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Notes on the Problems on the Motion of the Surface of an Elastic Solid produced by a Linear Source.

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Abstract

The integrals contained in the famous investigations by H. LAMB (1904) and H. NAKANO (1925) of the problems on the motion of the surface of an elastic solid produced by a linear source of disturbance, are evaluated systematically by the method developed by T. SAKAI (1934), and some remarks are given on the problems.

1. Introduction.

The propagation of tremors over the surface of a semi-infinite elastic solid produced by the vertical force concentrated at a line or a point on the surface was investigated by H. LAMB [1]. He concluded that three types of waves, with the velocities of the ordinary longitudinal, transverse and RAYLEIGH waves respectively, are propagated outward over the surface. H. NAKANO [2] investigated the case of a horizontal source of disturbance situated in the interior of the medium, and showed that the RAYLEIGH waves do not appear at places near the epicenter, and at large distances the RAYLEIGH waves are propagated with a phase retardation which takes place as if they are generated at the epicenter simultaneously with the original disturbance. The integrals contained in these famous and fundamental investigations were evaluated by considerably complicated processes. In 1934, T. SAKAI [3] studied the case of a point source of disturbance at an internal point of the body. The integrals involved in his theory were evaluated by the method of the steepest descent, which is similar to those used by A. SOMMERFELD and H. WEYL in the problems of wireless telegraphy. The method developed by SAKAI was so useful in the problems of the elastic waves that it has been used by many authors e.g. SAKAI and S. SYÔNO [4], SYÔNO [5], SAKAI [6], SYÔNO [7], T. HIRONO [8], H. HONDA and K. NAKAMURA [9] in attacking various problems.

In the present paper, we intend to show that the integrals involved in LAMB and NAKANO's papers relating to the problem of the motion of the surface of a semi-infinite elastic solid produced especially by a linear source of disturbance, can be evaluated very systematically by SAKAI's method, and state some remarks on the problems. No detailed exposition of the theories will be attempted here, since they are described fully in their papers.

2. Surface Linear Source. (LAMB's Problem).

We deal with a semi-infinite elastic solid, and take the rectangular coordinates

(x, y), so that $y = 0$ should coincide with the surface of the solid lying on the positive side of the plane. We suppose that the motion of the solid to be in two dimensions. The component displacements are denoted by u and v, the density by ρ, the period of the motion by $T = 2\pi/p$, the time by t, and the LAMÉ's constans by λ and μ. LAMB showed that the component displacements u and v of the surface of the solid produced by the concentrated force $Q \exp(ipt)$ acting parallel to y at points on the line $x = 0$, $y = 0$ on the surface, per unit length of it, can be expressed by the following formulae:

$$\left.\begin{aligned} u_0 &= -\frac{iQ}{2\pi\mu}\int_{-\infty}^{\infty}\frac{\xi(2\xi^2-k^2-2\alpha\beta)\exp(i\xi x)d\xi}{F(\xi)}, \\ v_0 &= -\frac{Q}{2\pi\mu}\int_{-\infty}^{\infty}\frac{k^2\alpha\exp(i\xi x)d\xi}{F(\xi)}, \end{aligned}\right\} \quad \cdots\cdots (1), \text{ [Lamb. (52)]}$$

$$F(\xi) = (2\xi^2-k^2)^2 - 4\xi^2\alpha\xi, \quad \cdots\cdots (2)$$

$$k = \sqrt{\rho/(\lambda+2\mu)}\; p = p/v_p, \qquad k = \sqrt{\rho/\mu}\; p = p/v_s.$$

α and β are the positive real, or positive imaginary quantities determined by

$$\alpha^2 = \xi^2 - h^2, \qquad \beta^2 = \xi^2 - k^2, \qquad \xi \text{ being real.}$$

v_p and v_s are the velocities of the longitudinal and the transverse waves respectively.

The time factor $\exp(ipt)$ is here and often in the sequel temporarily omitted.

Now let us consider a complex plane defined by $\zeta\,(\xi+i\eta)$, and represent it further upon $w\,(p+iq)$ plane by

$$\zeta = h\sin w, \qquad \text{where} \quad \sin w = \sin p\cosh q + i\cos p\sinh q,$$

and confine ourselves to the region bounded by two straight lines $p = -\pi/2$ and $\pi/2$, and a certain region adjoining it, if necessary. Then we have

$$\left.\begin{aligned} u_0 &= \frac{iQ}{2\pi\mu}\int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty}\sin w\, F_r(\gamma)\exp(ihx\sin w)\,dw, \\ v_0 &= -\frac{in^2Q}{2\pi\mu}\int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty}F_b(\gamma)\exp(ihx\sin w)\,dw, \end{aligned}\right\} \quad \cdots\cdots (3)$$

$$\begin{aligned} F_r(\gamma) &= \gamma\,[n^2 + 2(\gamma^2-1) - 2\gamma\sqrt{\gamma^2+n^2-1}\,]/D(\gamma), \\ F_b(\gamma) &= \gamma^2/D(\gamma), \\ D(\gamma) &= [n^2 + 2(\gamma^2-1)]^2 - 4\gamma(\gamma^2-1)\sqrt{\gamma^2+n^2-1}, \\ \gamma &= \cos w, \qquad n = k/h = v_p/v_s > 1. \end{aligned}$$

The integrals (3) belong to the type fully discussed by SAKAI. The branch points B, B′ ($\gamma = \pm i\sqrt{n^2-1}$) and the poles A, A′ ($\gamma_0 = \pm i\delta_0$, $\delta_0 = \sqrt{n^2|\zeta_0|-1}$, $\zeta_0 < -1$, after (Sakai) lie on the lines $p = \pm\pi/2$ on the w-plane, as are shown in the figure 1.

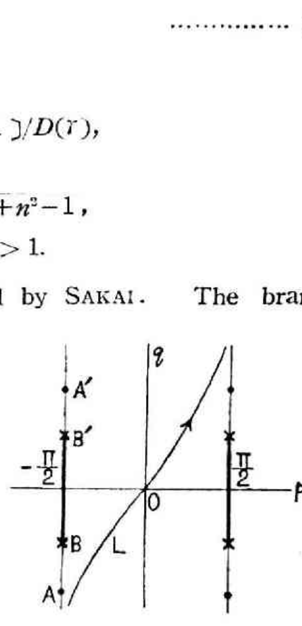


Fig. 1. w-plane.

For B ($\cos w = -i\sqrt{n^2-1}$; $\sin w = -n$) on the line $p = -\pi/2$, $\cosh q' = n$, $\sinh q' = -\sqrt{n^2-1}$, and for A ($\cos w = -i\delta_0$, $\sin w = -\sqrt{1+\delta_0^2}$) on the line $p = -\pi/2$, $\cosh q_0 = \sqrt{1+\delta_0^2}$, $\sinh q_0 = -\delta_0$. The velocity of the RAYLEIGH waves v_3 is represented as

$v_3 = v_1/\sqrt{1+\delta_0^2}$. Now the cut is the line BB'. We assume that in the upper sheet of the w-planes, $\sqrt{\gamma^2+n^2-1}$ tends to γ when γ tends to a infinitely large negative imaginary value, and the path of integration $L\,(-\pi/2-i\infty \to \pi/2+i\infty)$ lies in the upper sheet. The poles also lie on the upper sheet.

When hx is large, we can use the method of the steepest descent. The saddle points lie at $(p=\pm\pi/2,\ q=0)$. As $ihx\sin w = -hx\cos p\sinh q + ihx\sin p\cosh q$, the path L' of the steepest descent which passes through the saddle point $(p=-\pi/2,\ q=0)$ is defined by $\sin p\cosh q = -1$. $\cos p\sinh q$ tends to $+\infty$ when $(p\to -\pi,\ q\to-\infty)$ or $(p\to 0,\ q\to\infty)$.

If we deform the original path L of integtation into L', we must further take a contour integral along a path L_1L_2 going round the branch point B, and also take into account the residue at the pole A, as are shown in the figure 2. The path L_1L_2 along which the integrands diminish most rapidly is $\sin p\cosh q = -n$.

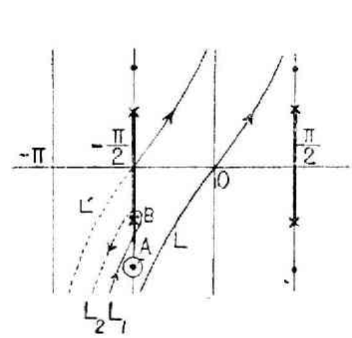


Fig. 2. w plane.

Putting $u_0 = u_0^{(1)}+u_0^{(2)}+u_0^{(3)}$,

$v_0 = v_0^{(1)}+v_0^{(2)}+v_0^{(3)}$,

we have

$$\left.\begin{aligned} u_0^{(1)} &= \frac{iQ}{2\pi\mu}\int_{L'}\sin w\,F_x(\gamma)\exp(ihx\sin w)\,dw,\\ v_0^{(1)} &= -\frac{in^2Q}{2\pi\mu}\int_{L'}F_y(\gamma)\exp(ihx\sin w)\,dw,\end{aligned}\right\} \quad\cdots\cdots\cdots (4)$$

$$\left.\begin{aligned} u_0^{(2)} &= \frac{iQ}{2\pi\mu}\int_{L_1L_2}\sin w\,F_x(\gamma)\exp(ihx\sin w)\,dw,\\ v_0^{(2)} &= -\frac{in^2Q}{2\pi\mu}\int_{L_1L_2}F_y(\gamma)\exp(ihx\sin w)\,dw,\end{aligned}\right\} \quad\cdots\cdots\cdots (5)$$

$$\left.\begin{aligned} u_0^{(3)} &= \frac{Q}{\mu}\left[\frac{\gamma\{n^2+2(\gamma^2-1)-2\gamma\sqrt{\gamma^2+n^2-1}\}}{D'(\gamma)}\exp(ihx\sin w)\right]_{\text{at A}},\\ v_0^{(3)} &= -\frac{n^2Q}{\mu}\left[\frac{\gamma^2}{D'(\gamma)\sin w}\exp(ihx\sin w)\right]_{\text{at A}}.\end{aligned}\right\} \quad\cdots\cdots\cdots (6)$$

The values of the integrals (4) and (5) are obtained by the methods similar to those described briefly in the succeeding article, θ therein being replaced by $\pi/2$ in the process of calculations, and they are expressed as follows, the time factor being inserted:

$$\left.\begin{aligned} u_0^{(1)} &= -\frac{Q}{\mu}\sqrt{\frac{2}{\pi}}\frac{n^2\sqrt{n^2-1}}{(n^2-2)^3}\frac{i\exp\{i(pt-hx-\pi/4)\}}{(hx)^{3/2}}\\ &= -\frac{Q}{\pi}\sqrt{\frac{2}{\pi}}\frac{h^3k^2\sqrt{k^2-h^2}}{(k^2-2h^2)^3}\frac{i\exp\{i(pt-hx-\pi/4)\}}{(hx)^{3/2}},\\ v_0^{(1)} &= \frac{Q}{2\mu}\sqrt{\frac{2}{\pi}}\frac{n^2}{(n^2-2)^2}\frac{i\exp\{i(pt-hx-\pi/4)\}}{(hx)^{3/2}}\\ &= \frac{Q}{2\mu}\sqrt{\frac{2}{\pi}}\frac{h^2k^2}{(k^2-2h^2)^2}\frac{i\exp\{i(pt-hx-\pi/4)\}}{(hx)^{3/2}},\end{aligned}\right\} \quad\cdots\cdots (7)$$

$$\left.\begin{aligned} u_0^{(2)} &= \frac{Q}{\mu}\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{n^2-1}}{n^{5/2}}\,\frac{i\exp\{i(pt-kx-\pi/4)\}}{(hx)^{3/2}} \\ &= \frac{Q}{\mu}\sqrt{\frac{2}{\pi}}\sqrt{1-\frac{h^2}{k^2}}\,\frac{1}{(kx)^{3/2}}\exp\{i(pt-kx-\pi/4)\}, \\ v_0^{(2)} &= \frac{2Q}{\mu}\sqrt{\frac{2}{\pi}}\,\frac{(n^2-1)}{n^{7/2}}\,\frac{i\exp\{i(pt-kx-\pi/4)\}}{(hx)^{3/2}} \\ &= \frac{2Q}{\mu}\sqrt{\frac{2}{\pi}}\left(1-\frac{h^2}{k^2}\right)\frac{i\exp\{i(pt-kx-\pi/4)\}}{(kx)^{3/2}}. \end{aligned}\right\} \quad \cdots\cdots (8)$$

As for $u_0^{(3)}$ and $v_0^{(3)}$, we have

$$\left.\begin{aligned} u_0^{(3)} &= -i\frac{Q}{\mu}\,\frac{\delta_0\{n^2-2\delta_0^2-2+2\,\delta_0\sqrt{\delta_0^2-n^2+1}\}}{D'(-i\delta_0)}\exp\{i(pt-hx\sqrt{1+\delta_0^2})\}, \\ v_0^{(3)} &= -\frac{n^2Q}{\mu}\,\frac{\delta_0^2}{\sqrt{1+\delta_0^2}\;D'(-i\delta_0)}\exp\{i(pt-hx\sqrt{1+\delta_0^2})\}. \end{aligned}\right\} \quad \cdots\cdots\cdots (9)$$

Putting $p/v_3=\kappa$, we have $h\sqrt{1+\delta_0^2}=\kappa$. From the relation $F(h\sin w) = h^4D(\gamma)$, we have $F'(h\sin w) = -h^3\sin w\; D'(\gamma)/\cos w$. Putting the values of $\cos w$ and $\sin w$ at A into above equation and taking into account the fact that $F'(\xi)$ is the odd function of ξ, we get

$$D'(-i\delta_0) = i\delta_0\;F'(\kappa)/h^3\sqrt{1+\delta_0^2}.$$

Also we have

$$\alpha_1 = \sqrt{\kappa^2-h^2} = h\delta_0, \qquad \beta_1 = \sqrt{\kappa^2-k^2} = h\sqrt{\delta_0^2-n^2+1}\,.$$

Taking these relations into consideration, we can transform (9) into following (9′):

$$\left.\begin{aligned} u_0^{(3)} &= -\frac{Q}{\mu}\;H\exp\{i(pt-\kappa x)\}, \\ v_0^{(3)} &= -i\frac{Q}{\mu}\;K\exp\{i(pt-\kappa x)\}, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots (9')$$

where

$$H = -\frac{\kappa\{2\kappa^2-k^2-2\alpha_1\beta_1\}}{F'(\kappa)}, \qquad K = -\frac{k^2\alpha_1}{F'(\kappa)}.$$

The results of the present calculation (7), (8) and (9′) together, are quite the same with those expressed as (90) and (91) in LAMB's paper. It should be noticed that in LAMB's calculation an adequate free standing waves of RAYLEIGH type had to be added for obtaining the progressive RAYLEIGH waves, whereas in the present calculation the progressive RAYLEIGH waves are obtained directly as the result of the evaluation of the integral.

Similar treatment of the case of a vertical force concentrated at a point on the surface may be found in HIRONO's papers as a special case of his investigation.

3. Internal Linear Source. (NAKANO's Problem)

The periodic longitudinal waves are assumed to be emitted from the line source at $x=0$, $y=f$ in the semi-infinite elastic solid. The same notations of the quantities concerned are adopted here as in the preceding article. Using the polar coordinates $r=\sqrt{x^2+(y-f)^2}$, $\varphi=\tan^{-1}\frac{y-f}{x}$ we put as the radial and the transverse components of displacement ϑ_r and ϑ_φ of the longitudinal waves

$$\theta_r = \frac{\partial\Phi}{\partial r} = -h\,H_{2,1}(hr), \quad \vartheta_\varphi = \frac{\partial\Phi}{r\partial\varphi} = 0, \quad \Phi = H_{2,0}(hr).$$

Then the components of displacement at the surface of the solid are expressed by NAKANO as follows:

$$\left.\begin{aligned} u_0 &= \frac{4k^2}{\pi}\int_{-\infty}^{\infty}\frac{\beta\xi}{F(\xi)}\exp(-\alpha f+i\xi x)\,d\xi, \\ v_0 &= -\frac{2ik^2}{\pi}\int_{-\infty}^{\infty}\frac{2\xi^2-h^2}{F(\xi)}\exp(-\alpha f+i\xi x)\,d\xi. \end{aligned}\right\} \quad \cdots\cdots\cdots (10), \text{〔NAKANO. (98)〕}$$

NAKANO transformed these expressions by using contour integration taking special paths, and obtained important results.

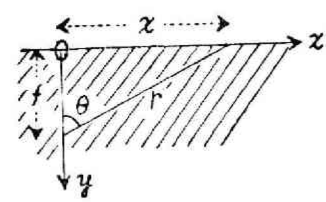


Fig. 3.

Now let us again consider a complex plane defined by $\zeta(\xi+i\eta)$, and repesent it further upon $w(p+iq)$ plane by $\zeta = h\sin w$, and put

$$r = \sqrt{x^2+f^2}, \quad f = r\cos\vartheta, \quad x = r\sin\theta. \text{ (See figure 3).}$$

We have

$$\left.\begin{aligned} u_0 &= \frac{4ik^2}{\pi h}\int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty}\sin w\; G_x(\gamma)\,\exp\{-ihr\cos(w+\theta)\}\,dw, \\ v_0 &= \frac{2ik^2}{\pi h}\int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty}G_y(\gamma)\,\exp\{-ihr\cos(w+\theta)\}dw, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots (11)$$

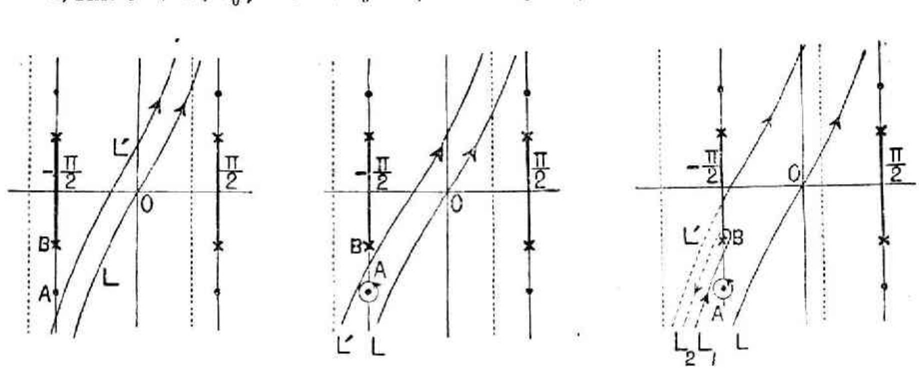
$$G_x = \gamma\sqrt{\gamma^2+n^2-1}/D(\gamma), \qquad G_y = \gamma[n^2+2(\gamma^2-1)]/D(\gamma).$$

When hr is large, the saddle point of the integrands of (11) lies at $w=-\theta$. As $-ihr\cos(w+\theta) = -hr\sin(p+\theta)\sinh q - ihr\cos(p+\theta)\cosh q$, the path L' of the steepest descent which passes through the saddle point ($p=-\theta$, $q=0$) is defined by $\cos(p+\theta)\cosh q = 1$. $\sin(p+\theta)\sinh q$ tends to infinity when ($p\to-\pi/2-\theta$, $q\to-\infty$) or ($p\to\pi/2-\theta$, $q\to\infty$). The point C of intersection of the path L' with the line $p=-\pi/2$ is given by $\cosh q = 1/\sin\theta$.

The problem is divided into three cases according as C lies below A or between A and B or above B on the line as are shown in the figure 4. These distinctions are expressed by the relations

$$1/\sin\theta > \sqrt{1+\delta_0^2}, \quad \sqrt{1+\delta_0^2} > 1/\sin\theta > n, \quad 1/\sin\theta < n. \qquad (12)$$

i.e.



(i) (ii) (iii)

Fig. 4. w-plane.

$$x < v_3 f/\sqrt{v_1^2-v_3^2}, \quad v_3 f/\sqrt{v_1^2-v_3^2} < x < v_2 f/\sqrt{v_1^2-v_2^2}, \quad v_2 f/\sqrt{v_1^2-v_2^2} < x.$$

These critical values of x i.e. $v_3 f/\sqrt{v_1^2-v_3^2}$, $v_2 f/\sqrt{v_1^2-v_2^2}$ are respectively equal to $hf/\sqrt{\kappa^2-h^2}$, $hf/\sqrt{k^2-h^2}$, which were given by NAKANO.

(i) $x < v_3 f/\sqrt{v_1^2-v_3^2}$. The original path L of integration is displaced simply into L', and we have $u_0 = u_0^{(1)}$, $v_0 = v_0^{(1)}$.

$$\left.\begin{aligned} u_0^{(1)} &= \frac{4ik^2}{\pi h}\int_{L'} \sin w\, G_x(\gamma)\ \exp\{-ihr\cos(w+\theta)\}\ dw, \\ v_0^{(1)} &= \frac{2ik^2}{\pi h}\int_{L'} G_y(\gamma)\ \exp\{-ihr\cos(w+\theta)\}\ dw. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (13)$$

Now $\displaystyle\int_{L'} dw = \int_{-\frac{\pi}{2}-\theta-i\infty}^{\frac{\pi}{2}-\theta+i\infty} dw = \int_{-\theta}^{\frac{\pi}{2}-\theta+i\infty} dw - \int_{-\theta}^{-\frac{\pi}{2}-\theta-i\infty} dw.$ Putting $s = w+\theta$, we get $\gamma = \cos w = \cos\theta\ \cos s + \sin\theta\ \sin s,\quad \beta = \sin w = \cos\theta\ \sin s - \sin\theta\ \cos s.$

and $$\int_{L'} dw = \int_0^{\frac{\pi}{2}+i\infty} ds - \int_0^{-\frac{\pi}{2}-i\infty} ds.$$

Take $-s$ in place of s in the latter integral, and put

$$\gamma^* = \cos\theta\ \cos s - \sin\theta\ \sin s, \quad \beta^* = -\cos\theta\ \sin s - \sin\theta\ \cos s.$$

If we take again $s = p+iq$, the path of integration becomes $\cos p\ \cosh q = 1$, and along it we can take $\cos s = 1-i\tau$, where τ is real and increases from zero to infinity. Taking into account the relations $\sin s\ ds = id\tau$, $\sin s = \sqrt{\tau}\sqrt{2i+\tau}$, (13) can be transformed as follows:

$$\left.\begin{aligned} u_0^{(1)} &= -\frac{4k^2}{\pi h}\exp(-ihr)\int_0^\infty [\beta G_x(\beta)+\beta^* G_x(\gamma^*)]\ \frac{\exp(-hr\tau)}{\sin s}\ d\tau, \\ v_0^{(1)} &= -\frac{2k^2}{\pi h}\exp(-ihr)\int_0^\infty [G_y(\gamma)+G_y(\gamma^*)]\frac{\exp(-hr\tau)}{\sin s}\ d\tau. \end{aligned}\right\} \quad \cdots\cdots\cdots (14)$$

Now, $[\beta G_x(\gamma)+\beta^* G_x(\gamma^*)]_{\tau=0} = -2\sin\theta\ G_x(\cos\theta)$, $[G_y(\gamma)+G_y(\gamma^*)]_{\tau=0} = 2G_y(\cos\theta)$, and $\int_0^\infty \{\exp(-hr\tau)/\sqrt{\tau}\}d\tau = \sqrt{\pi/hr}$. Transferring all terms besides $\exp(-hr\tau)/\sqrt{\tau}$, before the integral sign and putting $\tau = 0$, we get very approximately,

$$\left.\begin{aligned} u_0^{(1)} &= -(i-1)\frac{4k^2}{\sqrt{\pi}\,h^{3/2}}\ \frac{\sqrt{n^2-\sin^2\theta}\ \sin\theta\cos\theta}{D(\cos\theta)}\ \frac{\exp(-ihr)}{\sqrt{r}}, \\ v_0^{(1)} &= (i-1)\frac{2k^2}{\sqrt{\pi}\,h^{3/2}}\ \frac{(n^2-2\sin^2\theta)\cos\theta}{D(\cos\theta)}\ \frac{\exp(-ihr)}{\sqrt{r}}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (15)$$

Putting $h\sin\theta = hx/r = \xi$, we have $h^4 D(\cos\theta) = F(\xi)$, and inserting the time factor, (15) can be transformed into (16) :

$$\left.\begin{aligned} u_0^{(1)} &= \frac{4(1-i)k^2h^{1/2}f}{\sqrt{\pi}\ r^{3/2}}\ \frac{\xi\sqrt{k^2-\xi^2}}{F(\xi)}\ \exp\{i(pt-hr)\}, \\ v_0^{(1)} &= \frac{2(1-i)k^2h^{1/2}f}{\sqrt{\pi}\ r^{3/2}}\ \frac{2\xi^2-k^2}{F(\xi)}\ \exp\{i(pt-hr)\}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (16)$$

(16) is the results obtained by NAKANO [(91) of p. 37], but under the supposition that $hr^3/2f^2$, instead of hr, is large.

(ii) $v_3 f/\sqrt{v_1^2-v_3^2} < x < v_2 f/\sqrt{v_1^2-v_2^2}$ The path L' passes between A and B, and we must take into account the residue at A, and we have:

$$u_0 = u_0^{(1)}+u_0^{(3)}, \qquad v_0 = v_0^{(1)}+v_0^{(3)},$$

$$\left.\begin{aligned} u_0^{(3)} &= \frac{8k^2}{h}\left[\frac{\gamma\sqrt{\gamma^2+n^2-1}}{D'(\gamma)}\exp\{-ihr\cos(w+\theta)\}\right]_{atA}, \\ v_0^{(3)} &= \frac{4k^2}{h}\left[\frac{\gamma[n^2+2(\gamma^2-1)]}{D'(\gamma)\sin w}\exp\{-ihr\cos(w+\theta)\}\right]_{atA}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (17)$$

Now, $[-ihr\cos(w+\theta)]_{atA} = -ih[-i\delta_0 f+\sqrt{1+\delta_0^2}\,x]$.

Quite similarly as in the preceding section, (17) can be transformed into following (18) and (19) successively.

$$\left.\begin{aligned} u_0^{(3)} &= -\frac{8k^2}{h}\frac{\delta_0\sqrt{\delta_0^2-(n^2-1)}}{D'(-i\delta_0)}\exp\{-ihx\sqrt{1+\delta_0^2}-hf\delta_0\}, \\ v_0^{(3)} &= -\frac{4ik^2}{h}\frac{\delta_0[2(\delta_0^2+1)-n^2]}{\sqrt{1+\delta_0^2}\,D'(-i\delta_0)}\exp\{-ihx\sqrt{1+\delta_0^2}-hf\delta_0\}. \end{aligned}\right\} \quad \cdots\cdots\cdots (18)$$

And inserting the time factor, we have

$$\left.\begin{aligned} u_0^{(3)} &= 8ik^2\frac{\beta_1\kappa}{F'(\kappa)}\exp(-\alpha_1 f)\exp\{i(pt-\kappa x)\}, \\ v_0^{(3)} &= -4k^2\frac{2\kappa^2-k^2}{F'(\kappa)}\exp(-\alpha_1 f)\exp\{i(pt-\kappa x)\}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots (19)$$

(19) is the most important result obtained by NAKANO (114). Also in NAKANO's calculation, an adequate standing waves of RAYLEIGH type had to be added as in LAMB's case.

(iii) $v_2 f/\sqrt{v_1^2-v_2^2} < x$. The path L' lies above B, and we must take a contour integral along a path L_1L_2 going round B, besides the residue at A. The path L_1L_2 is defined by $\cos(p+\theta)\cosh q = n\sin\theta$, and is assumed to tends to $w = -\pi/2-\theta-i\infty$. We have,

$$u_0 = u_0^{(1)}+u_0^{(2)}+u_0^{(3)}, \qquad v_0 = v_0^{(1)}+v_0^{(2)}+v_0^{(3)},$$

$$\left.\begin{aligned} u_0^{(2)} &= \frac{4ik^2}{\pi h}\int_{L_1L_2}\sin w\, G_x(\gamma)\exp\{-ihr\cos(w+\theta)\}\,dw, \\ v_0^{(2)} &= \frac{2ik^2}{\pi h}\int_{L_1L_2}G_y(\gamma)\exp\{-ihr\cos(w+\theta)\}\,dw. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (20)$$

When w_1 is a value of w on L_2, $\sqrt{\cos^2 w_1+n^2-1}$ is $-\sqrt{\cos^2 w+n^2-1}$ on L_1, if w is the value on L_1 corresponding to w_1. Donoting $G_x(\cos w_1)$ and $G_y(\cos w_1)$ by $G_x^*(\cos w)$ and $G_y^*(\cos w)$ on L_1, we have

$$\left.\begin{aligned} u_0^{(2)} &= \frac{4ik^2}{\pi h}\int_{L_1}\sin w[G_x(\gamma)-G_x^*(\gamma)]\exp\{-ihr\cos(w+\theta)\}\,dw, \\ v_0^{(2)} &= \frac{2ik^2}{\pi h}\int_{L_1}[G_y(\gamma)-G_y^*(\gamma)]\exp\{-ihr\cos(w+\theta)\}\,dw, \end{aligned}\right\} \quad \cdots\cdots\cdots (21)$$

$$G_x(\gamma)-G_x^*(\gamma) = 2\gamma[n^2+2(\gamma^2-1)]^2\sqrt{\gamma^2+n^2-1}\,/D(\gamma)D^*(\gamma),$$

$$G_y(\gamma)-G_y^*(\gamma) = 8\gamma^2(\gamma^2-1)[n^2+2(\gamma^2-1)]\sqrt{\gamma^2+n^2-1}\,/D(\gamma)D^*(\gamma),$$

$$D(\gamma)D^*(\gamma) = [n^2+2(\gamma^2-1)]^4-16\gamma^2(\gamma^2-1)^2(\gamma^2+n^2-1).$$

Along $-L_1$, we can put $\cos(w+\theta) = n\sin\theta - i(\tau + \cos\theta\sqrt{n^2-1})$ and $-ihr\cos(w+\theta) = -\sqrt{k^2-h^2}\, f - ikx - hr\tau$ Where τ increases from zero at B to infinity. After Sakai, we have, along $-L_1$ near B,

$$\cos w + i\sqrt{n^2-1} = -\frac{in}{\cos(\theta+iq')}\,\tau.$$

Transferring the factors except $\sqrt{\tau}\,\exp(-hr\tau)$ before the integral sign, putting $\tau = 0$, we have, as $\int_0^\infty \sqrt{\tau}\,\exp(-hr\tau)\,d\tau = \sqrt{\pi}/2(hr)^{3/2}$:

$$\left.\begin{aligned} u_0^{(2)} &= -\frac{4\sqrt{2}\,(n^2-1)^{3/4}}{\sqrt{\pi k}(n\cos\theta+i\sqrt{n^2-1}\sin\theta)^{3/2}}\exp(-\sqrt{k^2-h^2}\,f)\frac{\exp\{i(pt-kx)\}}{r^{3/2}} \\ v_0^{(3)} &= -\frac{8\sqrt{2}\,i(n^2-1)^{5/4}}{\sqrt{\pi k}\,n(n\cos\theta+i\sqrt{n^2-1}\sin\theta)^{3/2}}\exp(-\sqrt{k^2-h^2}\,f)\frac{\exp\{i(pt-kx)\}}{r^{3/2}} \end{aligned}\right\} \quad \cdots\cdots (22)$$

(22) corresponds to NAKANO (112), i.e.

$$\left.\begin{aligned} u_0^{(3)} &= \frac{4k^2}{\pi}\exp\{i(pt-kx)\}\int_{(-k)}\left\{\frac{\sqrt{\zeta^2-k^2}\,\zeta}{F(\zeta)}+\frac{\sqrt{\zeta^2-k^2}\,\zeta}{f(\zeta)}\right\}e^X\,d\zeta, \\ v_0^{(3)} &= -\frac{2ik^2}{\pi}\exp\{i(pt-kx)\}\int_{(-k)}\left\{\frac{2\zeta^2-k^2}{F(\zeta)}-\frac{2\zeta^2-k^2}{f(\zeta)}\right\}e^X\,d\zeta, \end{aligned}\right\}$$

$$f(\zeta) = (2\zeta^2-k^2)^2 + 4\sqrt{\zeta^2-h^2}\sqrt{\zeta^2-k^2}\,\zeta^2,$$
$$X = \mathrm{Re}(-\sqrt{\zeta^2-h^2}\,f + i\zeta x),$$

which were left in the integral form.

When the periodic transverse waves are assumed to be emitted from the linear source at $x=0$, $y=f$, the motion of the surface of a semi-infinite elastic solid can be calculated by the methods similar to those stated above.

4. Summary

The integrals involved in the investigations "On the Propagation of Tremors over the Surface of an Elastic Solid" due to periodic vertical force concentrated at a line of the surface of the solid by H. LAMB (1904) as a part of his famous paper, and "On RAYLEIGH Waves" produced by the waves emitted from a horizontal linear source lying in the interior of the solid by H. NAKANO (1925), are shown to be evaluated quite systematically by the method developed by T. SAKAI (1934). The values of some integrals left in the integral form in NAKANO's paper are also obtained.

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