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Numerical Study of Water Vapour Transmission

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Abstract

The most accurate representation of the transmission for an infra-red spectral interval is, of course, obtained by a numerical computation like COWLING's with respect to a real band. In this paper it is shown numerically that such a computed transmission is expressed as a function of the variable $u\sigma\alpha\delta^{-1}\left(\alpha^2+\frac{u\sigma\alpha}{\pi}\right)^{-1/2}$, which was introduced by GOODY in expressing the transmission of a model spectrum, where u is the absorbing mass in g cm^{-2}, α the half-width, σ the mean line intensity, δ the mean line distance. So that we only need to compute the transmission against u for a given value of the half-width.

1 Introduction

The transmission of a model spectrum which assumed random line positions and an exponential distribution of line intensities has been investigated by GOODY [6] and shown to be in good agreement with COWLING's computation [1] of the transmission for several intervals of the rotational band of water vapour. GODSON [4,5] has also investigated the transmission of a similar model in which a logarithmic ogive distribution of line intensities was assumed, and has shown that it gave smaller errors than other transmission functions previously proposed in representing the transmission of a certain spectral interval of the rotational band of water vapour.

There is, however, one noteworthy point in adopting a random model. A random model assumes the presence of sufficient numbers of lines in a given interval, while the interval is usually so narrow that the black body energy curve can safely be regarded as constant in it, and that not so many lines as assumed by the random model can exist in it. In addition, as the distribution of line positions in narrow intervals may differ considerably from one interval to another, even if the transmission based on the random model is shown to be valid for certain intervals, it is not certain that the same will hold for other intervals.

Under these circumstances, if a higher approximate representation of the transmission than what is known, for example, as COWLING's "universal curve" is to be hoped for, we think it will be obtained by a numerical computation like COWLING's with respect to the real band. It is, however, needed to reduce labour of computation as much as possible. In the presnt paper it is shown numerically that the computed transmission is expressed as a function, not necessarily an exponential one, of the variable ; $u\sigma\alpha\,\delta^{-1}\left(\alpha^2+\frac{u\sigma\alpha}{\pi}\right)^{-1/2}$, which was introduced by GOODY and accordingly will be called the GOODY variable in this paper, where u is the absorbing mass in g.cm^{-2}, α the half-width, σ the mean line intensty, and δ the mean line distance. So that for a given interval we only need to compute the transmission against u for a given value of the half-width.

2 Transmission at Constant Pressure and Temperature

Whether a given distribution of line positions is allowed to be random or not is not so clearly guessed from the definition of randomness which implies that all arrangements of line positions are equally probable. The following consideration will be useful in guessing the problem. Let n lines be distributed at random in the interval $\Delta\nu$. The probability that no line exists in a narrow interval, d, of the interval will be given $\left(\frac{\Delta\nu-d}{\Delta\nu}\right)^n$. As $\Delta\nu=n\delta$, where δ is the mean line distance, the probability becomes $\left(1-\frac{1}{n}\frac{d}{\delta}\right)^n$ and tends to $e^{-\frac{d}{\delta}}$ as $n\rightarrow\infty$. This means that in case of random line positions the disribution of line distances is expressed as an exponental function of the line distance. In Fig. 1 are shown the distributions of line distance for the

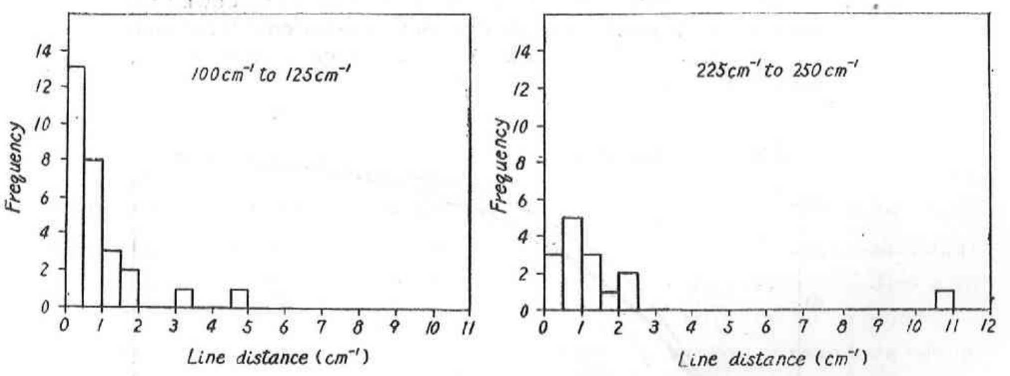

Fig. 1. Line distance histograms

intervals of 100–125 cm^{-1} and 225–250 cm^{-1} of the water vapour band, the transmission for both intervals being computed by COWLING. It will be seen that the distribution of line distance for 100–125 cm^{-1} is nearer to the exponential distribution than that for 225–250 cm^{-1}. By the preceeding discussion we infer that the line arrangement of the former interval is more likely to be at random than that of the latter interval.

Now, GOODY has shown (cf. Fig.1, (b) of his paper [6]), that the transmission of the 100–125 cm^{-1} interval computed by COWLING was well expressed by the transmission function based on his random model. The comparison between GOODY's transmission function and COWLING's data for this interval is reproduced here in Fig. 2 taking the

GOODY variable as abscissa.

Similar comparison of the GOODY's theory with COWLING's date for the 225–250 cm^{-1} interval, which GOODY did not show explicitly, is shown in Fig. 3. If the statistical

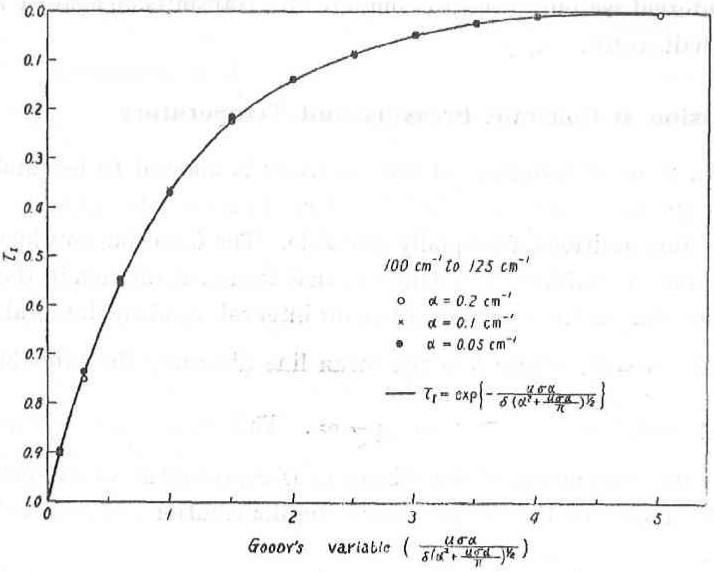

Fig. 2. Comparison between GOODY's transmission function and COWLING's data for 100–125 cm^{-1} interval. The solid curve is GOODY's function and the points are due to COWLING's data and $\sigma = 4.25 \times 10^3$ cm^2g^{-1}, $\delta = 1.48$ cm^{-1} recommended by GOODY.

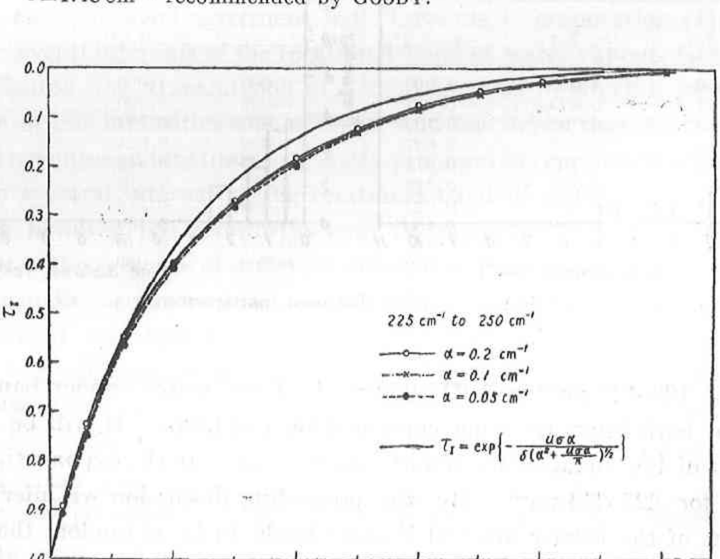

Fig. 3. Comparison between GOODY's transmission function and COWLING's data for 225–250 cm^{-1} interval. The heavy solid curve is GOODY's function and the points are due to COWLING's data and $\sigma = 6.93 \times 10^3$ cm^2g^{-1}, $\delta = 2.51$ cm^{-1} recommended by GOODY.

theory of GOODY were applicable to the transmission of this interval, the points in the figure, due to COWLING's data, should have coincided with the exponential curve representing the theory. Actually, some discrepancy between them is seen in Fig. 3.

In reviewing the random theories of transmission hitherto proposed [4,6] we see that the exponential representation of the transmission function is due to the randomness of the line positions, while the distribution of line intensities determines the functional form of the exponent. Now suppose an interval in which the distribution of line intensities is exponential like GOODY's model, but that the distribution of line positions differs from the random distribution. Then it may be possible to express the transmission of the interval as a certain function of $u\sigma\alpha\delta^{-1}\left(\alpha^2+\frac{u\sigma\alpha}{\pi}\right)^{-1/2}$, which may in general differ slightly from the exponential function. According to GODSON, a logarithmic ogive distribution of line intensities is more likely than an exponential one, so that the use of the GODSON variable will be more reasonable. However, as the GOODY variable is by far a simpler combination of $\sigma u/\delta$ and α/δ than GODSON variable, the GOODY variable is used in this paper from the standpoint of reducing the labour of computation.

Now in expressing the transmission as an empirical function of the GOODY variable, there is no need to use the empirical, or fictitious values of σ and δ recommended

Table 1

The transmission as a function of the GOODY variable for different values of α.

100–125 cm^{-1} interval

GOODY variable	τ_I ($\alpha=0.1$)	τ_I ($\alpha=0.2$)	$\tau_I(\alpha=0.1)$ $-\tau_I(\alpha=0.2)$	τ_I ($\alpha=0.05$)	$\tau_I(\alpha=0.1)$ $-\tau_I(\alpha=0.1)$
0.0	1.000	1.000	0.000	1.000	0.000
0.1	0.923	0.921	+0.002	0.920	+0.003
0.3	0.796	0.796	0.000	0.792	+0.004
0.6	0.636	0.627	+0.009	0.624	+0.012
1.0	0.467	0.472	−0.005	0.461	+0.006
1.5	0.321	0.326	−0.005	0.316	+0.005
2.0	0.219	0.221	−0.002	0.214	+0.005
2.5	0.150	0.150	0.000	0.150	0.000
3.0	0.103	0.104	−0.001	0.110	−0.007
3.5	0.072	0.072	0.000	0.080	−0.008
4.0	0.050	0.051	−0.001	0.051	−0.001
5.0	0.020	0.021	−0.001	0.030	−0.010

225–250 cm^{-1} interval

GOODY variable	τ_I ($\alpha=0.1$)	τ_I ($\alpha=0.2$)	$\tau_I(\alpha=0.1)$ $-\tau_I(\alpha=0.2)$	τ_I ($\alpha=0.05$)	$\tau_I(\alpha=0.1)$ $-\tau_I(\alpha=0.05)$
0.0	1.000	1.000	0.000	1.000	0.000
0.1	0.920	0.910	+0.010	0.920	0.000
0.3	0.771	0.773	−0.002	0.760	+0.012
0.6	0.594	0.607	−0.013	0.579	+0.015
1.0	0.435	0.447	−0.012	0.434	+0.001
1.5	0.319	0.326	−0.007	0.321	−0.002
2.0	0.236	0.241	−0.005	0.244	−0.008
2.5	0.175	0.173	+0.002	0.181	−0.006
3.0	0.129	0.127	+0.002	0.131	−0.002
3.5	0.095	0.091	+0.004	0.095	0.000
4.0	0.065	0.059	+0.006	0.063	−0.002
5.0	0.020	0.020	0.000	0.020	0.000

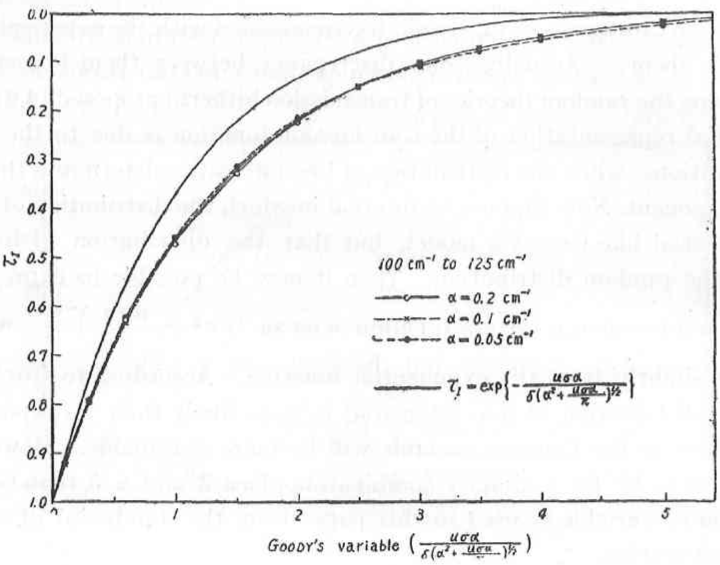

Fig. 4. The transmission as a function of the GOODY variable, for different values of α for 100–125 cm interval. The points represent the transmission computed by COWLING's method with use of the spectrum data of YAMAMOTO and ONISHI, corresponding to the GOODY variable composed from the same data. The heavy solid curve shows the GOODY's transmission function for the sake of comparison.

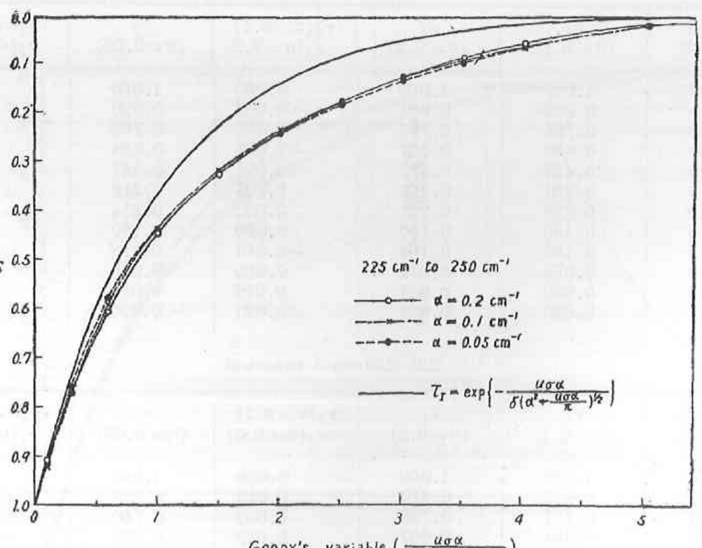

Fig. 5. The transmission as a function of the GOODY variable, for the different values of α, for 225–250 cm^{-1} interval. The points represent the transmission computed by COWLING's method with use of the spectrum data of YAMAMOTO and ONISHI, corresponding to the GOODY variable composed from the same data. The heavy solid curve shows the GOODY's transmission function for the sake of the comparison.

by GOODY instead of the real values of σ and δ available from the intensity-position data of absorption band. Using the intensity-position data by YAMAMOTO and ONISHI [10] the transmission for 100–125 cm^{-1} and 225–250 cm^{-1} intervals were recalculated by COWLING's method for $\alpha=0.2$, 0.1 and 0.05 cm^{-1} and they are shown as a function of the GOODY variable whose σ and δ values are obtained from YAMAMOTO-ONISHI's data. (Fig. 4 and Fig. 5) It will be seen that the transmission curves for both intervals differ from the exponential function of the GOODY variable, but the difference of curves of both intervals is rather small, suggesting that COWLING's "universal curve" approximation is a useful simplification. If the line which connects the point $\alpha = 0.1$ cm^{-1} is taken to be the transmission curve, the transmission errors due to different values of α are seen in Table 1 for $\alpha=0.2$ and 0.05 cm^{-1}. A systematic distribution of errors is seen in the table, but the errors are of a tolerable degree.

3 Transmission in Case of Variable Pressure

The pressure effect on the transmission has been studied chiefly on the cases of ELSASSER band and of a single line of the LORENTZ shape, and so we will consider the case of the ELSASSER band first.

1. *Transmission for the ELSSASSER band.*

The transmission function for the ELSASSER band is given by

$$\tau_f = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \exp\left[-\int_{p_1}^{p_2} \frac{Sq}{gd}\,\frac{\sinh\beta}{\cosh\beta - \cos s}\,dp\right] ds\,, \tag{1}$$

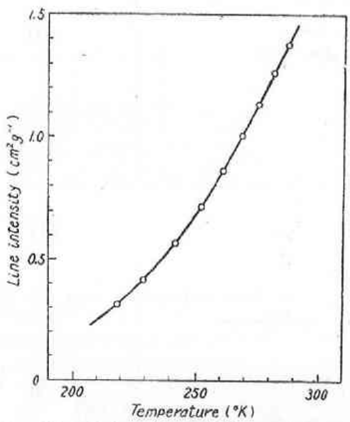

Fig. 6. The intensity of the water vapour absorption line at 852.51 cm^{-1} as a function of temperature. The marks from left to right indicate temperatures at 200, 300, 400, 500, 600, 700, 800, 900, and 1000 mb levels respectively of the standard atmosphere.

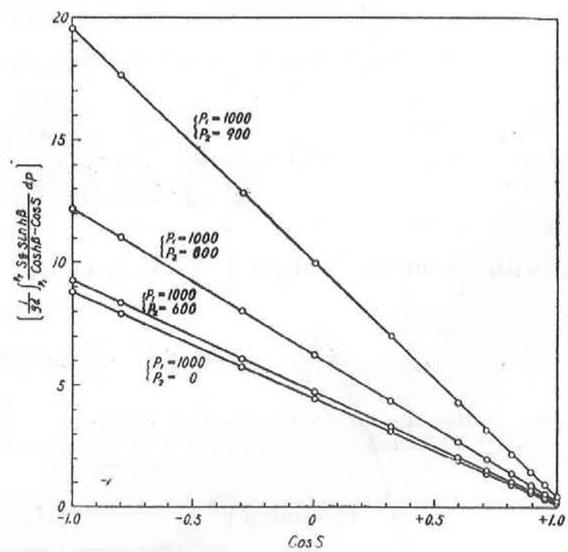

Fig. 7. The numerical values of $\left[\frac{1}{gd}\int_{p_1}^{p_2}\frac{Sq\sinh\beta}{\cosh\beta-\cos s}\,dp\right]^{-1}$ as a function of cos s.

where p_1 and p_2 are the pressure at the top and the bottom of the air column considered, S is the intensity of the line, d the line distance, q the specific humidity, g the acceleration of gravity, $\beta=\frac{2\pi\alpha}{d}$, α the half-width, $s=\frac{2\pi\nu}{d}$, and ν the wave number. PLASS and FIVEL [9] have studied theoretically the transmission of the ELSASSER band for model atmosphere in which q is constant with height. The following study, however, is based on numerical computation. In order to carry out a numerical study, a standard atmosphere with a relative humidity of 60% at all heights, which was originally used as a model atmosphere in estimating a nocturnal radiation by LÖNNQVIST [8], was assumed, and a weak absorption line at 852.51 cm^{-1}, whose intensity is shown in Fig. 6 as a function of temperature, was taken as an example. Then the values of $\int_{p_1}^{p_2}\frac{Sq}{gd}\frac{\sinh\beta}{\cosh\beta-\cos s}dp$ for various values of p_1, p_2 and $\cos s$ were computed by numercal integration, and the reciprocals of the values are shown in Fig. 7 as a function of $\cos s$. As can be seen from the figure, $\left[\int_{p_1}^{p_2}\frac{Sq}{gd}\frac{\sinh\beta}{\cosh\beta-\cos s}dp\right]^{-1}$ is nearly proportional to $\cos s$. It is reasonable that the linear relationship holds when the values of p_1, p_2 are nearly equal, but it is significant that the relationship still holds with high approximation even in the case of $p_1=0$ mb and $p_2=1000$ mb. Hence we can generally assume that,

$$\int_{p_1}^{p_2}\frac{Sq}{gd}\frac{\sinh\beta}{\cosh\beta-\cos s}dp=\frac{\left(\frac{\overline{Su}}{d}\right)\sinh\bar{\beta}}{\cosh\bar{\beta}-\cos s}. \tag{2}$$

Putting two special values for s, for instance, $\cos s=1$ and 0, we have

$$\cosh\bar{\beta}=\frac{\int_{p_1}^{p_2}\frac{Sq}{gd}\frac{\sinh\beta}{\cosh\beta-1}dp}{\int_{p_1}^{p_2}\frac{Sq}{gd}\frac{\sinh\beta}{\cosh\beta(\cosh\beta-1)}dp}, \tag{3}$$

and with use of the $\bar{\beta}$ of (3), $\left(\frac{\overline{Su}}{d}\right)$ is given by

$$\left(\frac{\overline{Su}}{d}\right)=\coth\bar{\beta}\int_{p_1}^{p_2}\frac{Sq}{gd}\tanh\beta\,dp, \tag{4}$$

Hence, quite similar to the case of constant pressure, the transmission can be expressed as

$$\tau_l=\sinh\bar{\beta}\int_{\frac{\overline{Su}}{d\sinh\bar{\beta}}}^{\infty}e^{-y\cosh\bar{\beta}}J_0(iy)\,dy. \tag{5}$$

The values of τ_l of (5) were computed by GODSON [3] as function of $\bar{\beta}$ and $\overline{lu}=\frac{\overline{Su}}{d}\bar{\beta}$. The numerical computation of KAPLAN [7] is also of use in evaluating (5). For the standard atmosphere and the ELSASSER band composed of the above mentioned lines the values of τ_l were obtained both by the above approximation with use of $\frac{\overline{Su}}{d}$ and

$\overline{\beta}$, and by a complete numerical integration of equation (1), as listed in Table 2. It will be seen that the agreement of both τ_f values is excellently good.

Table 2

Comparison of τ_f values obtained from equations (3), (4) and (5) with those obtained by a complete numerical integration of equation (1), for an ELSASSER band and a standard atmosphere. Conventionally the former is expressed as τ_f (Approx.) and the latter as τ_f(Exact).

p_2	p_1	τ_f (Exact)	τ_f (Approx.)	Error	p_2	p_1	τ_f (Exact)	τ_f (Approx.)	Error
1000	900	0.777	0.777	0.000	700	600	0.927	0.928	0.001
〃	800	0.703	0.703	0.000	〃	500	0.899	0.899	0.000
〃	700	0.670	0.670	0.000	〃	400	0.890	0.889	0.001
〃	600	0.654	0.654	0.000	〃	300	0.887	0.886	0.001
〃	500	0.647	0.649	0.002	〃	200	0.886	0.886	0.000
〃	400	0.646	0.647	0.001	〃	100	0.886	0.886	0.000
〃	300	0.645	0.646	0.001	〃	0	0.886	0.886	0.000
〃	200	0.645	0.646	0.001					
〃	100	0.645	0.646	0.001	600	500	0.959	0.958	0.001
〃	0	0.645	0.646	0.001	〃	400	0.944	0.944	0.000
					〃	300	0.940	0.940	0.000
900	800	0.834	0.834	0.000	〃	200	0.939	0.938	0.001
〃	700	0.777	0.777	0.000	〃	100	0.939	0.938	0.001
〃	600	0.753	0.753	0.000	〃	0	0.939	0.938	0.001
〃	500	0.742	0.743	0.001					
〃	400	0.739	0.739	0.000	500	400	0.981	0.981	0.000
〃	300	0.737	0.737	0.000	〃	300	0.975	0.975	0.000
〃	200	0.736	0.737	0.001	〃	200	0.974	0.974	0.000
〃	100	0.736	0.737	0.001	〃	100	0.974	0.974	0.000
〃	0	0.736	0.737	0.001	〃	0	0.974	0.974	0.000
800	700	0.885	0.885	0.000	400	300	0.993	0.994	0.001
〃	600	0.844	0.843	0.001	〃	200	0.992	0.992	0.000
〃	500	0.828	0.827	0.001	〃	100	0.992	0.992	0.000
〃	400	0.820	0.820	0.000	〃	0	0.992	0.992	0.000
〃	300	0.818	0.819	0.001					
〃	200	0.816	0.818	0.002	300	200	0.9987	0.9986	0.0001
〃	100	0.816	0.818	0.002	〃	100	0.9986	0.9985	0.0001
〃	0	0.816	0.818	0.002	〃	0	0.9984	0.9984	0.0001
					200	100	0.99992	0.99991	0.00001
					〃	0	0.99991	0.99990	0.00001
					100	0	0.99999	0.99999	0.00000

An easier, but less accurate way of estimating the value of $\overline{\beta}$ was suggested by CURTIS [2]. That is,

$$\overline{\beta}=\frac{\int_{p_1}^{p_2}\frac{Sq}{gd}\,\beta\,dp}{\int_{p_1}^{p_2}\frac{Sq}{gd}\,dp}\,, \tag{6}$$

and the corresponding $\frac{\overline{Su}}{d}$ is simply given by

$$\frac{\overline{Su}}{d}=\int_{p_1}^{p_2}\frac{Sq}{gd}\,dp\,. \tag{7}$$

The values of τ_f obtained from equation (6), (7) and (5) for several p_1 and p_2 of the model atmosphere are shown in Table 3. The agreement of the τ_f values with the τ_f (exact) values is fairly good.

Table 3
Comparison of τ_I values obtained from equations (6), (7) and (5) with τ_I(Exact).

p_2	p_1	τ_I (Approx.)	τ_I (Exact)	Error
1000	900	0.779	0.777	0.002
〃	800	0.706	0.703	0.003
〃	700	0.675	0.670	0.005
〃	600	0.658	0.654	0.004
〃	500	0.652	0.647	0.005
〃	400	0.649	0.646	0.003
〃	300	0.648	0.645	0.003
〃	200	0.647	0.645	0.002
〃	100	0.647	0.645	0.002
〃	0	0.647	0.645	0.002

2. *The GOODY variable in case of variable pressure.*

The next problem is to find out an effective GOODY variable in case of variable pressure. The preceeding derivation of the ELSASSER transmission function will be instructive for the purpose. Starting from the LORENTZ line case, we will assume

$$\frac{1}{\pi g}\int_{p_1}^{p_2}\frac{Sq\alpha}{\nu^2+\alpha^2}\,dp=\frac{\overline{Su}}{\pi}\,\frac{\overline{\alpha}}{\nu^2+\overline{\alpha}^2}\,. \tag{8}$$

Putting two special values of ν in equation (8), for instance $\nu=0$ and $\nu=\delta/4$, δ being the mean line distance, (corresponding to $\cos s=1$ and 0 of the case of the ELSASSER band) and computing numerically the left side of (8), we can determine the valus of $\overline{Su}$ and $\overline{\alpha}$. The effective GOODY variable is, accordingly, given by

$$\frac{(\overline{Su})_{av}\overline{\alpha}}{\delta\left(\overline{\alpha}^2+\dfrac{(\overline{Su})_{av}\overline{\alpha}}{\pi}\right)^{1/2}}\,, \tag{9}$$

where $(\overline{Su})_{av}$ means the average value of $\overline{Su}$ for the lines in the interval considered. If the transmission for a given interval at constant pressure is known as a function of the GOODY variable at constant pressure, the same transmission at variable pressures will be obtained as a function of the effective GOODY variable given by (9). The computation of the GOODY variable given by (9), however, will be very complicated because of the two-fold averaging with respect to pressures and lines. Instead of the quantity (9), using the average intensity in place of S in equation (8), we can derive the quantity $\overline{\sigma u}\,\overline{\alpha}\,\delta^{-1}\left(\overline{\alpha}^2+\dfrac{\overline{\sigma u}\;\overline{\alpha}}{\pi}\right)^{-1/2}$ to be the effective GOODY variable. The usefulness of this quantity in expressing the transmission at variable pressure is not examined numerically owing to an enormous amount of computation to be carried out. The usefulness of the effective $\dfrac{\overline{Su}}{d}$ and $\overline{\beta}$ for the case of the ELSASSER band is, at present, the only guarantee on this point.

The effective half-width proposed by CURTIS is given by

$$\bar{\alpha} = \frac{\int_{p_1}^{p_2} \alpha \frac{\sigma q}{g} dp}{\int_{p_1}^{p_2} \frac{\sigma q}{g} dp}, \tag{10}$$

and the effective thickness is given by

$$\overline{\sigma u} = \int_{p_1}^{p_2} \frac{\sigma q}{g} dp. \tag{11}$$

These will, of course, be of use in composing the effective GOODY variable with high accurancy.

Acknowledgement

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