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On the Probability Distribution of the Auto-Correlation Function in Atmospheric Turbulence

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Abstract

It is stated that the existence of the systematic difference of the velocity correlation functions due to meteorological conditions should not be discussed without consideration of their statistical errors. In the present paper, the probability distribution of the auto-correlation function obtained from the velocity fluctuation within a finite time interval is calculated for a model fluctuation.

1 Introduction

It seems that there are variations of the statistical properties of the turbulent velocity with meteorological conditions in the atmosphere. For example, given a long time series of observed data of the velocity fluctuation, and if we calculate the values of the auto-correlation function of velocity from the data of a finite time interval which is a part of the given time interval, we may perhaps find that the values differ according as the position of the part in the whole time interval. These variations of the statistical properties of the turbulent velocity in the atmosphere obtained from finite numbers of data cannot simply be attributed to inhomogeneity of the atmospheric conditions, since the statistical errors due to the use of limited numbers of data may inevitably obscure the results of calculation. Therefore, it is necessary to evaluate the dispersion of the statistical properties of the turbulent velocity composed of finite numbers of data in the case of homogeneous turbulence. Then the variations that exceed the statistical probable errors may be said to be due to inhomogeneity of the atmospheric conditions.

In the present paper, we evaluate the probability distribution of velocity correlation function obtained from the values of the velocity within a finite time interval under a statistically stationary condition.

2 A Model of the Velocity Fluctuations

We shall deal with the v riations of one component of velocity with time. To find the probability distribution of the auto-correlation function obtained from the velocity fluctuation within a finite time interval, we must know, for instance, the stochastic relation of the velocity at any two times (i.e. the joint-probability density function), rather than the velocity correlation function. The absence, at present, of any available representation obliges us to imagine a model of the velocity fluctuation which seems to be relevant to our problem. We shall assume that the turbulent velocity is statistically stationary, and define velocity u as the deviation from the mean velocity for an infinite time interval.

The joint-probability density $P(u_{t_1}, u_{t_2})$ of velocity at any two instances t_1, t_2 will have non-zero values only near a straight line $u_{t_2}=u_{t_1}$ in the (t_1, t_2) plane for $t_1 \fallingdotseq t_2$; and as $|t_1-t_2|\to\infty$, $P(u_{t_1}, u_{t_2})$ will have a distribution of a circular contour, of which the centre is a point $u_{t_1}=u_{t_2}=0$. Thus, we will assume that $P(u_{t_1}, u_{t_2})$ has the distribution of elliptical contours as shown in Fig. 1. The middle point of the chord of an ellipse at u_{t_1}=const. (see Fig. 1) is on a straight line $u_{t_2}=\{\varepsilon^2/(2-\varepsilon^2)\}u_{t_1}$, where ε is the eccentricity of the ellipse. If we assume that all the ellipses in Fig. 1 are similar in shape, we have ε=const, and u_{t_2} has the symmetrical distribution for every u_{t_1}, of which the centre is on the straight line given by $u_{t_2}=ru_{t_1}$, where $r=\varepsilon^2/(2-\varepsilon^2)<1$.

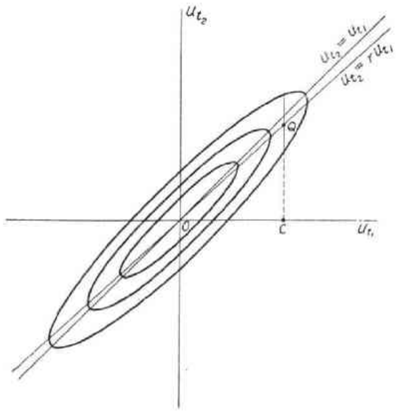

Fig. 1 Approximate density distribution of $P(u_{t_1}, u_{t_2})$ for a given value of $|t_1-t_2|$. Every ellipse (assumed to be similar) is the curve given by $P(u_{t_1}, u_{t_2})$=const. A middle point Q of a chord $u_{t_1}=c$ is on a line $u_{t_2}=ru_{t_1}$ independently of c.

Modifying the above intuitive consideration, we shall adopt the following model of the fluctuations of velocities.

(H. 1) $u_t=ru_{t-1}+X_t$, where u_t denotes the velocity at time t.

(H. 2) u_t has the expectation o and the dispersion $\overline{u^2}$, which are both independent of t.

(H. 3) r=const. $0<r<1$.

(H. 4) X_t is a random variable, which has a normal distribution $N(0, \sigma^2)$ independent of t.

(H. 5) For $t_1 \neq t_2$, X_{t_1} and X_{t_2} are statistically independent of each other.

Using the relation $\lim_{n\to\infty} r^n u_{t-n}=0$, (H. 1) becomes

$$u_t = \sum_{n=0}^{\infty} r^n X_{t-n}\,. \tag{1}$$

The representations of $\bar{u}$ and $\overline{u^2}$ are readily found from (1), (H. 4) and (H. 5):

$$\bar{u}=0\,,$$

$$\overline{u^2}=\frac{\sigma^2}{1-r^2}\,. \tag{2}$$

Bar above the letter denotes always the expected value. The relation $\bar{u}=0$ is nothing more than the definition of u.

The values of r and σ depend on the time-unit. If we assume that r and σ are changed to r' and σ' by the change of the time-unit from 1 to $1/n$, it is required from (H. 1) that

$$u_t = r'^n u_{t-1} + (X_t' + r' X'_{t-1/n} + r'^2 X'_{t-2/n} + \cdots\cdots + r'^{(n-1)} X_{t-(n-1)/n}) \,.$$

The second term of the right side of the above formula is independent of t, provided t is integer. Consequently, we can obtain from a comparison with (H. 1)

$$r = r'^n \quad \text{or} \quad a = na' \,, \tag{3}$$
$$\frac{\sigma'^2}{1-r'^2} = \frac{\sigma^2}{1-r^2} = \overline{u^2} \,,$$

where $a = -\log r$. From (3) we may find that a is proportional to time-unit, and therefore, $t_1 \equiv at$ and $k_1 \equiv k/a$ become non-dimensional values.

From (H. 5) we have

$$\overline{X_m X_n} = \delta_{m,n} \cdot \sigma^2 \,, \tag{4}$$

where $\delta_{m,n}$ is the Kronecker's delta. Unless any two of m, n, p and q are equal,

$$\left.\begin{aligned} &\overline{X_m X_n X_p X_q} = \overline{X_m^2 X_p X_q} = \overline{X_m^3 X_n} = 0 \,, \\ &\overline{X_m^2 X_n^2} = \sigma^4 \,, \\ &\overline{X_m^4} = 3\sigma^4 \,. \end{aligned}\right\} \tag{5}$$

3 Justification of the Model

We can check the validity of our model, by comp rison of the velocity correlation function or the energy spectrum function derived from the velocity of a infinite time interval by our model, and the generally admitted relations.

From our model, the velocity correlation function $R(t)$ and one-dimensional energy spectrum function $\phi(k)$ are easily found to be

$$\begin{aligned} R(t) &= \overline{u_\tau u_{\tau+t}} \,, \\ &= \overline{u^2}\, e^{-|t_1|} \end{aligned} \tag{6}$$

$$\begin{aligned} \phi(k_1) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R(t_1)\, e^{-ik_1 t_1}\, dt_1 \\ &= \frac{\overline{u^2}}{\pi} \cdot \frac{1}{1+k_1^2} \,. \end{aligned} \tag{7}$$

Y. Ogura (1953) has shown that when the velocity U of the mean flow is sufficiently large compared with the intensity $(\overline{u^2})^{1/2}$ of the turbulence, the Eulerian time-correlation function $R(t)$ has the same functional form as the space-correlation function, and that the departure from 1 of m in $1-R(t) \sim t^m$ has a maximum value in this case $(U/(\overline{u^2})^{1/2} \to \infty)$. Consequently, the exponential approximation for the Eulerian correlation function may have higher accuracy than that for the spatial longitudinal correlation function $f(r)$. We can, therefore, estim ite the maximum error of our model, by assuming that $f(r)$ is of the same functional form with (6), and examining the validity of the approximation. From this assumption for $f(r)$, we have the three-dimensional energy spectrum function $E(k_1)$:

$$E(k_1)=\frac{\overline{u^2}}{\pi}\cdot\frac{8k_1^4}{(1+k_1^2)^3} \tag{8}$$

(See G.K. BATCHELOR, p. 50). For the cases $k\ll 1$ and $k\gg 1$, the expression (8) becomes

$$E(k)\sim k^4 \qquad \text{for } k\ll 1\,,$$

$$E(k)\sim k^{-2} \qquad \text{for } k\gg 1\,.$$

The generally admitted relations are

$$E(k)\sim k^4 \qquad \text{for } k\ll 1\,,$$

$$E(k)\sim k^{-5/3} \qquad \text{for } k\gg 1\,.$$

Thus, there is the error of 20 percent in the power of k for the large wave number.

Consequently, the results in the present paper seem to be applicable to all the time- and space-correlation functions for the problems which do not require more than the above-mentioned accuracy.

4 The Probability Distribution of the Velocity Correlation Function

From the expression (1), the averaged value of the velocity u over the interval $[-T,\,T]$ (in terms of the random variable X) is found as

$$\{\bar u\}_T=\frac{1}{2T+1}\sum_{m=-T}^{T}u_m=\frac{1}{2T+1}\sum_{m=-T}^{T}\sum_{n=0}^{\infty}r^nX_{m-n}$$

$$=\frac{1}{2T+1}\left\{\sum_{s=1}^{\infty}\frac{r^s(1-r^{2T+1})}{1-r}X_{-T-s}+\sum_{s=-T}^{T}\frac{1-r^{T-s+1}}{1-r}X_s\right\}. \tag{9}$$

We cannot know the mean value of the velocity over an infinite interval, from the velocity within a finite interval. Consequently, we must use $u'=u-\{\bar u\}_T$ in place of u in $\{R(t)\}_T$.

$$u'_{-T+m}=u_{-T+m}-\{\bar u\}_T$$

$$=-\sum_{s=0}^{2T-m-1}\frac{1-r^{s+1}}{(2T+1)(1-r)}X_{T-s}+\sum_{s=2T-m}^{2T-1}\left(r^{s-2T+m}-\frac{1-r^{s+1}}{(2T+1)(1-r)}\right)X_{T-s}$$

$$+\sum_{s=0}^{\infty}r^s\left(r^m-\frac{1-r^{2T+1}}{(2T+1)(1-r)}\right)X_{-T-s} \tag{10}$$

$$(m=0,1,2,\ldots\ldots\ldots,2T)\,.$$

The correlation function between the velocities at time points of an interval t apart, obtained from u' within a finite time interval $2T$ can be written as

$$[R(t)]_T=[\overline{u_\tau'u'_{\tau+t}}]_T=[\overline{(u_\tau-\{\bar u\}_T)(u_{\tau+t}-\{\bar u\}_T)}]_T\,,$$

for $t \leqq T$

$$[R(t)]_T = \{\overline{u_\tau u_{\tau+t}}\}_T - \frac{1}{2T-t+1}\left[(2T+1)\{\bar{u}\}_T^2 + (2T-2t+1)\{\bar{u}\}_T\{\bar{u}\}_{T-t}\right] + \{\bar{u}\}_T^2$$

$$= \{\overline{u_\tau u_{\tau+t}}\}_T - \frac{t}{2T-t+1}\{\bar{u}\}_T^2 - \left(1 - \frac{t}{2T-t+1}\right)\{\bar{u}\}_T\{\bar{u}\}_{T-t}\,, \qquad (11)$$

for $t > T$

$$[R(t)]_T = \{\overline{u_\tau u_{\tau+t}}\}_T - \frac{t}{2T-t+1}\{\bar{u}\}_T^2 - \left(1 - \frac{t}{2T-t+1}\right)\{\bar{u}\}_T\{\bar{u}\}_{t-T-1}\,, \quad (12)$$

where $\{\ \}_T$ and $[\ ]_T$ denote the values obtained from u and u' within the interval of $[-T, T]$ respectively. We had the expression of $\{\bar{u}\}_T$ in (9), and $\{\overline{u_\tau u_{\tau+t}}\}_T$ is given by

$$\{\overline{u_\tau u_{\tau+t}}\}_T = \frac{1}{2T-t+1}\sum_{\tau=-T}^{T-t}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^{m+n} X_{\tau-n} X_{\tau+t-m}\,. \qquad (13)$$

The expectation $\overline{[R(t)]_T}$ and the dispersion $[\sigma(t)]_T^2$ of $[R(t)]_T$ are found from (4), (5), (11), (12) to be (See Appendix)*

$$\overline{[R(t)]_T} = \overline{u^2}\left\{e^{-t} + \frac{1}{2T-t}\left(-2 + \frac{t}{T} + \frac{t}{2T^2} + \frac{1}{T}e^{-t} - \frac{1}{T}e^{-2T+t} - \frac{t}{2T^2}e^{-2T}\right)\right\}, \qquad (14)^{**}$$

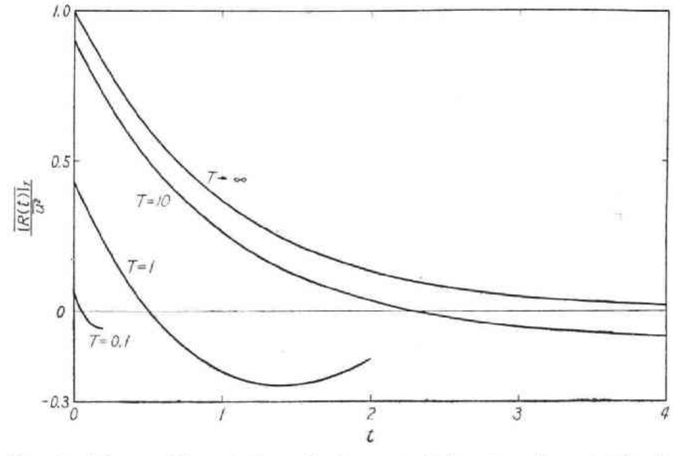

Fig. 2 The profiles of the velocity correlation function obtained from the velocity within a time interval $2T$.

* In the following expressions (also in Figs. and conclusions), $T_1(=aT)$ and $t_1(=at)$ under the limit as $a\to 0$ are, for simplicity, written as T and t. In this case, therefore, the unit of T and t are determined by the relation $\int_0^\infty R(t)dt = \overline{u^2}$.

** (14) is immediately found from the general formula by Y. OGURA (1957) by replacing $\overline{f_T(\tau)}$, T, τ and $f_\infty(\zeta)$ in his expression (11) by $\overline{[R(t)]_T}$, $2T$, t and $e^{-\zeta}$.

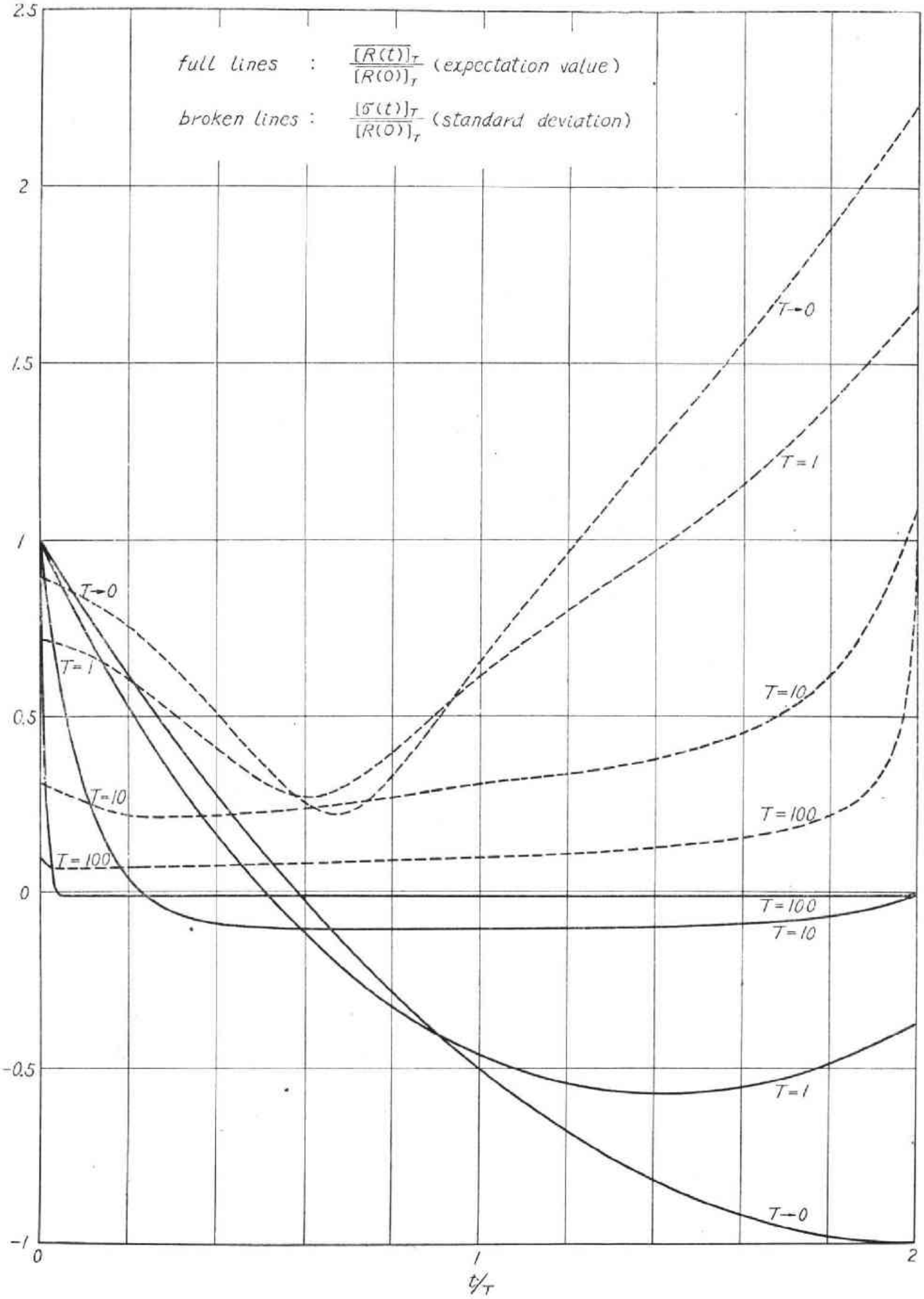

Fig. 3. Diagram of the expectation and standard deviation of the velocity correlation function $[R(t)]_T$ obtained from the velocity within a time interval $2T$.

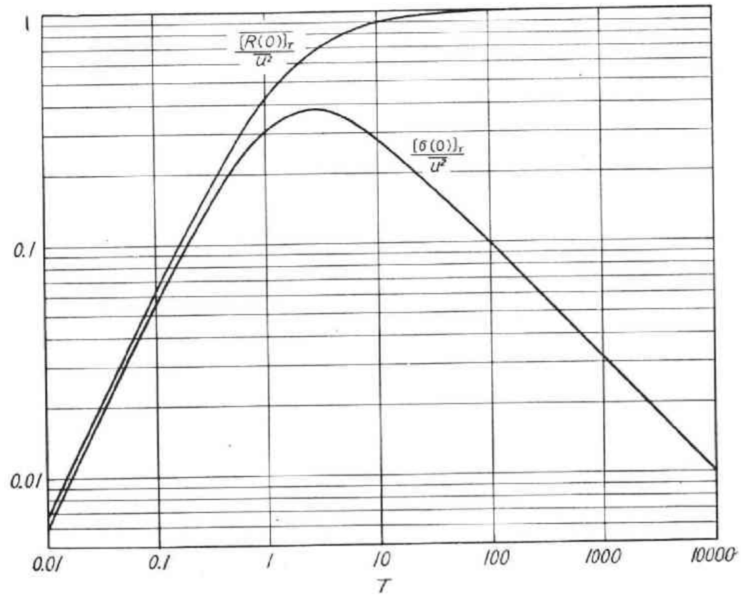

Fig. 4 Curves of the expectation and standard deviation at $t=0$ ($[R(0)]_T$ is equal to $\overline{u'^2}$).

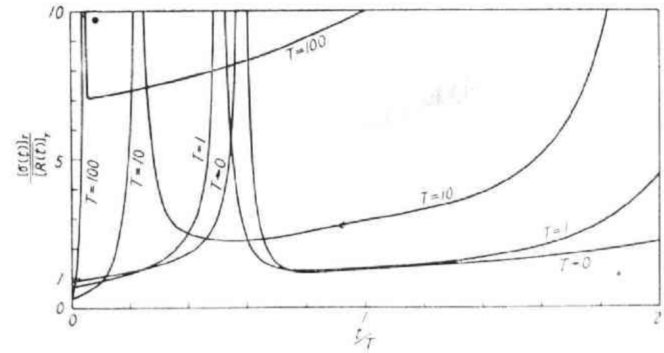

Fig. 5 Ratio of the standard deviation $[\sigma(t)]_T$ to the expectation $[\overline{R(t)}]_T$.

$$[\sigma(t)]_T^2=\frac{(\overline{u^2})^2}{(2T-t)^2}\left\{\left(2T-t-\frac{17}{2}+\frac{4t}{T}+\frac{3}{T}+\frac{2t^2}{T^2}+\frac{2t}{T^2}+\frac{1}{T^2}+\frac{2t^2}{T^3}+\frac{t^2}{2T^4}\right)\right.$$

$$+e^{-t}\left(\frac{3t}{T^2}+\frac{2t}{T^3}\right)+e^{-2t}\left(4Tt+2T-3t^2-2t-\frac{1}{2}+\frac{2t}{T}+\frac{3}{T}+\frac{1}{T^2}\right)$$

$$+e^{-2T+2t}\left(-4+\frac{4t}{T}-\frac{5}{T}-\frac{1}{T^2}\right)+e^{-2T+t}\left(-\frac{2t}{T}+\frac{t^2}{T^2}-\frac{4t}{T^2}-\frac{2t}{T^3}\right)$$

$$+e^{-2T}\left(-4+\frac{2t}{T}-\frac{3}{T}-\frac{2t}{T^2}-\frac{3}{T^2}-\frac{2t^2}{T^3}-\frac{t^2}{T^4}\right)+e^{-2T-t}\left(-\frac{2t}{T}+\frac{t^2}{T^2}-\frac{2t}{T^3}\right)$$

$$+e^{-4T+2t}\left(1+\frac{2}{T}+\frac{2}{T^2}\right)+e^{-4T+t}\left(\frac{t}{T^2}+\frac{2t}{T^3}\right)+e^{-4T}\left(\frac{t^2}{2T^4}\right)\Big\}$$

$$\text{for} \quad t \leqq T\,,$$

$$=\frac{(\overline{u^2})^2}{(2T-t)^2}\Big\{e^{2T-2t}\left(-\frac{1}{T}-\frac{1}{T^2}\right)+\left(2T-t-\frac{1}{2}-\frac{4t}{T}+\frac{3}{T}+\frac{2t^2}{T^2}-\frac{2t}{T^2}+\frac{1}{T^2}+\frac{2t^2}{T^3}+\frac{t^2}{2T^4}\right)$$

$$+e^{-t}\left(\frac{3t}{T^2}+\frac{2t}{T^3}\right)+e^{-2t}\left(4T^2-4Tt+t^2+4-\frac{2t}{T}+\frac{2}{T^2}\right)$$

$$+e^{-2T+t}\left(-\frac{2t}{T}+\frac{t^2}{T^2}-\frac{4t}{T^2}-\frac{2t}{T^3}\right)+e^{-2T}\left(-4+\frac{2t}{T}-\frac{3}{T}+\frac{2t}{T^2}-\frac{3}{T^2}-\frac{2t^2}{T^3}-\frac{t^2}{T^4}\right)$$

$$+e^{-2T-t}\left(-\frac{2t}{T}+\frac{t^2}{T^2}-\frac{2t}{T^3}\right)+e^{-4T+2t}\left(\frac{1}{2}+\frac{1}{T}+\frac{1}{T^2}\right)+e^{-4T+t}\left(\frac{t}{T^2}+\frac{2t}{T^3}\right)$$

$$+e^{-4T}\left(\frac{t^2}{2T^4}\right)\Big\} \qquad \text{for} \quad t > T\,. \qquad (15)$$

These results are illustrated in Figs. 2～5, with the ratio of the standard deviation $[\sigma(t)]_T$ to the expectation $\overline{[R(t)]_T}$.

5 Conclusions

Figs. 3, 4, 5 are usefull to estimate the statistical errors of the velocity correlation function obtained from the velocity within a finite interval. Moreover, we can find the following properties*.

(i) For $T \ll 1$. If we confine ourselves to the problems in inertial subrange, the expectation and the standard deviation of $[R(t)]_T$ have the nearly constant profiles, though their absolute values are proportional to T. Consequently, the relative errors are also independent of T.

(ii) For $T \approx 1\left(=\int_0^\infty (R(t)/\overline{u^2})\,dt\right)$. The expectation and the standared deviation vary largely with T.

(iii) For $T \gg 1$. $\overline{[R(t)]_T}$ is nearly equal to $R(t)$, but $[\sigma(t)]_T$ decreases as the $(-1/2)$ power of T. Consequently, the relative errors also decrease as the $(-1/2)$ power of T.

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Appendix

The derivation of the expressions (14), (15) from (11), (12)

We shall introduce matrixes R, U, V, W, of which elements are R_{ij}, U_{ij}, V_{ij}, W_{ij} defined by (c. f. (13) and (9))

$$(1-r^2)\sum_{\tau=-T}^{T-t}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^{m+n} X_{\tau+t-m}\, Y_{\tau-n}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} R_{ij} X_{T+1-i}\, Y_{T+1-j}\,,$$

* The reason why $\overline{[R(t)]_T}$ has negative region, may be easily understood from the fact that $[R(t)]_T$ is defined as the correlation of u' (but not of u).

$$\left\{\sum_{s=1}^{\infty} r^s(1-r^{2T+1})\,X_{-T-s}+\sum_{s=-T}^{T}(1-r^{T-s+1})\,X_s\right\}\left\{\sum_{s=1}^{\infty} r^s(1-r^{2T+1})\,Y_{-T-s}\right.$$

$$\left.+\sum_{s=-T}^{T}(1-r^{T-s+1})\,Y_s\right\}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}U_{ij}\,X_{T+1-i}\,Y_{T+1-j}\,,$$

$$\left\{\sum_{s=1}^{\infty} r^s(1-r^{2T+1})\,X_{-T-s}+\sum_{s=-T}^{T}(1-r^{T-s+1})\,X_s\right\}\left\{\sum_{s=1}^{\infty} r^s(1-r^{2T-2t+1})\,Y_{-T+t-s}\right.$$

$$\left.+\sum_{s=-T+t}^{T-t}(1-r^{T-t-s+1})\,Y_s\right\}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}V_{ij}\,X_{T+1-i}\,Y_{T+1-j}\qquad\text{for}\quad t\leqq T\,,$$

$$\left\{\sum_{s=1}^{\infty} r^s(1-r^{2T+1})\,X_{-T-s}+\sum_{s=-T}^{T}(1-r^{T-s+1})\,X_s\right\}\left\{\sum_{s=1}^{\infty} r^s(1-r^{2t-2T-1})\,Y_{-t+T+1-s}\right.$$

$$\left.+\sum_{s=-t+T+1}^{t-T-1}(1-r^{t-T-s})\,Y_s\right\}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}W_{ij}\,X_{T+1-i}\,Y_{T+1-j}\qquad\text{for}\quad t>T\,.$$

The elements of matrixes R, U, V, W may be written as follows:

$$R=\begin{pmatrix} R_{11}, & R_{12}, & R_{13}, & \cdots\cdots \\ R_{21}, & R_{22}, & R_{23}, & \cdots\cdots \\ R_{31}, & R_{32}, & R_{33}, & \cdots\cdots \\ & \cdots\cdots & & \cdots\cdots \\ & \cdots\cdots & & \cdots\cdots \end{pmatrix},$$

where elements R_{ij} are

$$\begin{aligned} R_{ij}&=0 &&\text{for}\quad j\leqq t\,,\\ &=r^{i+t-j}(1-r^{2j-2t}) &&\text{for}\quad t+1\leqq j\leqq i+t \text{ and } j\leqq 2T+1\,,\\ &=r^{j-i-t}(1-r^{2i}) &&\text{for}\quad i+t\leqq j\leqq 2T+1\,,\\ &=r^{j-2T-1}R_{i,2T+1} &&\text{for}\quad j>2T+1\,. \end{aligned}$$

Matrixes U, V, W can be separated in i and j:

$$U_{ij}=U_i\,U_j,\qquad V_{ij}=U_i\,V_j,\qquad W_{ij}=U_i\,W_j\,,$$

where

$$\begin{aligned} U_i&=(1-r^i) &&\text{for}\quad i\leqq 2T+1\,,\\ &=r^{i-2T-1}(1-r^{2T+1}) &&\text{for}\quad i>2T+1\,,\\ V_j&=0 &&\text{for}\quad j\leqq t\,,\\ &=(1-r^{j-t}) &&\text{for}\quad t+1\leqq j\leqq 2T-t+1\,,\\ &=r^{j-2T+t-1}(1-r^{2T-2t+1}) &&\text{for}\quad j>2T-t+1\,,\\ W_j&=0 &&\text{for}\quad j\leqq 2T-t+1\,,\\ &=(1-r^{j-2T+t-1}) &&\text{for}\quad 2T-t+2\leqq j\leqq t\,,\\ &=r^{j-t}(1-r^{2t-2T-1}) &&\text{for}\quad t<j\,. \end{aligned}$$

From the relations (4), (5), we may find expressions :

$$\overline{\{R(t)\}_T} = \frac{\overline{u^2}}{2T-t+1}\,\text{Spur}\,R\,,$$

$$\overline{\{\bar u\}_T^2} = \frac{\overline{u^2}}{(2T+1)^2}\left(\frac{1+r}{1-r}\right)\text{Spur}\,U\,,$$

$$\overline{\{\bar u\}_T\{\bar u\}_{T-t}} = \frac{\overline{u^2}}{(2T+1)\,(2T-2t+1)}\left(\frac{1+r}{1-r}\right)\text{Spur}\,V\,,$$

$$\overline{\{\bar u\}_T\{\bar u\}_{t-T-1}} = \frac{\overline{u^2}}{(2T+1)\,(2t-2T-1)}\left(\frac{1+r}{1-r}\right)\text{Spur}\,W\,,$$

$$\overline{\{R(t)\}_T^2} = \frac{(\overline{u^2})^2}{(2T-t+1)^2}\left\{(\text{Spur}\,R)^2+\text{Spur}\,R\widetilde{R}+\text{Spur}\,R^2\right\},$$

$$\overline{\{\bar u\}_T^4} = \frac{(\overline{u^2})^2}{(2T+1)^4}\left(\frac{1+r}{1-r}\right)^2\left\{(\text{Spur}\,U)^2+\text{Spur}\,U\widetilde{U}+\text{Spur}\,U^2\right\},$$

$$\overline{\{\bar u\}_T^2\{\bar u\}_{T-t}^2} = \frac{(\overline{u^2})^2}{(2T+1)^2\,(2T-2t+1)^2}\left(\frac{1+r}{1-r}\right)^2$$
$$\times\left\{(\text{Spur}\,V)^2+\text{Spur}\,V\widetilde{V}+\text{Spur}\,V^2\right\},$$

$$\overline{\{\bar u\}_T^2\{\bar u\}_{t-T-1}^2} = \frac{(\overline{u^2})^2}{(2T+1)^2\,(2t-2T-1)^2}\left(\frac{1+r}{1-r}\right)^2$$
$$\times\left\{(\text{Spur}\,W)^2+\text{Spur}\,W\widetilde{W}+\text{Spur}\,W^2\right\},$$

$$\overline{\{R(t)\}_T\{\bar u\}_T^2} = \frac{(\overline{u^2})^2}{(2T-t+1)\,(2T+1)^2}\left(\frac{1+r}{1-r}\right)$$
$$\times\left\{(\text{Spur}\,R)\,(\text{Spur}\,U)+\text{Spur}\,R\widetilde{U}+\text{Spur}\,RU\right\},$$

$$\overline{\{R(t)\}_T\{\bar u\}_T\{\bar u\}_{T-t}} = \frac{(\overline{u^2})^2}{(2T-t+1)\,(2T+1)\,(2T-2t+1)}\left(\frac{1+r}{1-r}\right)$$
$$\times\left\{(\text{Spur}\,R)\,(\text{Spur}\,V)+\text{Spur}\,R\widetilde{V}+\text{Spur}\,RV\right\},$$

$$\overline{\{R(t)\}_T\{\bar u\}_T\{\bar u\}_{t-T-1}} = \frac{(\overline{u^2})^2}{(2T-t+1)\,(2T+1)\,(2t-2T-1)}\left(\frac{1+r}{1-r}\right)$$
$$\times\left\{(\text{Spur}\,R)\,(\text{Spur}\,W)+\text{Spur}\,R\widetilde{W}+\text{Spur}\,RW\right\},$$

$$\overline{\{\bar u\}_T^3\{\bar u\}_{T-t}} = \frac{(\overline{u^2})^2}{(2T+1)^3\,(2T-2t+1)}\left(\frac{1+r}{1-r}\right)^2$$
$$\times\left\{(\text{Spur}\,U)\,(\text{Spur}\,V)+\text{Spur}\,U\widetilde{V}+\text{Spur}\,UV\right\},$$

$$\overline{\{\bar u\}_T^3\{\bar u\}_{t-T-1}} = \frac{(\overline{u^2})^2}{(2T+1)^3\,(2t-2T-1)}\left(\frac{1+r}{1-r}\right)^2$$
$$\times\left\{(\text{Spur}\,U)\,(\text{Spur}\,W)+\text{Spur}\,U\widetilde{W}+\text{Spur}\,UW\right\},$$

where Spur denotes the diagonal sum of the matrix, and $\sim$ transposed matrix. In general, the square matrixes A, B of the same order have the properties :

$$\text{Spur}\,AB = \text{Spur}\,BA\,\left(=\sum_i\sum_j A_{ij}B_{ji}\right),$$

$$\mathrm{Spur}\, \widetilde{A}B = \mathrm{Spur}\, \widetilde{B}A \left(= \sum_i \sum_j A_{ij} B_{ij}\right).$$

We shall also define R_i, R_j by

$$\begin{aligned}
R_i &= \sum_j R_{ij} U_j , \\
&= \sum_{n=0}^{i-1} r^{i-n-1} (1-r^{2(n+1)}) (1-r^{t+n+1}) + \sum_{n=0}^{2T-t-i} r^{n+1} (1-r^{2i}) (1-r^{t+n+i+1}) \\
&\quad + \sum_{n=0}^{\infty} r^{2T-t+2n-i+3} (1-r^{2i}) (1-r^{2T+1}) \\
&= -i\, r^{t+i} + \frac{1-r^i}{1-r^2} \left\{(1+r)^2 - r^{2T-t+2} - r^{2T-t-i+2}\right\} \\
&\qquad\qquad \text{for} \quad i \leqq 2T-t+1 , \\
&= r^{-2T+t+i-1} \left\{ \sum_{n=0}^{2T-t} r^{2T-t-n} (1-r^{2(n+1)}) (1-r^{t+n+1}) \right. \\
&\quad \left. + \sum_{n=0}^{\infty} r^{2(n+1)} (1-r^{2(2T-t+1)}) (1-r^{2T+1}) \right\} \\
&= r^{-2T+t+i-1} \left[-(2T-t+1)\, r^{2T+1} + \frac{1-r^{2T-t+1}}{1-r^2} \left\{(1+r)^2 - r^{2T-t+2} - r\right\} \right] \\
&\qquad\qquad \text{for} \quad i > 2T-t+1 ,
\end{aligned}$$

$$\begin{aligned}
R_j &= \sum_i R_{ij} U_i , \\
&= 0 \qquad\qquad \text{for} \quad j \leqq t , \\
&= \sum_{n=0}^{-t+j-1} r^{-t-n+j-1} (1-r^{2(n+1)}) (1-r^{(n+1)}) + \sum_{n=0}^{2T+t-j} r^{n+1} (1-r^{2(-t+j)}) (1-r^{-t+n+j+1}) \\
&\quad + \sum_{n=0}^{\infty} r^{2T+t-j+2n+3} (1-r^{2(j-t)}) (1-r^{2T+1}) \\
&= (t-j)\, r^{-t+j} + \frac{1-r^{-t+j}}{1-r^2} \left\{(1+r)^2 - r^{2T+2} - r^{2T+t-j+2}\right\} \\
&\qquad\qquad \text{for} \quad t+1 \leqq j \leqq 2T+1 , \\
&= r^{-2T+j-1} \left\{ \sum_{n=0}^{2T-t} r^{2T-t-n} (1-r^{2(n+1)}) (1-r^{n+1}) \right. \\
&\quad \left. + \sum_{n=0}^{t-1} r^{n+1} (1-r^{2(2T-t+1)}) (1-r^{2T-t+n+2}) + \sum_{n=0}^{\infty} r^{2n+t+2} (1-r^{2(2T-t+1)}) (1-r^{2T+1}) \right\} \\
&= r^{-2T+j-1} \left[-(2T-t+1)\, r^{2T-t+1} + \frac{1-r^{2T-t+1}}{1-r^2} \left\{(1+r)^2 - r^{2T+2} - r^{t+1}\right\} \right] \\
&\qquad\qquad \text{for} \quad j > 2T+1 .
\end{aligned}$$

Thus, we have

$$\mathrm{Spur}\, U = \sum_{n=0}^{2T} (1-r^{n+1})^2 + \frac{r^2}{1-r^2} (1-r^{2T+1})^2 ,$$

$$\text{Spur } V = \sum_{n=0}^{2T-2t} (1-r^{t+n+1})(1-r^{n+1}) + \sum_{n=0}^{t-1} (1-r^{2T-t+n+2})\, r^{n+1} (1-r^{2T-2t+1})$$

$$+ \frac{r^{t+2}}{1-r^2} (1-r^{2T+1})(1-r^{2T-2t+1}) ,$$

$$\text{Spur } W = \sum_{n=0}^{-2T+2t-2} (1-r^{2T-t+n+2})(1-r^{n+1}) + \sum_{n=0}^{2T-t} (1-r^{t+n+1})\, r^{n+1} (1-r^{-2T+2t-1})$$

$$+ \frac{r^{2T-t+3}}{1-r^2} (1-r^{2T+1})(1-r^{-2T+2t-1}) ,$$

$$\text{Spur } R = r^t \left\{ \sum_{n=0}^{2T-t} (1-r^{2n+2}) + \frac{r^2}{1-r^2} (1-r^{2(2T-t+1)}) \right\} ,$$

$$\text{Spur } U\widetilde{U} = \text{Spur } U^2 = (\text{Spur } U)^2 ,$$

$$\text{Spur } V\widetilde{V} = (\text{Spur } U) \times \left\{ \sum_{n=0}^{2T-2t} (1-r^{n+1})^2 + \frac{r^2}{1-r^2} (1-r^{2T-2t+1})^2 \right\} ,$$

$$\text{Spur } V^2 = (\text{Spur } V)^2 ,$$

$$\text{Spur } W\widetilde{W} = (\text{Spur } U) \times \left\{ \sum_{n=0}^{-2T+2t-2} (1-r^{n+1})^2 + \frac{r^2}{1-r^2} (1-r^{-2T+2t-1})^2 \right\} ,$$

$$\text{Spur } W^2 = (\text{Spur } W)^2 , \qquad \text{Spur } U\widetilde{V} = \text{Spur } UV = (\text{Spur } U)(\text{Spur } V) ,$$

$$\text{Spur } U\widetilde{W} = \text{Spur } UW = (\text{Spur } U)(\text{Spur } W) ,$$

$$\text{Spur } R\widetilde{U} = \sum_j R_j U_j$$

$$= \sum_{j=t+1}^{2T+1} (1-r^j) \left[(t-j) r^{j-t} + \frac{1-r^{j-t}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{2T+t-j+2} \right\} \right]$$

$$+ \sum_{j=2T+2}^{\infty} r^{2(j-2T-1)} (1-r^{2T+1}) \Big[-(2T-t+1)\, r^{2T-t+1}$$

$$+ \frac{1-r^{2T-t+1}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{t+1} \right\} \Big] ,$$

$$\text{Spur } RU = \text{Spur } R\widetilde{U} ,$$

$$\text{Spur } R\widetilde{V} = \sum_j R_j V_j$$

$$= \sum_{j=t+1}^{2T-t+1} (1-r^{j-t}) \left[-(j-t) r^{j-t} + \frac{1-r^{j-t}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{2T+t-j+2} \right\} \right]$$

$$+ \sum_{j=2T-t+2}^{2T+1} r^{j-2T+t-1} (1-r^{2T-2t+1}) \Big[-(j-t)\, r^{j-}$$

$$+ \frac{1-r^{j-t}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{2T+t-j+2} \right\} \Big]$$

$$+ \sum_{j=2T+2}^{\infty} r^{2j-4T+t-2} (1-r^{2T-2t+1}) \Big[-(2T-t+1)\, r^{2T-t+1}$$

$$+ \frac{1-r^{2T-t+1}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{t+1} \right\} \Big] ,$$

$$
\begin{aligned}
\text{Spur } RV &= \sum_i R_i V_i \\
&= \sum_{i=t+1}^{2T-t+1} (1-r^{i-t}) \Big[-i r^{i+t} + \frac{1-r^i}{1-r^2} \left\{ (1+r)^2 - r^{2T-t+2} - r^{2T-t-i+2} \right\} \\
&\quad + \sum_{i=2T-t+2}^{\infty} r^{2(i-2T+t-1)} (1-r^{2T-2t+1}) \Big[-(2T-t+1) r^{2T+1} \\
&\quad + \frac{1-r^{2T-t+1}}{1-r^2} \left\{ (1+r)^2 - r^{2T-t+2} - r \right\} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\text{Spur } R\widetilde{W} &= \sum_j R_j W_j \\
&= \sum_{j=t+1}^{2T+1} r^{j-t} (1-r^{2t-2T-1}) \Big[-(j-t) r^{j-t} + \frac{1-r^{j-t}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{2T+t-j+2} \right\} \Big] \\
&\quad + \sum_{j=2T+2}^{\infty} r^{j-t} (1-r^{2t-2T-1}) r^{j-2T-1} \Big[-(2T-t+1) r^{2T-t+1} \\
&\quad + \frac{1-r^{2T-t+1}}{1-r^2} \left\{ (1+r)^2 - r^{2T+2} - r^{t+1} \right\} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\text{Spur } RW &= \sum_i R_i W_i \\
&= \sum_{i=2T-t+2}^{t} (1-r^{i-2T+t-1}) r^{i-2T+t-1} \Big[-(2T-t+1) r^{2T+1} \\
&\quad + \frac{1-r^{2T-t+1}}{1-r^2} \left\{ (1+r)^2 - r^{2T-t+2} - r \right\} \Big] \\
&\quad + \sum_{i=t+1}^{\infty} r^{2i-2T-1} (1-r^{2t-2T-1}) \Big[-(2T-t+1) r^{2T+1} \\
&\quad + \frac{1-r^{2T-t+1}}{1-r^2} \left\{ (1+r)^2 - r^{2T-t+2} - r \right\} \Big],
\end{aligned}
$$

$$
\text{Spur } R\widetilde{R} = \sum_{n=0}^{2T-t} \frac{1+r^2}{1-r^2} (1-r^{2n+2})^2 + \frac{r^4}{(1-r^2)^2} \left\{ 1-r^{2(2T-t+1)} \right\}^2 ,
$$

$$
\begin{aligned}
\text{Spur } R^2 &= 2 \Big[\sum_{s=0}^{2T-2t} \Big\{ \sum_{n=0}^{t} (1-r^{2s+2}) r^{2t} (1-r^{2(n+s+1)}) \\
&\quad + \sum_{n=0}^{\infty} (1-r^{2s+2}) (1-r^{2(t+s+1)}) r^{2t+2n+2} \Big\} \\
&\quad + \sum_{s=0}^{t-1} \Big\{ \sum_{n=0}^{t-s-1} r^{2t} (1-r^{2(2T-2t+s+2)}) (1-r^{2(2T-2t+s+n+2)}) \\
&\quad + \sum_{n=0}^{\infty} (1-r^{2(2T-2t+s+2)}) (1-r^{2(2T-t+1)}) r^{2t+2n+2} \Big\} \\
&\quad + \frac{r^{2(t+2)}}{(1-r^2)^2 (1+r^2)} \left\{ 1-r^{2(2T-t+1)} \right\}^2 \Big] \\
&\quad - r^{2t} \Big\{ \sum_{n=0}^{2T-t} (1-r^{2n+2})^2 + \frac{r^4}{1-r^4} (1-r^{2(2T-t+1)})^2 \Big\} \qquad \text{for} \quad t \leqq T ,
\end{aligned}
$$

$$\text{Spur } R^2 = 2\Big[\sum_{s=0}^{2T-t}\Big\{\sum_{n=0}^{2T-t-s} r^{2t}(1-r^{2s+2})(1-r^{2(n+s+1)}) + \frac{r^{2(t+1)}}{1-r^2}(1-r^{2(s+1)})(1-r^{2(2T-t+1)})\Big\} + \frac{r^{2(t+2)}}{(1-r^2)^2(1+r^2)}\{1-r^{2(2T-t+1)}\}^2\Big]$$
$$-r^{2t}\Big\{\sum_{n=0}^{2T-t}(1-r^{2n+2})^2 + \frac{r^4}{1-r^4}(1-r^{2(2T-t+1)})^2\Big\} \qquad \text{for } t > T.$$

Calculating the serieses, we have

$$\text{Spur } U = (2T+1) - 2r\frac{1-r^{2T+1}}{1-r^2},$$

$$\text{Spur } V = (2T-2t+1) - 2r^{t+1}\frac{1-r^{2T-2t+1}}{1-r^2},$$

$$\text{Spur } W = (2t-2T-1) - 2r^{2T-t+2}\frac{1-r^{2t-2T-1}}{1-r^2},$$

$$\text{Spur } R = (2T-t+1)\, r^t,$$

$$\frac{\text{Spur } V\widetilde{V}}{\text{Spur } U} = (2T-2t+1) - 2r\frac{1-r^{2T-2t+1}}{1-r^2},$$

$$\frac{\text{Spur } W\widetilde{W}}{\text{Spur } U} = (2t-2T-1) - 2r\frac{1-r^{2t-2T-1}}{1-r^2},$$

$$\text{Spur } R\widetilde{U} = \frac{2T-t+1}{1-r^2}\{(1+r)^2 + r^{2T-t+2} + r^{2T+t+2}\}$$
$$-2r(1+r^t)\frac{1-r^{2T-t+1}}{(1-r)^2} + 2r^{t+2}\frac{1-r^{2(2T-t+1)}}{(1-r^2)^2},$$

$$\text{Spur } R\widetilde{V} = \text{Spur } RV$$
$$= \frac{2T-2t+1}{1-r^2}\{(1+r)^2 + r^{2T-2t+2} + r^{2T+2}\} - \frac{t}{1-r^2}r^{2t+1}(1-r^{2T-2t+1})$$
$$-\frac{1-r^{2T-2t+1}}{(1-r)^2}(2r + r^{t+1} + r^{2t+1}) + \frac{1-r^{2T-2t+1}}{(1-r^2)^2}(r^2 + r^{2t+2} + 2r^{2T+3}),$$

$$\text{Spur } R\widetilde{W} = \text{Spur } RW$$
$$= r\frac{1-r^{-2T+2t-1}}{1-r^2}\Big[-(2T-t+1)r^{2T+1} + \frac{1-r^{2T-t+1}}{1-r^2}\{(1+r)^2 - r^{2T-t+2} - r\}\Big],$$

$$\text{Spur } R\widetilde{R} = \frac{1+r^2}{1-r^2}(2T-t+1) - 2r^2\frac{1-r^{2(2T-t+1)}}{(1-r^2)^2},$$

$$\text{Spur } R^2 = r^{2t}\Big\{\frac{2(2T-2t+1)}{1-r^2} + (4Tt - 3t^2 - 2T + 4t - 1)$$
$$-\frac{2r^2}{(1-r^2)^2}(1-r^{2(2T-2t+1)})\Big\} \qquad \text{for } t \leqq T,$$
$$= r^{2t}(2T-t+1)^2 \qquad \text{for } t \geqq T.$$

From the above results, we find that the expectation and the dispersion of (11) and (12) become

$$\overline{[R(t)]_T} = \overline{u^2}\,(S_1+S_2+S_3)\,,$$

$$\begin{aligned}\overline{[\sigma(t)]_T^2} &= \overline{([R(t)]_T-\overline{[R(t)]_T})^2}\\ &= \overline{[R(t)]_T^2}-(\overline{[R(t)]_T})^2\\ &= (\overline{u^2})^2\left\{2\,(S_1+S_2)^2+(S_1S_4-S_2{}^2)-4\,(S_5+S_6)-(S_7+S_8)\right\}\,,\end{aligned}$$

where

$$S_1=-\frac{t}{(2T+1)^2\,(2T-t+1)}\left(\frac{1+r}{1-r}\right)\cdot \text{Spur}\, U\,,$$

$$S_2=-\frac{1}{(2T+1)\,(2T-t+1)}\left(\frac{1+r}{1-r}\right)\cdot \text{Spur}\, V \qquad \text{for} \quad t\leqq T\,,$$

$$=\frac{1}{(2T+1)\,(2T-t+1)}\left(\frac{1+r}{1-r}\right)\cdot \text{Spur}\, W \qquad \text{for} \quad t> T\,,$$

$$S_3=\frac{1}{2T-t+1}\cdot \text{Spur}\, R\,,$$

$$S_4=-\frac{1}{(2T-t+1)\,t}\left(\frac{1+r}{1-r}\right)\cdot \frac{\text{Spur}\, V\widetilde{V}}{\text{Spur}\, U} \qquad \text{for} \quad t\leqq T\,,$$

$$=-\frac{1}{(2T-t+1)\,t}\left(\frac{1+r}{1-r}\right)\cdot \frac{\text{Spur}\, W\widetilde{W}}{\text{Spur}\, U} \qquad \text{for} \quad t> T\,,$$

$$S_5=-\frac{t}{(2T+1)^2\,(2T-t+1)^2}\left(\frac{1+r}{1-r}\right)\cdot \text{Spur}\, R\widetilde{U}\,,$$

$$S_6=-\frac{1}{(2T+1)\,(2T-t+1)^2}\left(\frac{1+r}{1-r}\right)\cdot \text{Spur}\, R\widetilde{V} \qquad \text{for} \quad t\leqq T\,,$$

$$=\frac{1}{(2T+1)\,(2T-t+1)^2}\left(\frac{1+r}{1-r}\right)\cdot \text{Spur}\, R\widetilde{W} \qquad \text{for} \quad t> T\,,$$

$$S_7=\frac{1}{(2T-t+1)^2}\,\text{Spur}\, R\widetilde{R}\,,$$

$$S_8=\frac{1}{(2T-t+1)^2}\,\text{Spur}\, R^2\,.$$

In order to find the expressions for continuous variation, replacing T, t by $T_1=aT$, $t_1=at$, tending $a\to 0$, and for simplicity omitting the subscript 1, we find the expressions (14), (15) derived from

$$S_1=\frac{1}{2T-t}\left(-\frac{t}{T}+\frac{t}{2T^2}-\frac{t}{2T^2}\,e^{-2T}\right),$$

$$S_2=\frac{1}{2T-t}\left(-2+\frac{2t}{T}+\frac{1}{T}\,e^{-t}-\frac{1}{T}\,e^{-2T+t}\right),$$

$$S_3=e^{-t}\,,$$

$$S_4=\frac{1}{2T-t}\left(-\frac{4T}{t}+4+\frac{2}{t}-\frac{2}{t}\,e^{-2T+2t}\right) \qquad \text{for} \quad t\leqq T\,,$$

$$=\frac{1}{2T-t}\left(\frac{4T}{t}-4+\frac{2}{t}-\frac{2}{t}\,e^{2T-2t}\right) \qquad \text{for} \quad t> T\,,$$

$$S_5=\frac{1}{4\,(2T-t)^2}\left\{\left(-\frac{8t}{T}+\frac{4\,t^2}{T^2}+\frac{4\,t}{T^2}\right)+\frac{3t}{T^2}\,e^{-t}+\left(-\frac{2t}{T}+\frac{t^2}{T^2}-\frac{4\,t}{T^2}\right)e^{-2T+t}\right.$$

$$\left.-\frac{4\,t}{T^2}\,e^{-2T}+\left(-\frac{2t}{T}+\frac{t^2}{T^2}\right)e^{-2T-t}+\frac{t}{T^2}\,e^{-4T+t}\right\},$$

$$S_6=\frac{1}{4\,(2T-t)^2}\left\{\left(-16+\frac{16\,t}{T}+\frac{7}{T}\right)+\frac{4}{T}\,e^{-t}+\left(\frac{2\,t}{T}+\frac{3}{T}\right)e^{-2t}\right.$$

$$+\left(-4+\frac{4\,t}{T}-\frac{7}{T}\right)e^{-2T+2t}-\frac{4}{T}\,e^{-2T+t}+\left(-4+\frac{2t}{T}-\frac{5}{T}\right)e^{-2T}$$

$$\left.+\frac{2}{T}\,e^{-4T+2t}\right\} \qquad \text{for}\quad t\leqq T\,,$$

$$=\frac{1}{4\,(2T-t)^2}\left\{\frac{3}{T}-\frac{3}{T}\,e^{2T-2t}+\frac{4}{T}\,e^{-t}+\left(4-\frac{2t}{T}\right)e^{-2t}-\frac{4}{T}\,e^{-2T+t}\right.$$

$$\left.+\left(-4+\frac{2t}{T}-\frac{1}{T}\right)e^{-2T}+\frac{1}{T}\,e^{-4T+2t}\right\} \qquad \text{for}\quad t>T\,,$$

$$S_7=\frac{1}{(2T-t)^2}\left\{\left(2T-t-\frac{1}{2}\right)+\frac{1}{2}\,e^{-4T+2t}\right\},$$

$$S_8=\frac{1}{(2T-t)^2}\left\{\left(4\,T\,t+2T-3\,t^2-2\,t-\frac{1}{2}\right)e^{-2t}+\frac{1}{2}\,e^{-4T+2t}\right\} \qquad \text{for}\quad t\leqq T\,,$$

$$=\frac{1}{(2T-t)^2}\left\{(4T^2-4Tt+t^2)e^{-2t}\right\} \qquad \text{for}\quad t>T\,.$$

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