\cos(k\,H\cos w)=0$. As can be seen from (11)′, the former gives the symmetric and the latter the antisymmetric motion with respect to the median plane, $z=0$.

As is well known, if the phase velocity is a function of the frequency, the energy of shock-type waves is transmitted with the group velocity U which is formally defined as $U = dw/d\,(k \sin w)$. In our case this can be expressed for the propagation mode,

$$U = \frac{d\,\omega}{d\,(k \sin w)} = v_s \frac{d\,\bar{\gamma}}{d\,(\bar{\gamma}\,\bar{v})} = v_s \frac{d\,(\sec\theta)}{d\,(\tan\theta)} = v_s \sin\theta \,. \tag{18}$$

Similarly, we have for the attenuation mode

$$U = i\, v_s \sinh\tilde{\theta} \,. \tag{19}$$

If we put

$$\bar{U} = \frac{U}{v_s}\,, \tag{20}$$

it follows that

$$\bar{U} = \bar{v}\,. \tag{21}$$

We note that the group velocity defined by (18) represents the speed of energy transportation in the simple harmonic motion as well as in a transient motion. (M.A. Biot (1958)).

The dispersive properties of eigenmotion as given by (15), (16) and (21) are shown in Fig. 3, in the form of $\bar{v}-\bar{\gamma}$ or $\bar{U}-\bar{\gamma}$ diagram. The upper part $0<\bar{v}$, $\bar{U}<1$ of the figure represents the propagation modes, and the lower part, where the ordinates are taken as $i\bar{v}$, $i\bar{U}$ instead of $-\bar{v}, -\bar{U}$, corresponds to the attenuation mode. It can be seen that for any value of $\bar{\gamma}$ there is a set of infinite values for $\bar{v}$, and also there is a set of infinite values for $\bar{\gamma}$ corresponding to a given value of $\bar{v}$. The fact shows that general motion due to a simple harmonic source can be represented by superposing infinite eigenmotions, and particularly, the propagated wave motion can be expressed with a set of finite eigenmotions of propagation mode.

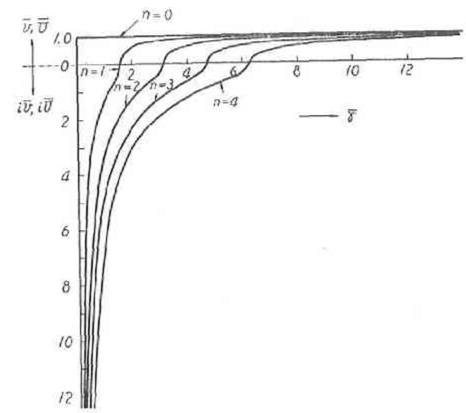

Fig. 3. Dispersion curves for *SH*-type normal mode waves in an elastic plate. Symmetric modes are represented by $n=0, 2, 4, \cdots\cdots$, and antisymmetric ones are by $n=1, 3, 5, \cdots\cdots$.

3 Normal Mode Solution

To evaluate the integral in (11)$'$, the original path of integration must be deformed so as to make a closed path. In view of the fact that the integrand is an odd function of $\cos w$, it is convenient to use a particular path L', as shown in Fig. 2, which extends from $w=-i\infty$ to $w=\pi+i\infty$, and is drawn in such a way that the two parts of the path halved at the point M $(\theta=\pi/2, \tilde{\theta}=0)$ are mutually obtained by twice reflections with respect to the two lines $\theta=\pi/2$ and $\tilde{\theta}=0$. As with the case of the original path L, we make a small indentation around M, which is ultimately made to infinitely small after the calculation of residues.

At the same distances from M upward and downward along the path L', the function $\cos w$ has the same absolute value, but with the opposite signs, so that the integral along the path L' vanishes. Also the integral vanishes along the lines joining the paths L and L' at infinite distances. Thus, the integral in (11′) reduces to residue integrals only. It is to be noted that the first term in the integrand has no normal mode. From the residue calculation it follows that;

$$v = v_s + v_a + \tilde{v}_s + \tilde{v}_a \,, \tag{22}$$

$$\left.\begin{aligned} &\begin{Bmatrix} v_s \\ \tilde{v}_s \end{Bmatrix} = 2 \sum_m \frac{1}{\bar{v}_m \sqrt{\bar{\gamma}}} \cos(m\pi \bar{z}) \cos(m\pi \bar{d})\, e^{i\omega t - i\xi_m x}, \\ &\bar{v}_m = \sin\theta_m \,, \qquad \bar{\gamma} = m\pi \sec\theta_m \,, \qquad \text{for } v_s \\ &\bar{v}_m = -i \sinh\tilde{\theta}_m \,, \quad \gamma = m\pi\, \mathrm{sech}\, \tilde{\theta}_m \,, \qquad \text{for } \tilde{v}_s \\ &m = 0, 1, 2, \ldots\ldots \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} &\begin{Bmatrix} v_a \\ \tilde{v}_a \end{Bmatrix} = 2 \sum_m \frac{1}{\bar{v}_m \sqrt{\bar{\gamma}}} \sin\left\{(m-\tfrac{1}{2})\pi \bar{z}\right\} \sin\left\{(m-\tfrac{1}{2})\pi \bar{d}\right\} e^{i\omega t - i\xi_m x} \\ &\bar{v}_m = \sin\theta_m \,, \qquad \bar{\gamma} = (m-\tfrac{1}{2})\pi \sec\theta_m \,, \qquad \text{for } v_a \\ &\bar{v}_m = -i \sinh\tilde{\theta}_m \,, \quad \bar{\gamma} = (m-\tfrac{1}{2})\pi\, \mathrm{sech}\, \tilde{\theta}_m \,, \qquad \text{for } \tilde{v}_a \\ &m = 1, 2, 3, \ldots\ldots \end{aligned}\right\} \tag{24}$$

where

$$\bar{x} = x/H \,, \quad \bar{d} = d/H \,, \quad \bar{z} = z/H \,, \quad \bar{\xi}_m = \bar{\gamma}\, \bar{v}_m \,. \tag{25}$$

Subscripts s and a show respectively the symmetric and antisymmetric motion with respect to the plane $z=0$, and v_s, v_a represent the propagation modes, $\tilde{v}_s$, $\tilde{v}_a$ the atenuation modes. The symmetric motion expressed by (23) corresponds to the case $n=2m$ in the relations (15) and (16), the antisymmetric motion (24) to the case $n=2m-1$ in the same relations. The summation must be carried out for all the possible modes determined from a prescribed frequency.

It is to be noted that (23) and (24) may be considered as the series expansion of the displacement in terms of normalized orthogonal eigenfunctions. The reciprocity relation between the source and receiver may be suggested from a general property of Green's function, by use of which the problem can be solved otherwise easily.

In the sequel, we will restrict the problem to the motion at relatively large distances from the epicenter. Then, the attenuation modes need not to be considered, since these modes are damped out exponentially with the horizontal distance. We will denote by Σ_m^+ and Σ_m^- the symmetric and antisymmetric types of propagated mode, respectively.

To discuss the amplitude it is convenient to write it in terms of a paramenter θ as follows;

$$\begin{Bmatrix} v_s \\ v_a \end{Bmatrix} = 2\sqrt{\frac{2}{\pi}} \sum_m \begin{Bmatrix} Q_{s,m}(\bar{z}, \bar{d}, \theta_m) \\ Q_{a,m}(\bar{z}, \bar{d}, \theta_m) \end{Bmatrix} e^{i(\omega t - \xi_m x)}, \tag{26}$$

$$\left.\begin{aligned} Q_{s,m} &= \frac{\sqrt{\cos\theta_m}}{\sqrt{2\,m}\sin\theta_m}\cos(m\,\pi\,\bar{z})\cos(m\,\pi\,\bar{d}) \\ Q_{a,m} &= \frac{\sqrt{\cos\theta_m}}{\sqrt{2\,m-1}\sin\theta_m}\sin\left\{(m-\tfrac{1}{2})\,\pi\,\bar{z}\right\}\sin\left\{(m-\tfrac{1}{2})\,\bar{d}\right\}, \end{aligned}\right\} \quad (27)$$

where $Q_{s,m}$ and $Q_{a,m}$ represent the excitation function of respective modes for steady state waves. They consist of three factors, that is, (1) the factor due to the location of receiver $\cos(m\pi\bar{z})$, $\sin\{(m-1/2)\,\pi\bar{z}\}$, (2) the factor due to the location of source $\cos(m\pi\bar{d})$, $\sin\{(m-1/2)\,\pi\bar{d}\}$, (3) the relative amplitude of each mode, $\sqrt{\cos\theta_m}/\{\sqrt{2m}\sin\theta_m$, $\sqrt{\cos\theta_m}/[\sqrt{2\,m-1}\,\sin\theta_m]$. The factors (1) and (2) vary with the mode number m, but are independent of the frequency. Whereas, the relative amplitude of each mode varies with both the frequency and the mode number through the relations

$$\left.\begin{aligned} \frac{1}{\sqrt{2\,m}}\frac{\sqrt{\cos\theta_m}}{\sin\theta_m} &= \frac{\sqrt{\pi}}{\sqrt{\bar{\gamma}^2-4\,m^2\,\pi^2}}, \qquad \text{for } \Sigma_m{}^+, \\ \frac{1}{\sqrt{2\,m-1}}\frac{\sqrt{\cos\theta_m}}{\sin\theta_m} &= \frac{\sqrt{\pi}}{\sqrt{\gamma^2-(2\,m-1)^2\,\pi^2}}, \qquad \text{for } \Sigma_m{}^-. \end{aligned}\right\} \quad (28)$$

Since the radiation due to the line source (10) is uniform in all direction, and is also independent of the frequency, the effect due to the source is not introduced.

The factors (1) and (2) are respectively shown in Figs. 4 and 5, and the relative amplitude (3) in Fig. 6. It is seen from Fig. 4 that the symmetric and the antisymmetric motions have respectively a loop in the median plane, a node on the boundary surfaces, and both have always loops on the boundary planes. Fig. 6 shows that resonance occurs at cut-off frequencies, and for a prescribed frequency, the amplitude increases with the mode number although the differences are small except near the cut-

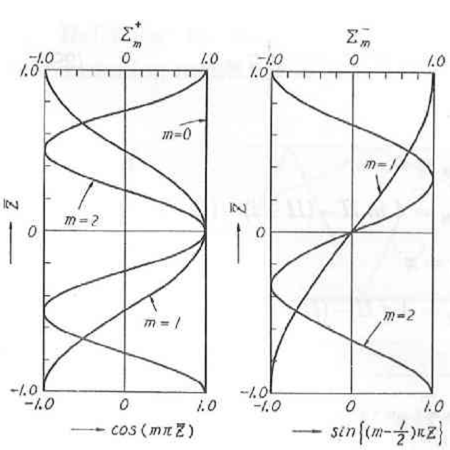

Fig. 4. Effect of the location of observing point, or amplitude distribution.

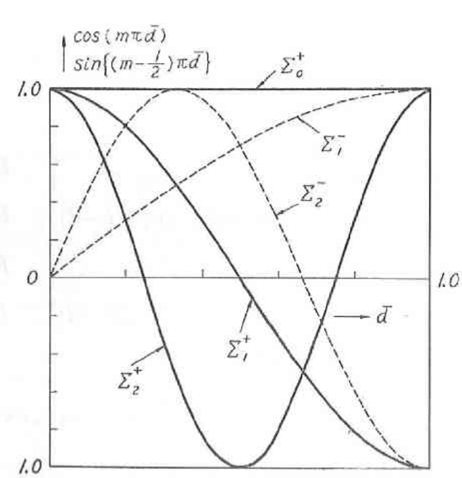

Fig. 5. Effect of the location of source.
Σ_m^+ : $\cos(m\,\pi\,\bar{d})$
Σ_m^- : $\sin\{(m-\frac{1}{2})\,\pi\,\bar{d}\}$

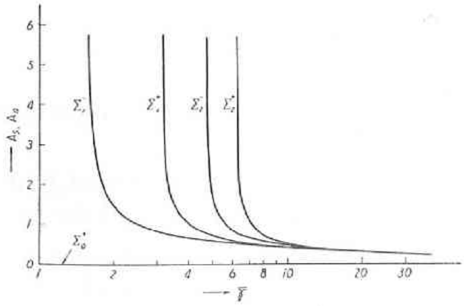

Fig. 6. Relative amplitude

$$\Sigma_m^+ : \ A_s = \frac{1}{\sqrt{2m}} \frac{\sqrt{\cos\theta_m}}{\sin\theta_m} \qquad \Sigma_m^- : \ A_a = \frac{1}{\sqrt{2m-1}} \frac{\sqrt{\mathrm{con}\,\theta_m}}{\sin\theta_m} .$$

off frequencies. It is shown that the relative amplitude of Σ_0^+ becomes zero.

4 Ray Solution

On the path L in Fig. 2, the imaginary part of $\cos w$ is not zero, so that the absolute value of $e^{-i4kH\cos w}$ is less than unity. Therefore, in the expression (11), the factor $1/\sin(2kH\cos w)$ can be expanded in the power series of $e^{-i4kH\cos w}$. We thus have a different expression for the displacement as follows;

$$\begin{aligned} v = \frac{\sqrt{kH}}{\pi} \int_{-\pi/2-i\infty}^{\pi/2+i\infty} \Bigg[& \sum_{n=0}^{\infty} e^{-ik[x\sin w + \{4nH-(H-z)+(H-d)\}\cos w]} \\ & + \sum_{m=1}^{\infty} e^{-ik[x\sin w + \{4mH-(H-z)-(H-d)\}\cos w]} \\ & + \sum_{p=1}^{\infty} e^{-ik[x\sin w + \{4pH+(H-z)-(H-d)\}\cos w]} \\ & + \sum_{q=0}^{\infty} e^{-ik[x\sin w + \{4qH-(H-z)+(H-d)\}\cos w]} \Bigg] dw . \end{aligned} \tag{29}$$

If we put

$$\left.\begin{aligned} R_n \sin\theta_n &= x \\ R_n \cos\theta_n &= 4nH+(H-z)+(H-d) \end{aligned}\right\}, \quad \left.\begin{aligned} R_m \sin\theta_m &= x \\ R_m \mathrm{ccs}\,\theta_m &= 4mH-(H-z)-(H-d) \end{aligned}\right\},$$
$$\left.\begin{aligned} R_p \sin\theta_p &= x \\ R_p \cos\theta_p &= 4pH+(H-z)-(H-d) \end{aligned}\right\}, \quad \left.\begin{aligned} R_q \sin\theta_q &= x \\ R_q \cos\theta_q &= 4qH-(H-z)+(H-d) \end{aligned}\right\}, \tag{30}$$

(1,29) can be written as

$$\begin{aligned} v = \frac{\sqrt{kH}}{\pi} \int_{-\pi/2-i\infty}^{\pi/2+i\infty} \Bigg[& \sum_{n=0}^{\infty} e^{-ikR_n\cos(w-\theta_n)} + \sum_{m=1}^{\infty} e^{-ikR_m\cos(w-\theta_m)} \\ & + \sum_{p=1}^{\infty} e^{-ikR_p\cos(w-\theta_p)} + \sum_{q=0}^{\infty} e^{-ikR_q\cos(w-\theta_q)} \Bigg] dw . \end{aligned} \tag{31}$$

If we assume that kR_n, kR_m, kR_p, $kR_q \gg 1$, the method of saddle point can give approximate values for the integrals in (31).

Consider, for example, the integral

$$I=\int_{-\pi/2-i\infty}^{\pi/2+i\infty} e^{-ikR_n\cos(w-\theta_n)}\,dw\,. \tag{32}$$

The saddle point becomes $w=\theta_n$, from the relation $(\partial/\partial w)\{\cos(w-\theta_n)\}=0$, and the path of steepest descent is determined from the equation $Im\,[-ikR_n\cos(w-\theta_n)]=\text{const}=-kR_n$, to be $\cos(\theta-\theta_n)\cosh\tilde{\theta}=1$. In Fig. 7 are shown the saddle point $S\,(w=\theta_n)$, the path of steepest descent L_s, and the original path of integration L. Convergency of the integral along L_s is secured by the relation $Im\,[\cos(w-\theta_n)]=\sin(\theta-\theta_n)\sinh\tilde{\theta}<0$.

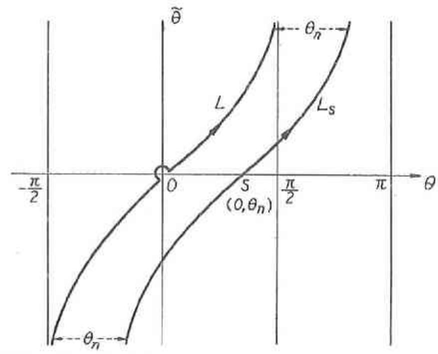

Fig. 7. Saddle point S, and the path of steepest descent L_s.

In the vicinity of the saddle point, we may put $w=\theta_n+\rho\, e^{\pi/4i}$, and the integral (32) can be evaluated approximately as

$$Im=\int_{-\infty}^{\infty} e^{\pi/4i-ikR_n-(1/2)kR_n\rho^2}\,d\rho=\sqrt{\frac{2\pi}{kR_n}}\,e^{-ikR_n+\pi/4i}. \tag{33}$$

By virtue of (32), an approximate value for (31) becomes

$$v=\sqrt{\frac{2}{\pi}}\,e^{\pi/4i}\left[\sum_{n=0}^{\infty}\frac{e^{-ikR_n}}{\sqrt{R_n/H}}+\sum_{m=1}^{\infty}\frac{e^{-ikR_m}}{\sqrt{R_m/H}}+\sum_{p=1}^{\infty}\frac{e^{-ikR_p}}{\sqrt{R_p/H}}+\sum_{q=0}^{\infty}\frac{e^{-ikR_q}}{\sqrt{R_q/H}}\right]. \tag{34}$$

Referring to the relations (30), we can identify the respective terms under the four summing notations in (34), with the direct or reflected bodily waves. Obviously,

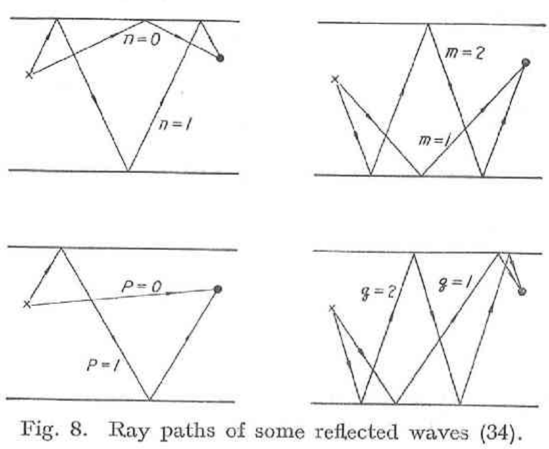

Fig. 8. Ray paths of some reflected waves (34).

the reflection coefficient at the free surfaces is unity. Some ray paths are indicated in Fig. 8. In (30), R_n, R_m, R_p, R_q represent the distances of the ray path of respective waves, and θ_n, θ_m, θ_p, θ_q, the angles between the rays and the vertical.

5 Relation between Normal Mode Solution and Ray Solution

In the normal mode solution (26), the summation is taken over m, which implies the mode number, i.e. a number showing the order in series of decreasing eigenvalues $\bar{v}_m$ for a prescribed frequency. (Fig. 3) On the other hand, the ray solution indicates the summation over the numbers of reflection of rays at a boundary surface. (Fig. 8) Although the physical meaning of the summation indexes are different from each other, it can be shown that the normal mode solution is derived directly from the ray solution if the frequency is relatively large.

Returning to (29), if the summation indexes of both the second and third sums are replaced by the negative numbers, and the same letter n is used in each sum, we have the formula,

$$v=\frac{\sqrt{kH}}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty}\Big[e^{-ik[x\sin w+(e+f)\cos w]}\sum_{n=0}^{\infty}e^{-i\gamma n}+e^{-ik[x\sin w-(e+f)\cos w]}\sum_{n=-1}^{\infty}e^{i\gamma n}$$

$$+e^{-ik[x\sin w+(e+f)\cos w]}\sum_{n=1}^{\infty}e^{in\gamma}+e^{-ik[x\sin w-(e-f)\cos w]}\sum_{n=0}^{\infty}e^{-in\gamma}\Big]dw\,, \tag{35}$$

where

$$e=H-z,\qquad f=H-d\,,\qquad \gamma=4kH\cos w\,. \tag{36}$$

Using the Poisson sum-formula (cf. MORSE and FESHBACH (1953)), we can write

$$\sum_{n=0}^{\infty}e^{-in\gamma}=\frac{\sqrt{2\pi}}{\gamma}\sum_{m=-\infty}^{\infty}F\Big(\frac{2m\pi}{\gamma}\Big), \tag{37}$$

where $F(n\gamma)$ is the Fourier transform of $f(n\gamma)=e^{-in\gamma}$. We have the relation

$$F(\nu\gamma)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\gamma\tau)\,e^{-i\gamma^2\tau\nu}\gamma\,d\tau=\frac{\gamma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\tau(\gamma^2\nu+\gamma)}\,d\tau\,. \tag{38}$$

At any point on the path L in Fig. 2, $Re(\cos w)$ is not small, so that, if γ is large, $Re(\gamma)$ becomes relatively large. Thus the integral in (38) may be written formally as $2\pi\,\delta(\gamma^2\nu+\gamma)$, where δ is the delta function. Accordingly, one obtains

$$F(2m\pi/\gamma)=\sqrt{2\pi}\,\gamma\,\delta(2m\pi+\gamma)\,. \tag{39}$$

Substituting (39) into (37) yields

$$\sum_{n=0}^{\infty}e^{-in\gamma}=2\pi\sum_{m=-\infty}^{\infty}\delta(2m\pi+\gamma)\,. \tag{40}$$

Similarly, one obtains

$$\sum_{n=-1}^{\infty}e^{-in\gamma}=2\pi\sum_{m=-\infty}^{\infty}\delta(2m\pi-\gamma)\,. \tag{41}$$

Substituting (40) and (41) into (35), it follows that

$$v = 4 \sum_{m=-\infty}^{\infty} \sqrt{\bar{\gamma}} \int_{-\pi/2-i\infty}^{\pi/2+i\infty} \Big[\delta(2m\pi+\gamma) \cos(k e \cos w)\, e^{-ik[x \sin w + f \cos w]} + \delta(2m\pi-\gamma) \cos(k e \cos w)\, e^{-ik[x \sin w - f \cos w]} \Big] dw . \tag{42}$$

Since no singular point is present in the integrand in (42), the path L may be shifted to the right by an amount of $\pi/2$. Then (42) becomes

$$v = \sum_{m=-\infty}^{\infty} \int_{-i\infty}^{\pi+i\infty} \sqrt{\bar{\gamma}} \Big[\delta(2m+\gamma)\, e^{-ikf\cos w} + \delta(2m-\gamma)\, e^{ikf\cos w} \Big] \cos(k e \cos w)\, e^{-ikx\sin w}\, dw . \tag{43}$$

If we put $2m\pi \pm \gamma = \lambda$, where + sign is to be used in the first term of the integrand, − sign in the second, (43) is transformed as

$$v = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{2\,\delta(\lambda)\, d\lambda}{\sqrt{\bar{\gamma}}\, \sin w} \cos(k f \cos w) \cos(k e \cos w)\, e^{-ikx\sin w} \right]. \tag{44}$$

Using the relation

$$\int_{-\infty}^{\infty} f(\lambda)\, \delta(\lambda)\, d\lambda = f(0), \tag{45}$$

we have from (44)

$$v = \sum_{m=0}^{\infty} \frac{\varepsilon_m}{\sqrt{\bar{\gamma}}\, \sqrt{1-g^2}} \cos(g k e) \cos(g k f)\, e^{-ikx\sqrt{1-g^2}}, \tag{46}$$

where $g = m\pi/2kH$, $\varepsilon_m = 2$, *if* $m \neq 0$, $\varepsilon_m = 1$, *if* $m = 0$.

By virtue of the relation (15), it follows that

$$\left.\begin{aligned} v &= v_s + v_a \\ v_s &= 2\sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \frac{\sqrt{\cos\theta_m}}{\sqrt{2m}\, \sin {}_m\theta} \cos(m\pi\bar{z}) \cos(m\pi\bar{d})\, e^{-im\pi\tan\theta_m x} \\ v_a &= 2\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{\sqrt{\cos\theta_m}}{\sqrt{2m-1}\, \sin\theta_m} \sin\{(m-\tfrac{1}{2})\pi\bar{z}\} \sin\{(m-\tfrac{1}{2})\pi\bar{d}\}\, e^{-i(m-1/2)\pi\tan\theta_m x} \end{aligned}\right\} \tag{47}$$

The expression (47) coincides with that of the normal mode solution (26), provided only that the summation in the latter is carried out for all the normal modes. Since this situation may be realized when $\bar{\gamma}$ is large, it may be said that the normal mode solution can be derived from the ray path solution if the frequency is large. Of course, when $\bar{\gamma}$ is not large, the approximation by the saddle point method becomes inaccurate, and the equivalency of the two kinds of solution is no more valid.

6 Aperiodic Solution

In the preceding sections we have obtained the steady state solution due to a line source with the time factor $e^{i\omega t}$. Now we will discuss an aperiodic case in which the primary waves are shock type. Generally, if the time variation of the displacement due to primary waves is given by the function

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega t}F(\omega)\,d\omega\,, \tag{48}$$

where $F(\omega)$ is the Fourier transform of $f(t)$, then the aperiodic solution corresponding to (26) is given by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)\,v_{s,a}\,d\omega\,. \tag{49}$$

We will consider the case in which $f(t)$ is given by

$$f(t)=e^{-t^2/2a^2}. \tag{50}$$

where a is a parameter to specify the time interval in which an appreciable amount of displacement occurs.

In view of the expression of the initial displacement (10), it is convenient to use, instead of (50), the function

$$f(t)=e^{-t^2/2a^2-\pi/4i}\,. \tag{51}$$

Since the Fourier transform of $f(t)$ is

$$F(\omega)=a\,e^{-a^2\omega^2/2-\pi/4i}, \tag{52}$$

the initial displacement V_0 for the aperiodic case is obtained from (49) and (52), as follows;

$$\left.\begin{aligned}V_0&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a\,e^{-a^2\omega^2/2-\pi/4i}\left[\sqrt{\frac{2}{\pi(r/H)}}\,e^{i\omega t-ikr+\pi/4i}\right]d\omega\\&=a\sqrt{\frac{2}{\pi}}\,Re\int_0^{\infty}e^{-a^2\omega^2/2-\pi/4i}\left[\sqrt{\frac{2}{\pi(r/H)}}\,e^{i\omega t-ikr+\pi/4i}\right]d\omega=\sqrt{\frac{2}{\pi(r/H)}}\,e^{-(t-r/v_s)^2/2a^2}\end{aligned}\right\} \tag{53}$$

The calculation of (53) shows that the aperiodic source can be derived by performing the operation

$$a\sqrt{\frac{2}{\pi}}\,Re\int_0^{\infty}e^{-a^2\omega^2/2-\pi/4i}\,d\omega\,, \tag{54}$$

to the expressions for a simple harmonic source. Accordingly, the same operation yields the aperiodic solutions V_s, V_a as

$$\begin{Bmatrix}V_s\\V_a\end{Bmatrix}=a\sqrt{\frac{2}{\pi}}\,Re\int_0^{\infty}e^{-a^2\omega^2/2-\pi/4i}\begin{Bmatrix}v_s\\v_a\end{Bmatrix}d\omega\,. \tag{55}$$

If we use non-dimensional quantities, (55) can be written

$$\begin{Bmatrix} V_s \\ V_a \end{Bmatrix} = 2\sqrt{\frac{2}{\pi}}\,\bar{a} \sum_{\binom{m=0}{m=1}}^{\infty} Re \int_0^{\infty} \begin{Bmatrix} G_s \\ G_a \end{Bmatrix} e^{if(\bar{\gamma},\bar{v},\bar{t},\bar{x})}, \tag{56}$$

$$\begin{Bmatrix} G_s \\ G_a \end{Bmatrix} = \frac{1}{\sqrt{\bar{\gamma}}\ \bar{v}} \begin{Bmatrix} \cos(m\pi\bar{z})\cos(m\pi\bar{d}) \\ \sin(m-\frac{1}{2})\pi\bar{z}\}\sin\{(m-\frac{1}{2})\pi\bar{d}\} \end{Bmatrix} e^{-(\bar{a}\bar{\gamma})^2/2-\pi/4i}, \tag{57}$$

$$f(\bar{\gamma},\bar{v},\bar{t},\bar{x}) = (\bar{\gamma}\,\bar{t}-\bar{\gamma}\,\bar{v}\,\bar{x}) = (\bar{\gamma}\,\bar{t}-\bar{\xi}\,\bar{x}), \tag{58}$$

$$\bar{t} = t/(H/v_s)\,, \qquad \bar{a} = a/(H/v_s)\,, \qquad \bar{\xi} = \bar{\gamma}\,\bar{v} = \frac{\omega H}{c}\,, \tag{59}$$

where the subscripts m in $\bar{v}_m$, $\bar{\xi}_m$ are omitted hereafter.

When we consider the case $\bar{x}$, $\bar{t} \gg 1$, the integral in (56) can be evaluated approximately by the method of stationary phase. A contribution to the integral from the immediate neighbourhood of the stationary point of the function f can give approximate value of the integral, since the contributions from other parts of the path are cancelled by interference. (cf. PEKERIS (1948)).

The stationary point $\bar{\gamma}_0$ of the function f is determined from the relation

$$\frac{df}{d\bar{\gamma}} = \bar{t} - \bar{x}\frac{d\bar{\xi}}{d\bar{\gamma}} = \bar{t} - \frac{\bar{x}}{\bar{U}} = 0\,. \tag{60}$$

The function f can be expanded in power series in the vicinity of the stationary point as

$$f = (\bar{\gamma}_0\bar{t}-\bar{\xi}_0\bar{x}) - \frac{1}{2}(\ddot{\bar{\xi}}_0)\,\bar{x}\,u^2 - \frac{1}{6}(\dddot{\bar{\xi}})\,\bar{x}\,u^3 - \frac{1}{24}(\ddddot{\bar{\xi}}_0)\,\bar{x}\,u^4 - \ldots\ldots, \tag{61}$$

where $u=\bar{\gamma}-\bar{\gamma}_0$, and the dots over $\bar{\xi}$ indicate successive differentiation with respect to $\bar{\gamma}$. The suffixed factors are to be evaluated at $\bar{\gamma}=\bar{\gamma}_0$. If the slowly varying terms G_s, G_a are approximated by their values at $\bar{\gamma}=\bar{\gamma}_0$, and are taken out from the integrand, it follows that,

$$\begin{aligned} \begin{Bmatrix} V_s \\ V_a \end{Bmatrix} &= 2\sqrt{\frac{2}{\pi}}\,\bar{a}\sum_m Re\left[\begin{Bmatrix} G_s \\ G_a \end{Bmatrix}_{\gamma_0} e^{i(\bar{\gamma}_0\bar{t}-\bar{\xi}_0\bar{x})}\int_{-\infty}^{\infty} e^{-1/2(\ddot{\bar{\xi}}_0u^2+1/3\dddot{\bar{\xi}}_0u^3+1/12\ddddot{\bar{\xi}}_0u^4-\ldots\ldots)\bar{x}}\,du\right] \\ &= 2\sqrt{\frac{2}{\pi}}\,\bar{a}\sum_m Re\left[\begin{Bmatrix} G_s \\ G_a \end{Bmatrix} e^{i(\bar{\gamma}_0\bar{t}-\bar{\xi}_0\bar{x})}\sqrt{\frac{2\pi}{|\ddot{\xi}_0|\bar{x}}}\,e^{\pm\pi/4i}\left(1+i\left\{-\frac{5}{24}\frac{(\dddot{\bar{\xi}}_0)^2}{(\ddot{\bar{\xi}}_0)^3}+\frac{(\ddddot{\bar{\xi}}_0)}{8(\ddot{\bar{\xi}}_0)^2}\right\}+0\left(\frac{1}{x^2}\right)\right)\right], \end{aligned} \tag{62}$$

where $+$ sign is to be used if $\ddot{\bar{\xi}}_0 < 0$, and $-$ sign if $\ddot{\bar{\xi}} > 0$. In our case, $\ddot{\bar{\xi}}$ can be calculated as

$$\ddot{\bar{\xi}} = \frac{d^2\bar{\xi}}{d\bar{\gamma}^2} = \frac{d^2(\tan\theta)}{d(\sec\theta)^2} = -\cot^3\theta\,. \tag{63}$$

which is always negative, so that $+$ sign must be used.

Taking the first term in the bracketed one following $e^{\pm\pi/4i}$ in (62), and retaining the second as a correction term, we have

$$\begin{Bmatrix} V_s \\ V_a \end{Bmatrix} = 2\sqrt{\frac{2}{\pi}}\,\frac{\bar{a}}{\sqrt{\bar{x}}}\sum_m \frac{1}{\sqrt{|\ddot{\xi}_0|}}\,e^{-(\bar{a}\bar{\gamma}_0)^2/2}\begin{Bmatrix} \cos(m\pi\bar{z})\cos(m\pi\bar{d}) \\ \sin\{(m-\frac{1}{2})\pi\bar{z}\}\sin\{(m-\frac{1}{2})\pi\bar{d}\} \end{Bmatrix}\cos(\bar{\gamma}_0\bar{t}-\bar{\xi}_0\bar{x})\,. \tag{64}$$

The condition of the validity of this approximation is expressed by

$$\frac{1}{\bar{x}}\left[-\frac{5}{24}\frac{(\dddot{\xi}_0)^2}{(\ddot{\xi}_0)^3}+\frac{(\ddddot{\xi}_0)}{8\,(\ddot{\xi}_0)^2}\right]\ll 1\,. \tag{65}$$

The condition (65) means that (64) is valid when the horizontal distance of the receiver is relatively large, and the time is sufficiently removed from that which is associated with the maximum value $\bar{U}=1$ of the group velocity.

The analysis above given is summarized as follows; in the case of shock-type waves, principal energy is transmitted with the group velocity, and for specified values of $\bar{x}$, $\bar{z}$ and $\bar{t}$, the approximate value of the displacement is given by (64), and this expression is characterized by the frequency which corresponds to the group velocity determined from the relation (60).

The parametric representation of (64) is given by

$$V_s=\frac{4\sqrt{2}\,\bar{a}}{\sqrt{\bar{x}}}\sum_m\frac{\sqrt{\sin\theta_0}}{\sqrt{2m}\cos\theta_0}\,e^{-1/2\,(a\,m\,\pi\sec\theta_0)^2}$$

$$\cdot\cos(m\pi\bar{z})\cos(m\pi\bar{d})\cos[\sin\theta_0\,\bar{t}-m\pi\tan\theta_0\,\bar{x}]$$

$$V_a=\frac{4\sqrt{2}\,\bar{a}}{\sqrt{\bar{x}}}\sum_m\frac{\sqrt{\sin\theta_0}}{\sqrt{2m-1}\cos\theta_0}\,e^{-1/2\,\{a\,(m-1/2)\,\pi\sec\theta_0\}^2}$$

$$\cdot\sin\left\{(m-\tfrac{1}{2})\,\pi\bar{z}\right\}\sin\left\{(m-\tfrac{1}{2})\,\pi\bar{d}\right\}\cos\left[\sin\theta_0\,\bar{t}-(m-\tfrac{1}{2})\,\pi\tan\theta_0\,\bar{x}\right] \tag{66}$$

This expression is useful for numerical calculation. The condition (65) can be written also in terms of θ. For this purpose, we write

$$\left.\begin{aligned}\dddot{\xi}&=\frac{d^3\bar{\xi}}{d\,\bar{\gamma}^3}=\frac{d^3(\tan\theta)}{d\,(\sec\theta)^3}=\frac{3\cos^4\theta}{\sin^5\theta}\\ \ddddot{\xi}&=\frac{d^4\bar{\xi}}{d\,\bar{\gamma}^4}=\frac{d^4(\tan\theta)}{d\,(\sec\theta)^4}=-\frac{3\,(\cos^7\theta+4\cos^5\theta)}{\sin^7\theta}\end{aligned}\right\} \tag{67}$$

Substitution of (63) and (67) into (65) gives

$$\tan\theta\ll\frac{8}{3}\,\bar{x}\,. \tag{68}$$

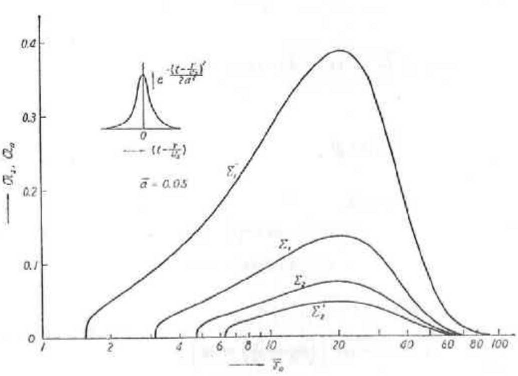

Fig. 9, a.

If this condition is violated, $\tan\theta$ or $\bar{\gamma}$ must be very large. But in this case, the integrand in (56) becomes very small on account of the factor $\bar{a}\,e^{-(\bar{a}\,\bar{\gamma})^2/2}$, so that this case needs not to be considered. Thus, the calculation of Airy phase becomes unnecessary. Figs. 9, a, b, c. show the variation of relative amplitudes a_s, a_a of first few modes with $\bar{\gamma}_0$,

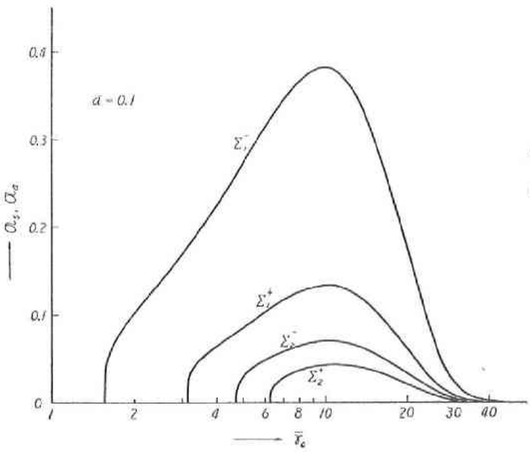

Fig. 9, b.

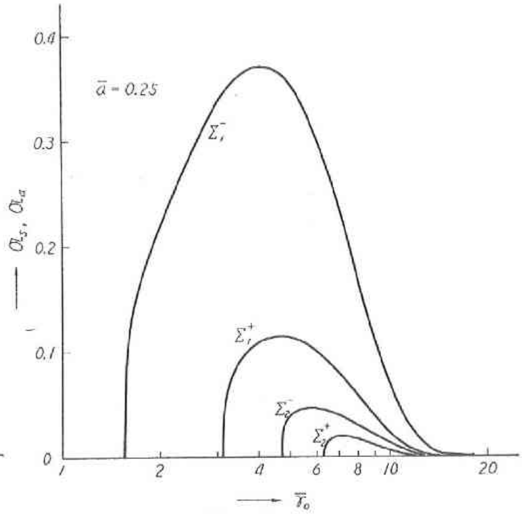

Fig. 9, c.

Fig. 9. Relative amplitude in the case of shock-type waves.

where

$$\alpha_s = \frac{\overline{a}\sqrt{\sin\theta_0}}{\sqrt{2m}\cos\theta_0}\, e^{-1/2[am\pi\sec\theta_0]^2}, \qquad \alpha_a = \frac{\overline{a}\sqrt{\sin\theta_0}}{\sqrt{2m-1}\,\cos\theta_0}\, e^{-1/2[a(m-1/2)\pi\sec\theta_0]^2} \tag{1,69}$$

and $\overline{\gamma}_0$ is the frequency satisfying the relation $\overline{U}\overline{t}=\overline{x}$. The inserted small figure in Fig. 9,a represents the error-functional pulse from the source. Fig. 9, a corresponds to the most sharp pulse from the source, and Figs. 9, b and c correspond to less sharp pulses in order. As the amplitude of Σ_0^+ always vanishes, this mode is omitted. In Figs. 10 and 11 are shown respectively the variation of amplitudes α_s, α_a, and the frequency $\overline{\gamma}_0$, with the factor $(t-t_0)/t_0$, where t_0 is given by x/v_s. The behaviours of the curves in Fig. 11 are not changed for any value of $\overline{a}$.

In Fig. 12 are shown the motions due to the first two modes when the source and the receiver lie in the median plane ($\overline{z}=\overline{d}=0$), the horizontal distances of the receiver

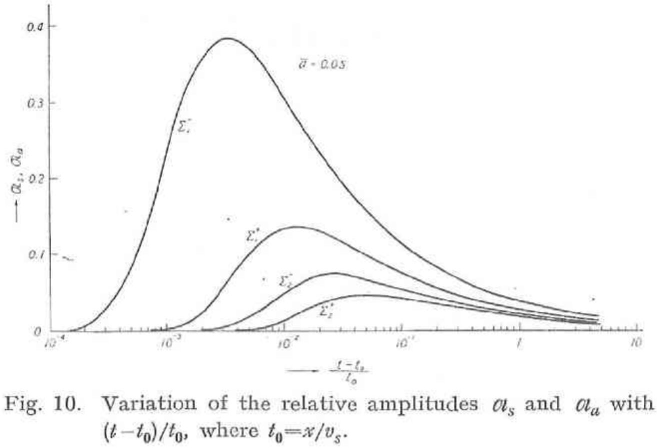

Fig. 10. Variation of the relative amplitudes α_s and α_a with $(t-t_0)/t_0$, where $t_0=x/v_s$.

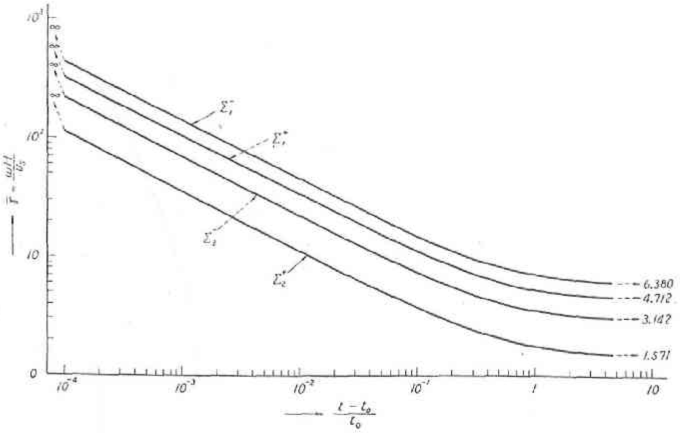

Fig. 11. Variation of $\bar{\gamma}=wH/v_s$ with $(t-t_0)/t_0$.

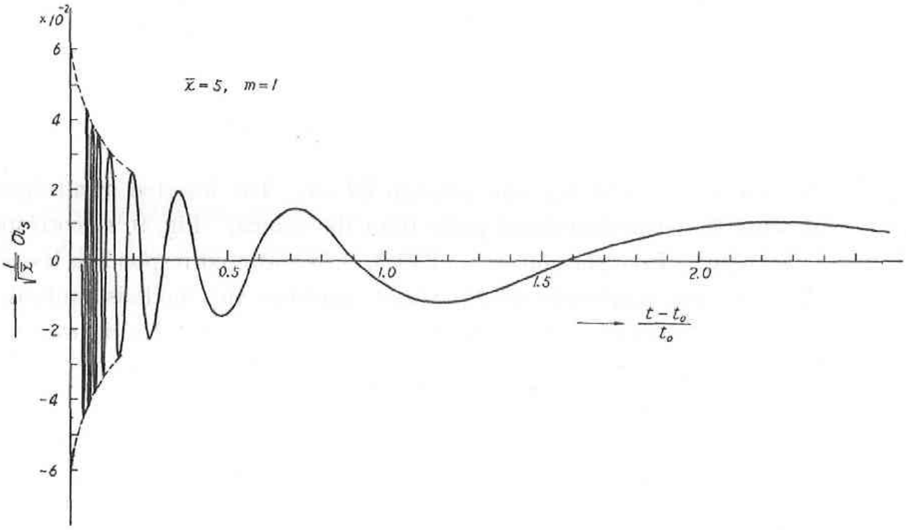

Fig. 12, a.

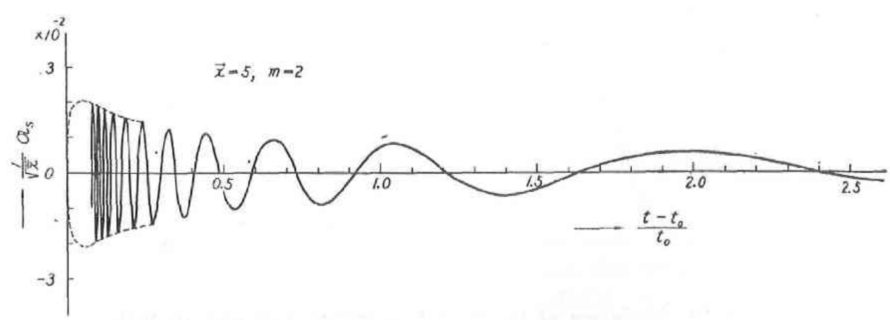

Fig. 12, b.

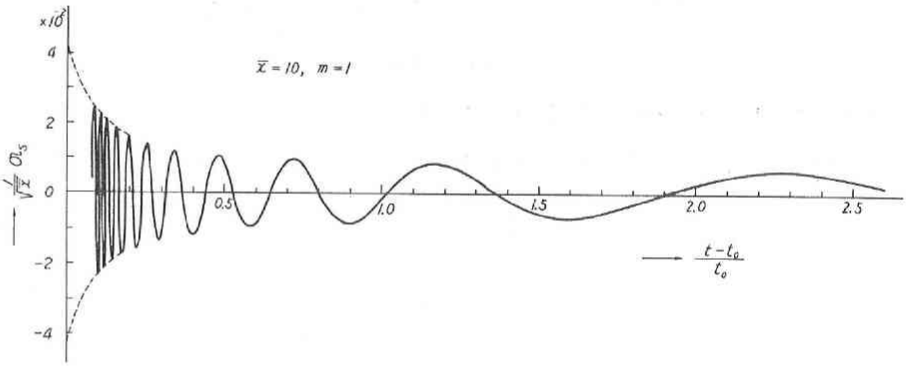

Fig. 12, c.

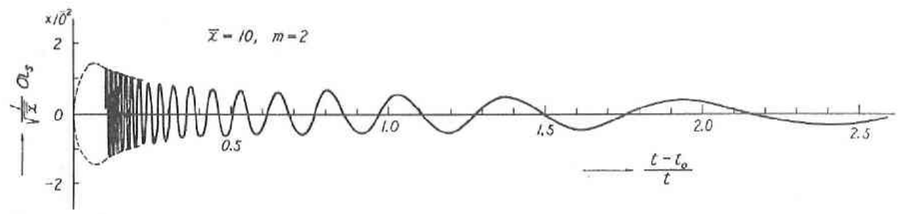

Fig. 12, d.

Fig. 12. Calculated records for Σ_1^+ and Σ_2^+, when both the source and receiver lie in the median plane. Figs. 12, a, b correspond to $\bar{x}=5$, Figs. 12, c, d to $\bar{x}=10$.

being $\bar{x}=5$ and 10. These figures show that the normal mode solution is inadequate for the precise prediction of the first arrival, but at later stages, a superposition of the first few modes may provides rather correct aspect of the record.

7 Concluding Remarks

In the case of simple harmonic motion, there occurs resonance at cut-off frequencies, as shown in Fig. 6. But, when we assume a source which radiates waves of varied amplitude in azimuthal directions, a new factor in the relative amplitude will be introduced, and there might be cases where the resonance above mentioned disappears.

In shock-type waves, the relative amplitude becomes the largest in Σ_1^- mode, which suggests the existence of predominant antisymmetric motion.

In the earlier stage of the record, especially at the beginning of it, the superposition of component motions due to respective modes becomes practically impossible owing to the ever increasing frequency, so that we must resort to the ray solution. But, in later stages, since the relative amplitude decreases considerably with increasing mode number, a superposition of first few modes may provides rather good aspect of the record.

If the formula (35), from which originates the ray solution, is obtained, the normal mode solution can be found directly from this formula when the frequency is relatively large. This result may be useful for the problems of other layered media, in which the formula equivalent to (35) can easily be constructed.

Appendix

1. An Integral Representation of $H_0^{(2)}(kr)$.

The Hankel function of the second kind and zeroth order can be written

$$H_0^{(2)} = -\frac{1}{\pi i}\int_{-\infty+\pi/2i}^{\infty-\pi/2i} e^{-iz\cosh w'}\,dw', \tag{1}$$

provided that $|\arg z|<\pi/2$ (cf. WATSON (1922))

Putting $w'=i(w-\theta)$ in (1), we have

$$H_0^{(2)}(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2+\theta+i\infty} e^{-iz\cos(w-\theta)}\,dw \tag{2}$$

If we put $w=p+iq$, the condition of convergency, $[\cos(w-\theta)]<0$ can be expressed by

$$\left.\begin{array}{ll} q>0, & \pi+\theta>p>\theta \\ q<0, & \theta>p>-\pi+\theta \end{array}\right\} \tag{3}$$

Since the integrand in (2) has no singular point, the path may be shifted to the left by an amount θ, so that

$$H_0^{(2)}(z) = \frac{1}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty} e^{-iz\cos(\omega-\theta)}\,dw\,. \tag{4}$$

In this expression, the convergency of the integral is secured by the relation (3).

If we put

$$z=kr, \qquad r\sin\theta=x, \qquad r\,\mathrm{coe}\,\theta=|z-d| \tag{5}$$

it follows that

$$H_0^{(2)}(kr) = \frac{1}{\pi}\int_{-\pi/2-i\infty}^{\pi/2+i\infty} e^{-ik[x\sin w+|z-d|\cos w]}\,dw \tag{6}$$

This expression may be considered to be a superposition of plane waves.

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