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On the Propagation of Elastic Waves in an Inhomogeneous Sphere (II)

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Abstract.

In this paper we treat the propagation of elastic waves in an elastic sphere of large radius assumed to have a concentric fluid spherical core whose radius is also large. Both the media of the elastic shell and fluid core are supposed to be inhomogeneous and the velocities of the elastic waves propagating in them are proportional to r^{-m} and r^{-s} respectively, where m and s are arbitrary real number and both are larger than -1.

The values of the elastic constants are taken from the observational data in the seismic phenomena. The numerical calculations are applied to the reflection and refraction coefficients at the boundary of the core and the displacements of the free surface at the points of emergence of various elastic waves.

I. The Reflection and Refraction Coefficients at the Boundary of the Fluid Core

An elastic sphere of radius a is assumed to have a concencentric fluid spherical core, the radius, density and the LAME's constant of the latter being c, ρ' and λ'. The velocity v_i of the waves propagating in the fluid core is supposed to be proportional to r^{-s};

$$\sqrt{\frac{\rho'}{\lambda'}} = \frac{pr^s}{\sigma} = \frac{1}{v_i}, \tag{1}$$

where s is real and > -1, p is a real constant and the motion is assumed to be simple harmonic and to be expressed by $\exp\{-i\sigma t\}$.

In such a fluid the particular solutions of the wave equation;

$$-\frac{\rho'}{\lambda'}\sigma^2 I = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial I}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial I}{\partial\theta}\right), \tag{2}$$

are
$$I^{(1),(2)} = Y^{(1),(2)}_{n,s}(p,r)\ P_n(\cos\theta), \tag{3}$$

where $Y_{n,s}$ is a particular solution of

$$\frac{d^2Y}{dr^2} + \frac{2}{r}\frac{dY}{dr} + \left(p^2r^{2s} - \frac{n(n+1)}{r^2}\right)Y = 0,$$

and is expressed by means of the Hankel function:

$$Y^{(1),(2)}_{n,s}(p,r) = \sqrt{\frac{\pi}{2r}}\ H^{(1),(2)}_{\frac{n+1/2}{s+1}}\left(\frac{pr^{s+1}}{s+1}\right). \tag{4}$$

$Y^{(1)}$ is a diverging wave which propagates outward, and $Y^{(2)}$ a converging, remembering the time factor $\exp(-i\sigma t)$.

The outer elastic shell $c \leqq r \leqq a$ is assumed to be the same medium as discussed in the previous paper [1], and a converging dilatational wave in the shell is expressed as follows;

$$\Delta = Y^{(2)}_{n,m}(k,r)\ P_n(\cos\theta). \tag{5}$$

This wave is accompanied at the boundary of the fluid core $r = c$ by the reflected waves:

$$\Delta_p = E_n \frac{Y^{(2)}_{n,m}(k,c)}{Y^{(1)}_{n,m}(k,c)} Y^{(1)}_{n,m}(k,r)\ P_n(\cos\theta), \tag{6}$$

$$\omega_p = \frac{1}{2} F_n \frac{Y^{(2)}_{n,m}(k,c)}{Y^{(1)}_{n,m}(\xi,c)} Y^{(1)}_{n,m}(\xi,r) \frac{dP_n(\cos\theta)}{d\theta}, \tag{7}$$

and by the refracted wave:

$$I_p = G_n \frac{Y^{(2)}_{n,m}(k,c)}{Y^{(1)}_{n,m}(p,c)} Y^{(2)}_{n,s}(p,r)\ P_n(\cos\theta). \tag{8}$$

In the case of a converging distortional wave;

$$\omega = \frac{1}{2} Y^{(2)}_{n,m}(\xi,r) \frac{dP_n(\cos\theta)}{d\theta}, \tag{6}$$

which propagates in the elastic shell, the reflected waves;

$$\Delta_s = V_n \frac{Y^{(2)}_{n,m}(\xi,c)}{Y^{(1)}_{n,m}(k,c)} Y^{(1)}_{n,n}(k,r)\ P_n(\cos\theta), \tag{10}$$

$$\omega_s = \frac{1}{2} W_n \frac{Y^{(2)}_{n,m}(\xi,c)}{Y^{(1)}_{n,m}(\xi,c)} Y^{(1)}_{n,m}(\xi,r) \frac{dP_n(\cos\theta)}{d\theta}, \tag{11}$$

and the refracted wave;

$$I_s = U_n \frac{Y^{(2)}_{n,m}(\xi,c)}{Y^{(2)}_{n,s}(p,c)} Y^{(2)}_{n,s}(p,r)\ P_n(\cos\theta), \tag{12}$$

appear at the boundary.

A diverging wave in the fluid core;

$$I = Y^{(1)}_{n,s}(p,r)\ P_n(\cos\theta) \tag{13}$$

produces at the boundary $r = c$ the reflected converging wave;

$$I' = L_n \frac{Y^{(1)}_{n,s}(p,c)}{Y^{(2)}_{n,s}(p,c)} Y^{(2)}_{n,s}(p,r)\ P_n(\cos\theta), \tag{14}$$

and the waves refracted into the outer elastic shell;

$$\Delta = M_n \frac{Y^{(1)}_{n,s}(p,c)}{Y^{(1)}_{n,m}(k,c)} Y^{(1)}_{n,m}(k,r)\ P_n(\cos\theta), \tag{15}$$

$$\omega = \frac{1}{2} N_n \frac{Y^{(1)}_{n,s}(p,c)}{Y^{(1)}_{n,m}(\xi,c)} Y^{(1)}_{n,m}(\xi,r) \frac{dP_n(\cos\theta)}{d\theta}. \tag{16}$$

When we denote the radial and tangential eomponents of the displacements just outside the boundary $r = c$ by u_e and v_e and those just inside the boundary by u_i and v_i, the boundary conditions, which assure that the displacements and the normal component of the pressure are continuous at the boundary and there is no shearing force in the fluid, are as follows;

$$u_e = u_i, \qquad \lambda\Delta_e + 2\mu\frac{\partial u_e}{\partial r} = \lambda'\Delta_i, \qquad \frac{\partial v_e}{\partial r} - \frac{v_e}{r} + \frac{1}{r}\frac{\partial u_e}{\partial\theta} = 0. \tag{17}$$

With the help of these relations the reflection and refraction cofficients can be obtained easily. After a short calculation we get:

$$F_n = \frac{2i(m+1)}{c^2} \frac{\psi_n}{D_n}, \tag{18}$$

$$G_n = \frac{2i(m+1)}{c^2}$$
$$\times \frac{\frac{m+1}{c}\left\{U^{(1)}_{n,m}(\xi,c)\overline{U^{(1)}_{n,m}}(k,c)-n(n+1)V^{(1)}_{n,m}(\xi,c)\overline{V^{(1)}_{n,m}}(k,c)\right\}-\left\{\frac{\xi^2c^{2m+1}}{2}+2(m+1)^2-n(n+1)\right\}\phi^{(1)}_n}{Y^{(1)}_{n,m}(k,c)\ Y^{(2)}_{n,m}(k,c)\ \overline{V^{(1)}_{n,m}}(k,c)\ D_n}, \tag{19}$$

$$E_n = \frac{2i(m+1)}{c^2}\ \frac{k^2}{\xi^2}\ \frac{U^{(1)}_{n,m}(\xi,c)\psi_n}{Y^{(1)}_{n,m}(k,c)\ Y^{(2)}_{n,m}(k,c)\ D_n} - \frac{\overline{V^{(2)}_{n,m}}(k,c)}{\overline{V^{(1)}_{n,m}}(k,c)}, \tag{20}$$

$$U_n = -\frac{i(m+1)n(n+1)k^2c^{2m-1}\overline{V^{(1)}_{n,m}}(k,c)}{Y^{(1)}_{n,m}(\xi,c)Y^{(2)}_{n,m}(\xi,c)\ D_n}, \tag{21}$$

$$V_n = \frac{k^2}{\xi^2}\ \frac{U^{(1)}_{n,m}(\xi,c)\ W_n + U^{(2)}_{n,m}(\xi,c)}{\overline{V^{(1)}_{n,m}}(k,c)}, \tag{22}$$

$$W_n = \frac{1}{D_n}\Bigg[\frac{\lambda'}{2\mu}k^2c^{2m+1}\ \phi^{(2)}_n - \frac{k^2c^{2m}}{p^2c^{2s}}\ \frac{Y^{(1)\prime}_{n,s}(p,c)}{Y^{(1)}_{n,s}(p,c)}\ \Big\{\overline{U^{(1)}_{n,m}}(k,c)\ U^{(2)}_{n,m}(\xi,c)$$
$$- n(n+1)\ \overline{V^{(1)}_{n,m}}(k,c)\ V^{(2)}_{n,m}(\xi,c)\Big\}\Bigg], \tag{23}$$

$$L_n = \frac{1}{D_n}\Bigg[\frac{\lambda'k^2c^{2m+1}}{2\mu}\ \phi^{(1)}_n - \frac{k^2c^{2m}}{p^2c^{2s}}\ \frac{Y^{(1)\prime}_{n,s}(p,c)}{Y^{(1)}_{n,s}(p,c)}\ \Big\{\overline{U^{(1)}_{n,m}}(k,c)\ U^{(1)}_{n,m}(\xi,c)$$
$$- n(n+1)\ \overline{V^{(1)}_{n,m}}(k,c)\ V^{(1)}_{n,m}(\xi,c)\Big\}\Bigg], \tag{24}$$

$$M_n = -\frac{i(s+1)\lambda'}{\mu}\ \frac{k^4c^{4m}\ U^{(1)}_{n,m}(\xi,c)}{p^2c^{2s+1}\ Y^{(1)}_{n,s}(p,c)\ Y^{(2)}_{n,s}(p,c)\ D_n}, \tag{25}$$

$$N_n = -\frac{i(s+1)\lambda'}{\mu}\ \frac{k^2\xi^2c^{4m}\ \overline{V^{(1)}_{n,m}}(kc)}{p^2c^{2s+1}\ Y^{(1)}_{n,s}(p,c)\ Y^{(2)}_{n,s}(p,c)\ D_n}, \tag{26}$$

where
$$U^{(s)}_{n,m}(k,c) = (m+1)\ \frac{Y^{(s)\prime}_{n,m}(k,c)}{Y^{(s)}_{n,m}(k,c)} + \frac{\xi^2c^{2m+1}}{2} + \frac{(m+1)\ -\ n(n+1)}{c}, \tag{27}$$

$$\overline{U^{(s)}_{n,m}}(k,c) = 2(m+1)\ \frac{Y^{(s)\prime}_{n,m}(k,c)}{Y^{(s)}_{n,m}(k,c)} + \frac{\xi^2C^{2m+1}}{2} - \frac{n(n+1)}{c}, \tag{28}$$

$$V^{(s)}_{n,m}(k,c) = \frac{Y^{(s)\prime}_{n,m}(k,c)}{Y^{(s)}_{n,m}(k,c)} - \frac{2m+1}{c}, \tag{29}$$

$$\overline{V^{(s)}_{n,m}}(k,c) = \frac{Y^{(s)\prime}_{n,m}(k,c)}{Y^{(s)}_{n,m}(k,c)} - \frac{m+1}{c}, \tag{30}$$

$$\phi^{(s)}_n = U^{(s)}_{n,m}(\xi,c)\ \frac{Y^{(1)\prime}_{n,m}(k,c)}{Y^{(1)}_{n,m}(k,c)} + \frac{n(n+1)}{c}\ \overline{Y^{(1)}_{n,m}}(k,c), \tag{31}$$

$$\psi_n = \frac{\lambda'}{2\mu}\ \xi^2c^{2m}(m+1) - \frac{\xi^2c^{2m}}{p^2c^{2s}}\ \frac{Y^{(2)\prime}_{n,s}(p,c)}{Y^{(2)}_{n,s}(p,c)}\ \left\{\frac{\xi^2c^{2m+1}}{2} + \frac{2(m+1)^2-n(n+1)}{c},\right\} \tag{32}$$

$$D_n = \frac{k^2c^{2m}}{p^2c^{2s}}\ \frac{Y^{(2)\prime}_{n,s}(p,c)}{Y^{(2)}_{n,s}(p,c)}\ \Big\{U^{(1)}_{n,m}(\xi,c)\ \overline{U^{(1)}_{n,m}}(k,c)$$
$$- n(n+1)\ V^{(1)}_{n,m}(\xi,c)\ \overline{V^{(1)}_{n,m}}(k,c)\Big\} - \frac{\lambda'}{2\mu}k^2c^{2m+1}\ \phi^{(1)}_n. \tag{33}$$

In these equations the dash $'$ denotes the differentiation d/dr.

2. The Reflected Waves at the Boundary of the Fluid Core

When we consider a small sphere of radius R_0, whose center is the source Q $(b, 0, 0)$ of the elastic waves, the medium in this small sphere may be regarded as homogeneous and isotropic approximately. Thus the dilatational and distortional waves at the surface of this sphere may be expressed by

$$\underset{\circ}{\Delta} \sim R_0^{-1} \exp(ikR_0) \quad \text{and} \quad \underset{\circ}{\omega} \sim \frac{1}{2} \frac{\partial}{\partial \theta} \{ (R_0^{-1} \exp(i\xi R_0) \}, \tag{34}$$

where θ_0 is the angle between the directions of R_0 and (OQ). The displacements and the corresponding external forcer of the $\underset{\circ}{\Delta}$-and $\underset{\circ}{\omega}$-waves according to [2], are

$$u_{R_0} \sim \frac{1}{k^2} \frac{\exp(ikR_0)}{R_0^2}, \qquad u_{\theta_0} = 0, \qquad u_{\varphi_0} = 0, \tag{35}$$

$$F_{R_0} \sim -\frac{4\mu}{k^2} \frac{\exp(ikR_0)}{R_0^3}, \qquad F_{\theta_0} = 0, \qquad F_{\varphi_0} = 0, \tag{36}$$

and

$$u_{R_0} \sim \frac{\cos\theta_0}{\xi^2} \frac{\exp(i\xi R_0)}{R_0^3}, \qquad u_{\theta_0} \sim \frac{\sin\theta_0}{2\xi^2} \frac{\exp(i\xi R_0)}{R_0^3}, \qquad u_{\varphi_0} = 0, \tag{37}$$

$$F_{R_0} \sim \frac{6\mu \cos\theta_0}{\xi^2} \frac{\exp(i\xi R_0)}{R_0^4}, \qquad F_{\theta_0} \sim \frac{3\mu \sin\theta_0}{\xi^2} \frac{\exp(i\xi R_0)}{R_0^4}, \qquad F_{\varphi_0} = 0. \tag{38}$$

In the above relations and all through this paper a constant factor, which assures that the displacements have the dimension of length, is omitted.

The dilatational and distortional waves propagating from such a source are assumed to take the form:

$$\Delta = \frac{1}{2} \int \frac{n dn}{(m+1)\cos n\pi} Y_\nu^{(1)}(k,b)\, Y_\nu^{(2)}(k,r)\, P_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}, \tag{39}$$

and

$$\omega = \frac{1}{4} \int \frac{n dn}{(m+1)\cos n\pi} Y_\nu^{(1)}(\xi,b)\, Y_\nu^{(2)}(\xi,r) \frac{dP_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}}{d\theta}. \tag{40}$$

These expressions coincide asymptotically $(R\to\infty)$ with $R^{-1}\exp(ikR)$ and $\frac{1}{2} \frac{\partial}{\partial\theta} \{R^{-1} \cdot \exp(i\xi R)\}$ respectively, when the medium is homogeneous $m = 0$, and take the same form as (34) in the vicinity of the source. These waves propagate inward through the elastic shell. The reflected waves at the boundary, $r = c$, are

$$\Delta_P = \frac{1}{2} \int \frac{n dn}{(m+1)\cos n\pi} \frac{Y_\nu^{(1)}(k,b)\, Y_\nu^{(2)}(k,c)\, Y_\nu^{(2)}(k,c)\, Y_\nu^{(1)}(k,r)}{Y_\nu^{(1)}(k,c)\, Y_\nu^{(2)}(k,c)} \times E_{n-\frac{1}{2}} P_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}, \tag{41}$$

$$\omega_P = \frac{1}{4} \int \frac{n dn}{(m+1)\cos n\pi} \frac{Y_\nu^{(1)}(k,b)\, Y_\nu^{(2)}(k,c)\, Y_\nu^{(2)}(\xi,c)\, Y_\nu^{(1)}(\xi,r)}{Y_\nu^{(1)}(\xi,c)\, Y_\nu^{(2)}(\xi,c)} \times F_{n-\frac{1}{2}} \frac{dP_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}}{d\theta}, \tag{42}$$

$$\Delta_S = \frac{1}{2} \int \frac{n dn}{(m+1)\cos n\pi} \frac{Y_\nu^{(1)}(\xi,b)\, Y_\nu^{(2)}(\xi,c)\, Y_\nu^{(2)}(k,c)\, Y_\nu^{(1)}(k,r)}{Y_\nu^{(1)}(k,c)\, Y_\nu^{(2)}(k,c)} \times V_{n-\frac{1}{2}} P_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}, \tag{43}$$

$$\omega_S = \frac{1}{4} \int \frac{n dn}{(m+1)\cos n\pi} \frac{Y_\nu^{(1)}(\xi,b)\, Y_\nu^{(2)}(\xi,c)\, Y_\nu^{(2)}(\xi,c)\, Y_\nu^{(1)}(\xi,c)}{Y_\nu^{(1)}(\xi,c)\, Y_\nu^{(2)}(\xi,c)} \times W_{n-\frac{1}{2}} \frac{dP_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}}{d\theta}, \tag{44}$$

where Δ_P and ω_P are reflected dilatational and distortional waves of the primary dilatational wave Δ, and Δ_S and ω_S are those of the primary distortional one ω.

The approximate expressions of these waves can be obtained by the same method of saddle point as in the previous paper. For this purpose we substitute the expressions:

$$Y_\nu^{(1),(2)}(k,r) = \frac{1}{\sqrt{2\pi r}} \int_{C_1,C_2} \exp\left\{\pm i\frac{kr^{m+1}}{m+1}\cos\tau \pm \frac{in}{m+1}\left(\tau-\frac{\pi}{2}\right)\right\},$$

and

$$P_{n-\frac{1}{2}}(\cos\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\exp\{n\log(\cos\theta+i\sin\theta\cos\phi)\}}{(\cos\theta+i\sin\theta\cos\phi)^{\frac{1}{2}}}\,d\phi, \tag{45}$$

into the four Y-functions in the numerator and the spherical harmonics.

On the other hand the asymptotic formula of the denominator is

$$Y_\nu^{(1)}(k,c)\ Y_\nu^{(2)}(k,c) \sim \frac{m+1}{c\sqrt{k^2c^{2m+2}-n^2}} \quad (c\to\infty).$$

Remembering that the reflection coefficients have no exponential character, the saddle point can be easily obtained. If we denote the values at this point by bar, we have the first order approximation as follows;

$$\Delta_P \sim \frac{-\bar{n}\cos\bar{\tau}_2\, E_{\bar{n}}\exp\left\{i\sigma\int_Q^A\frac{ds}{v_\Delta}+i\sigma\int_A^P\frac{ds}{v_\Delta}\right\}}{\sqrt{br\bar{n}\,\sigma\sin\theta}\left\{\left(\frac{r}{c}\right)^{m+1}\cos\bar{\tau}_3\cos\bar{\tau}_4\int_Q^A\frac{ds}{v_\Delta}+\left(\frac{b}{c}\right)^{m+1}\cos\bar{\tau}_1\cos\bar{\tau}^2\int_A^P\frac{ds}{v_\Delta}\right\}^{\frac{1}{2}}}, \tag{46}$$

where

$$\bar{n} = kb^{m+1}\sin\bar{\tau}_1 = kc^{m+1}\sin\bar{\tau}_2 = kc^{m+1}\sin\bar{\tau}_3 = kr^{m+1}\sin\bar{\tau}_4,$$

$$\theta = \frac{-\bar{\tau}_1+\bar{\tau}_2+\bar{\tau}_3-\bar{\tau}_4}{m+1},$$

$$\sigma\int_Q^A\frac{ds}{v_\Delta} = \frac{kb^{m+1}}{m+1}\cos\bar{\tau}_1 - \frac{kc^{m+1}}{m+1}\cos\bar{\tau}_2,$$

$$\sigma\int_A^P\frac{ds}{v_\Delta} = -\frac{kc^{m+1}}{m+1}\cos\bar{\tau}_3 + \frac{kr^{m+1}}{m+1}\cos\bar{\tau}_4. \tag{47}$$

The path of the integration is taken along the ray determined by

$$kr^{m+1}\sin\bar{\tau}_4 = \bar{n}.$$

In the following the "bar" and the word "in the first approximation" will be omitted where no confusion occurs.

$$\omega_P \sim -\frac{in^2\cos\tau_3\, F_n\exp i\sigma\left\{\int_Q^A\frac{ds}{v_\Delta}+\int_A^P\frac{ds}{v_\omega}\right\}}{2kc^m\sqrt{brn\,\sigma\sin\theta}\left\{\left(\frac{r}{c}\right)^{m+1}\frac{\cos\tau_3\cos\tau_4}{k^2c^{2m}}\int_Q^A\frac{ds}{v_\Delta}+\left(\frac{b}{c}\right)^{m+1}\frac{\cos\tau_1\cos\tau_2}{\xi^2c^{2m}}\int_A^P\frac{ds}{v_\omega}\right\}^{\frac{1}{2}}}, \tag{48}$$

where

$$n = kb^{m+1}\sin\tau_1 = kc^{m+1}\sin\tau_2 = \xi c^{m+1}\sin\tau_3 = \xi r^{m+1}\sin\tau_4,$$

$$\theta = \frac{-\tau_1+\tau_2+\tau_3-\tau_4}{m+1},$$

$$\sigma\int_A^P\frac{ds}{v_\omega} = -\frac{\xi c^{m+1}}{m+1}\cos\tau_3 + \frac{\xi r^{m+1}}{m+1}\cos\tau_4. \tag{49}$$

$$\Delta_S \sim \frac{-n\cos\tau_3\, V_n\exp i\sigma\left\{\int_Q^A\frac{ds}{v_\omega}+\int_A^P\frac{ds}{v_\Delta}\right\}}{\xi c^m\sqrt{brn\,\sigma\sin\theta}\left\{\left(\frac{r}{c}\right)^{m+1}\frac{\cos\tau_3\cos\tau_4}{\xi^2c^{2m}}\int_Q^A\frac{ds}{v_\omega}+\left(\frac{b}{c}\right)^{m+1}\frac{\cos\tau_1\cos\tau_2}{k^2c^{2m}}\int_A^P\frac{ds}{v_\Delta}\right\}^{\frac{1}{2}}}, \tag{50}$$

where $$n = \xi b^{m+1}\sin\tau_1 = \xi c^{m+1}\sin\tau_2 = kc^{m+1}\sin\tau_3 = kr^{m+1}\sin\tau_4.$$

$$\omega_S \sim -\frac{in^2\cos\tau_2\, W_n \exp i\sigma\left\{\int_{QAP}\frac{ds}{v_\omega}\right\}}{\sqrt{brn\sigma\sin\theta}\left\{\left(\frac{r}{c}\right)^{m+1}\cos\tau_3\cos c_4\int_Q^A\frac{ds}{v_\omega}+\left(\frac{b}{c}\right)^{m+1}\cos\tau_1\cos\tau_2\int_A^P\frac{ds}{v_\omega}\right\}^{\frac{1}{2}}}, \tag{51}$$

where $$\sigma\int_Q^A\frac{ds}{v_\omega} = \frac{\xi b^{m+1}}{m+1}\cos\tau_1 - \frac{\xi c^{m+1}}{m+1}\cos\tau_2.$$

3. The Waves through the Fluid Core.

The primary waves (39) and (40) are refracted partly into the fluid core at the boundary $r=c$. The refracted waves pass again into the outer elastic shell and propagate outward. They are denoted by Δ_{PK} and ω_{PK}, when the primary wave is the dilatational Δ (39), and by Δ_{SK}, ω_{SK} in the case when the primary wave is the distortional ω (40). If we write

$$Y_\beta^{(t)}(p,c) \equiv Y_{n-\frac{1}{2},s}^{(t)}(p,c),\quad t=1,2,$$

then according to the well known relations of the Bessel functions, [3],

$$Y_\beta^{(1)}\left(p,\, c\exp\frac{i\pi}{s+1}\right) = -\exp\left(-i\pi\frac{n}{s+1}\right)Y_\beta^{(2)}(p,c),$$

$$Y_\beta^{(2)}\left(p, c\exp\frac{-i\pi}{s+1}\right) = -\exp\left(i\pi\frac{n}{s+1}\right)Y_\beta^{(1)}(p,c). \tag{52}$$

With the help of these relations and the above refraction coefficients, we obtain

$$\Delta_{PK} = \frac{1}{2}\int\frac{ndn}{(m+1)\cos n\pi}\,\frac{Y_\nu^{(1)}(k,b)\,Y_\nu^{(2)}(k,c)\,Y_\beta^{(1)}(k,c)\,Y_\beta^{(2)}(p,ce^{-\frac{\pi i}{s+1}})\,Y_\nu^{(2)}(k,c)\,Y_\nu^{(1)}(k,r)}{\{Y_\nu^{(1)}(k,c)\,Y_\nu^{(2)}(k,c)\}^2\{Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,c)\}^2}$$
$$\times G_{n-\frac{1}{2}}\,M_{n-\frac{1}{2}}\,e^{-\frac{n}{s+1}\pi i}\,P_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}, \tag{53}$$

$$\omega_{PK} = \frac{1}{4}\int\frac{ndn}{(m+1)\cos n\pi}\,\frac{Y_\nu^{(1)}(k,b)\,Y_\nu^{(2)}(k,c)\,Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,ce^{-\frac{\pi i}{s+1}})\,Y_\nu^{(2)}(\xi,c)\,Y_\nu^{(1)}(\xi,r)}{Y_\nu^{(1)}(k,c)\,Y_\nu^{(2)}(k,c)\{Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,c)\}^2\,Y_\nu^{(1)}(\xi,c)\,Y_\nu^{(2)}(\xi,c)}$$
$$\times G_{n-\frac{1}{2}}\,N_{n-\frac{1}{2}}\,e^{-\frac{n}{s+1}\pi i}\,\frac{dP_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}}{d\theta}, \tag{54}$$

$$\Delta_{SK} = \frac{1}{2}\int\frac{ndn}{(m+1)\cos n\pi}\,\frac{Y_\nu^{(1)}(\xi,b)\,Y_\nu^{(2)}(\xi,c)\,Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,ce^{-\frac{\pi i}{s+1}})\,Y_\nu^{(2)}(k,c)\,Y_\nu^{(1)}(k,r)}{Y_\nu^{(1)}(\xi,c)\,Y_\nu^{(2)}(\xi,c)\{Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,c)\}^2\,Y_\nu^{(1)}(k,c)\,Y_\nu^{(2)}(k,c)}$$
$$\times U_{n-\frac{1}{2}}\,M_{n-\frac{1}{2}}\,e^{-\frac{n}{s+1}\pi i}\,P_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}, \tag{55}$$

$$\omega_{SK} = \frac{1}{4}\int\frac{ndn}{(m+1)\cos n\pi}\,\frac{Y_\nu^{(1)}(\xi,b)\,Y_\nu^{(2)}(\xi,c)\,Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,ce^{-\frac{\pi i}{s+1}})\,Y_\nu^{(2)}(\xi,c)\,Y_\nu^{(1)}(\xi,r)}{\{Y_\beta^{(1)}(p,c)\,Y_\beta^{(2)}(p,c)\}^2\{Y_\nu^{(1)}(\xi,c)\,Y_\nu^{(2)}(\xi,c)\}^2}$$
$$\times U_{n-\frac{1}{2}}\,N_{n-\frac{1}{2}}\,e^{-\frac{n}{s+1}\pi i}\,\frac{dP_{n-\frac{1}{2}}\{\cos(\theta-\pi)\}}{d\theta}. \tag{56}$$

The same method of approximation as above shows that

$$\Delta_{PK} \sim \frac{npc^{m+s+1}\cos^2\tau_2\cos^2\tau_3\,G_n\,M_n\exp if_1}{(m+1)(s+1)\sqrt{(br)^{m+2}n\sin\theta\cos\tau_1\cos\tau_2\cdots\cos\tau_6}\sqrt{\dfrac{\partial\theta}{\partial n}}}, \tag{57}$$

where

$$n = kb^{m+1}\sin\tau_1 = kc^{m+1}\sin\tau_2 = pc^{s+1}\sin\tau_3 = pc^{s+1}\sin\tau_4 = kc^{m+1}\sin\tau_5 = kr^{m+1}\sin\tau_6,$$

$$\theta = \frac{\tau_2-\tau_1}{m+1} + \frac{\pi-\tau_3-\tau_4}{s+1} + \frac{\tau_5-\tau_6}{m+1},$$

$$-\frac{\partial\theta}{\partial n} = \frac{1}{m+1}\left(\frac{1}{kb^{m+1}\cos\tau_1} - \frac{1}{kc^{m+1}\cos\tau_2}\right) + \frac{1}{s+1}\left(\frac{1}{pc^{s+1}\cos\tau_3} + \frac{1}{pc^{s+1}\cos\tau_4}\right) + \frac{1}{m+1}\left(-\frac{1}{kc^{m+1}\cos\tau_5} + \frac{1}{kr^{m+1}\cos\tau_6}\right),$$

$$f_1 = \frac{kb^{m+1}}{m+1}\cos\tau_1 - \frac{kc^{m+1}}{m+1}\cos\tau_2 + \frac{pc^{s+1}}{s+1}\cos\tau_3 + \frac{pc^{s+1}}{s+1}\cos\tau_4 - \frac{kc^{m+1}}{m+1}\cos\tau_5 + \frac{kr^{m+1}}{m+1}\cos\tau_6$$
$$= \sigma\left\{\int_Q^{A_1}\frac{ds}{v_\Delta} + \int_{A_1}^{A_2}\frac{ds}{v_l} + \int_{A_2}^{P}\frac{ds}{v_\Delta}\right\}. \tag{58}$$

The path of integration must be taken along the geometrical ray determined by the relations $kr^{m+1}\sin\tau = n =$ const. in the outer shell, and $pr^{s+1}\sin\tau = n$ in the fluid core. v_l is the velocity of the dilatational wave in the fluid, as is given in (1). The points A_1 and A_2 on the boundary $r = c$ are determined by the above relations of the ray curve, so that the angles $\angle QOA_1$, $\angle A_1OA_2$ and $\angle A_2OP$ are equal to $\frac{\tau_2-\tau_1}{m+1}$, $\frac{\pi-\tau_3-\tau_4}{s+1}$ and $\frac{\tau_5-\tau_6}{m+1}$ respectively, and the sum of these three angles is θ itself, as is seen from the relation (58).

When the refracted wave is distortional, we get :

$$\omega_{PK} \sim \frac{in^2 pc^{m+s+4}\cos\tau_2\cos^2\tau_3\cos\tau_5\, G_nN_n\exp if_2}{(m+1)(s+1)\sqrt{(br)^{m+2}\, n\sin\theta\cos\tau_1\cos\tau_2\cdots\cos\tau_6}\sqrt{-\frac{\partial\theta}{\partial n}}}, \tag{59}$$

where

$$n = kb^{m+1}\sin\tau_1 = kc^{m+1}\sin\tau_2 = pc^{s+1}\sin\tau_3 = pc^{s+1}\sin\tau_4 = \xi c^{m+1}\sin\tau_5 = \xi r^{m+1}\sin\tau_6,$$

$$-\frac{\partial\theta}{\partial n} = \frac{1}{m+1}\left(\frac{1}{kb^{m+1}\cos\tau_1} - \frac{1}{kc^{m+1}\cos\tau_2}\right) + \frac{1}{s+1}\left(\frac{1}{pc^{s+1}\cos\tau_3} + \frac{1}{pc^{s+1}\cos\tau_4}\right) + \frac{1}{m+1}\left(\frac{-1}{\xi c^{m+1}\cos\tau_5} + \frac{1}{\xi r^{m+1}\cos\tau_6}\right),$$

$$f_2 = \frac{kb^{m+1}}{m+1}\cos\tau_1 - \frac{kc^{m+1}}{m+1}\cos\tau_2 + \frac{pc^{s+1}}{s+1}\cos\tau_3 + \frac{pc^{s+1}}{s+1}\cos\tau_4 - \frac{\xi c^{m+1}}{m+1}\cos\tau_5 + \frac{\xi r^{m+1}}{m+1}\cos\tau_6$$
$$= \sigma\left\{\int_Q^{A_1}\frac{ds}{v_\Delta} + \int_{A_1}^{A_2}\frac{ds}{v_l} + \int_{A_2}^{P}\frac{ds}{v_\omega}\right\}. \tag{60}$$

In the case when the primary converging wave is distortional ω (40),

$$\Delta_{SK} \sim \frac{npc^{m+s+4}\cos\tau_2\cos^2\tau_3\cos\tau_5\, U_nM_n\exp if_3}{(m+1)(s+1)\sqrt{(br)^{m+2}\, n\sin\theta\cos\tau_1\cos\tau_2\cdots\cos\tau_6}\sqrt{-\frac{\partial\theta}{\partial n}}}, \tag{61}$$

and

$$\omega_{SK} \sim \frac{in^2 pc^{m+s+4}\cos^2\tau_2\cos^2\tau_3\, U_nN_n\exp if_4}{(m+1)(s+1)\sqrt{(br)^{m+2}\, n\sin\theta\cos\tau_1\cos\tau_2\cdots\cos\tau_6}\sqrt{-\frac{\partial\theta}{\partial n}}}, \tag{62}$$

where

$$f_3 = \sigma\left\{\int_Q^{A_1}\frac{ds}{v_\omega} + \int_{A_1}^{A_2}\frac{ds}{v_l} + \int_{A_2}^{P}\frac{ds}{v_\Delta}\right\},$$

and

$$f_4 = \sigma\left\{\int_Q^{A_1}\frac{ds}{v_\omega} + \int_{A_1}^{A_2}\frac{ds}{v_l} + \int_{A_2}^{P}\frac{ds}{v_\omega}\right\}.$$

These waves have a focal surface

$$\frac{\partial \theta}{\partial n} = 0. \tag{63}$$

The intensity of the waves in the vicinity of this surface can not be obtained by the above approximation of the first order. It depends mainly upon the numerical values of the elastic constants at the boundary of the fluid core whether the equation (63) has a real root or not.

4. Approximate Values of the Reflection and Refraction Coefficients.

The behaviour of the various coefficients of reflection and refraction at the boundary $r = c$ are considerably complicated. Now as the radius c and the order n are assumed to be sufficiently large, the asymptotic formulas

$$\frac{Y^{(1)\prime}_{n,m}(k,c)}{Y^{(1)}_{n,m}(k,c)} \sim i\frac{\sqrt{k^2c^{2m+2}-n^2}}{c}, \qquad \frac{Y^{(2)\prime}_{n,m}(k,c)}{Y^{(2)}_{n,m}(k,c)} \sim -i\frac{\sqrt{k^2c^{2m+2}-n^2}}{c}, \tag{64}$$

and $\quad Y^{(1)}_{n,m}(k,c)\, Y^{(2)}_{n,m}(k,c) \sim -\dfrac{m+1}{c\sqrt{k^2c^{2m+3}-n^2}}$

are useful for the approximate evaluation of the coefficients. Moreover with the help of the values at the saddle point obtained in the last section, they can be much simplified and depend explicitly upon the incident, reflected and refracted angles. The coefficients coincide with those of plane boundary in this approximation, as is expected naturally. It is convenient to write the values at the saddle point

$$n = kc^{m+1}\sin\tau_2 = \xi c^{m+1}\sin\tau_3 = pc^{s+1}\sin\tau' \tag{65}$$

for the calculation of all the coefficients together. Then we obtain the following results :

$$E = D^{-1}\left\{\sin 2\tau'\left(\sin 2\tau_2 \sin 2\tau_3 - \frac{\xi^2}{k^2}\cos^2 2\tau_3\right) + \frac{\lambda'}{\mu}\sin 2\tau_2\right\}, \tag{66}$$

$$F = D^{-1}\left\{\frac{4i\xi^2}{k^3c^{m+1}}\sin 2\tau' \cos\tau_2 \cos 2\tau_3\right\}, \tag{67}$$

$$G = D^{-1}\left\{2\frac{\xi^2}{k^2}\sin 2\tau_2 \cos 2\tau_3\right\}, \tag{68}$$

$$W = D^{-1}\left\{\sin 2\tau'\left(\sin 2\tau_2 \sin 2\tau_3 - \frac{\xi^2}{k^2}\cos^2 2\tau_3\right) - \frac{\lambda'}{\mu}\sin 2\tau_2\right\}, \tag{69}$$

$$U = D^{-1}\left\{2ipc^{s+1}\sin\tau' \sin 2\tau_2 \sin 2\tau_3\right\}, \tag{70}$$

$$V = D^{-1}\left\{-ikc^{m+1}\sin 2\tau' \sin\tau_2 \sin 4\tau_3\right\}, \tag{71}$$

$$L = D^{-1}\left\{\sin 2\tau'\left(\sin 2\tau_2 \sin 2\tau_3 + \frac{\xi^2}{k^2}\cos^2 2\tau_3\right) - \frac{\lambda'}{\mu}\sin 2\tau_2\right\}, \tag{72}$$

$$M = D^{-1}\left\{\frac{2\lambda'}{\mu}\sin 2\tau' \cos 2\tau_3\right\}, \tag{73}$$

$$N = D^{-1}\left\{\frac{4i\lambda'}{\mu kc^{m+1}}\sin 2\tau' \cos\tau_2\right\}, \tag{74}$$

where

$$D = \sin 2\tau'\left\{\sin 2\tau_2 \sin 2\tau_3 + \frac{\xi^2}{k^2}\cos^2 2\tau_3\right\} + \frac{\lambda'}{\mu}\sin 2\tau_2. \tag{75}$$

It is interesting to notice that there are simple relations

$$\frac{GM}{1-L} + E = 1, \quad \frac{GN}{1-L} = F, \quad \frac{UN}{1-L} + W = -1, \quad \frac{UM}{1-L} = -V. \tag{76}$$

among the above coefficients.

5. Numerical calculation.

We will calculate the above results numerically with the help of observational data in the seismic phenomena, which is in fact so complicated (see [4], [5]) that we must take only the mean values of the data. We assume that the densities of the media and the velocities of dilatational (P-) and distortional (S-) waves just outside and inside of the boundary of the fluid core, according to [6], are as shown in the Table 1.

Table 1.

	densities	velocities	
		P-wave	S wave
outside	$\rho = 6.0$	$v_\Delta = 1.30$	$v_\omega = 7.25$
inside	$\rho' = 9.5$	$v_l = 8.5$	

Then

$$\frac{v_l}{v_\Delta} = \frac{kc^m}{pc^s} = \frac{8.5}{13.0} = 0.654, \qquad \frac{v_l}{v_\omega} = \frac{\xi c^m}{pc^s} = \frac{8.5}{7.25} = 1.17,$$

and

$$\frac{k}{\xi} = \frac{7.25}{13.0} = 0.5576.$$

Since

$$\sqrt{\frac{\mu}{\rho}} = v_\omega \text{ and } \sqrt{\frac{\lambda'}{\rho'}} = v_l, \qquad \frac{\lambda'}{\mu} = \frac{\rho' v_l^2}{\rho v_\omega^2} = 2.18.$$

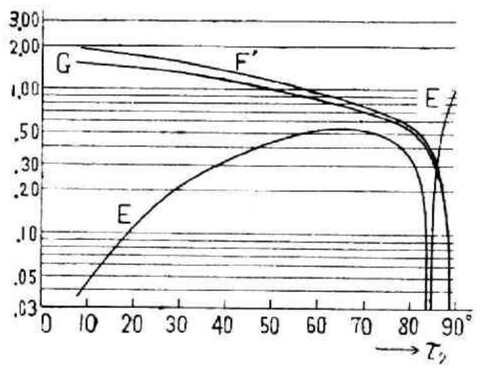

Fig. 1 Coefficients of reflection and refraction at the boundary of the fluid core (E; P−P reflection, F; P−S reflection, G; P−P refraction), and the incident angle τ_2.

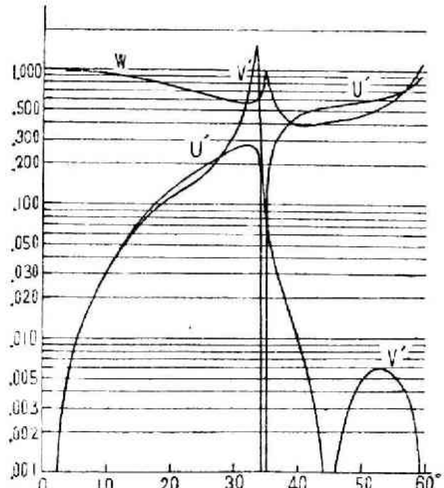

Fig. 2 Coefficients of reflection and refraction at the boundary of the fluid core (U'; S-P refraction, V'; S P reflection, W; S-S reflecton), and the incident angle τ_3.

With the help of these values, the reflection and refraction coefficients can be calculated numerically for the angle of incidence, as in the figures 1, 2 and 3, where τ_2, τ_3 and τ' are the angles of incidence of P-, S- in the outer elastic shell and P-wave in the fluid core respectively. F', U', V', and N' are given by the relations:

$$F = \frac{i}{kc^{m+1}} F', \qquad U = ipc^{s+1} U', \qquad V = -ikc^{m+1} V' \quad \text{and} \quad N = \frac{i}{kc^{m+1}} N',$$

in places of (67), (70), (71) and (74) respectively.

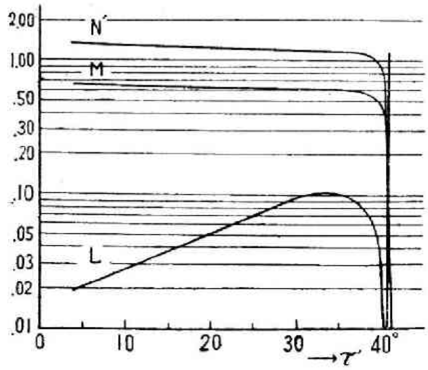

Fig. 3 Coefficients of reflection and refraction at the boundary of the fluid core against the mautle (L; P-P reflection, M; P-P refraction, N, P-S refraction), and the incident angle τ'.

It depends upon the assumed mechanical structure (37) of the source of the distortional waves that the displacements caused by once reflected SS-wave are larger than the ones caused by the primary S-wave, as shown in Fig. 7, and that the displacements caused by the incidence of the ScS-wave behave themselves in peculiar appearance in Fig. 8.

In the previous paper the constant m was assumed to take the value 3 in the numerical calculation, where it was shown that the assumption was valid where the depth was not so large. It seems, however, to be better to take the value $m = 0.3$ so far as our attention is concerned to the waves propagating to a large distance from the source. The displacements of the free surface (i. e. the surface of the earth) caused by an incident wave have the radial (perpendicular to the surface) and the tangential (horizontal) components u_p and v_p :

$$u_p = -\frac{2\xi^2}{R^2}\,\rho\,\frac{\cos\tau_2\cos2\tau_3}{ka^m}\,\Delta, \qquad v_p = -\frac{2\xi^2}{k^2}\,\rho\,\frac{\cos\tau_2\sin2\tau_3}{ka^m}\,\Delta, \tag{77}$$

when the incident wave is dilatational Δ. On the other hand when the incident wave is distortional ω, its components are

$$u_s = -\frac{4i}{\xi a^m}\,\rho\cos\tau_3\sin2\tau_2\,\omega, \qquad v_s = -\frac{4i\xi}{k^2a^m}\,\rho\cos\tau_3\cos2\tau_3\,\omega, \tag{68}$$

where
$$\rho = \left\{\sin2\tau_2\sin2\tau_3 + \frac{\xi^2}{k^2}\cos^2 2\tau_3\right\}^{-1},$$

and a is the radius of the earth. The relation between τ_2 and τ_3 in the above formulas has been given in (45). If we write the above (77) and (78)

$$\begin{cases} u_p = \alpha_p\,\Delta, \\ v_p = \beta_p\,\Delta, \end{cases} \quad \text{and} \quad \begin{cases} u_s = \alpha_s\,\omega, \\ v_s = \beta_s\,\omega, \end{cases}$$

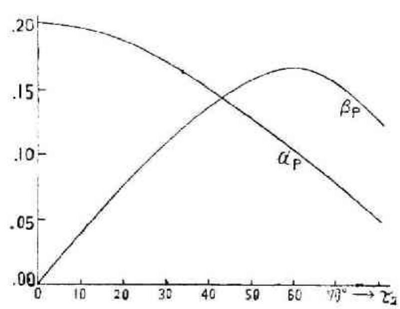

Fig. 4 Displacement coefficients of the free surface due to P-wave (α_P; vertical, β_P; horizontal) and the incident angle τ_2.

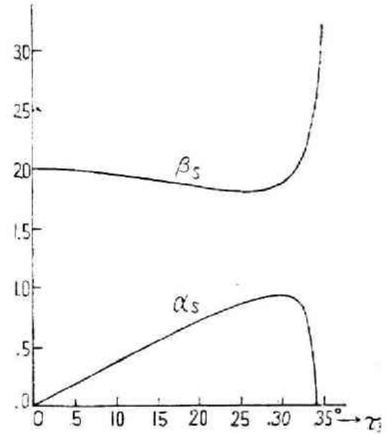

Fig. 5 Displacement coefficients of the free surface due to S-wave (v_S; vertical, β_S: horizontae) and the incident angle τ_3.

α_p, β_p, α_s and β_s are the measure of the components of the displacements for the given angle of incidence, as shown in Fig. 4 and 5, see [7]. So far as the waves at a large distance from the source is concerned, the depth of the source may be neglected as compared to the radius of the earth. Then the expressions of various waves become extremely simple.

Using the ordinary notations in the seismology, we get at the free surface, $r=a$, the following formulas:

$$|P| = \sqrt{\frac{m+1}{2}}\,\frac{1}{a}\sqrt{\frac{\tan\tau_2}{\sin\theta}} \quad \text{from (19) in the previous paper,}$$

$$|PP| = \sqrt{\frac{m+1}{2}}\,\frac{1}{a}\,\frac{\sin 2\tau_2 \sin 2\tau_3 - \frac{\xi^2}{k^2}\cos^2 2\tau_3}{\sin 2\tau_2 \sin 2\tau_3 + \frac{\xi^2}{k^2}\cos^2 2\tau_3}\sqrt{\frac{\tan\tau_2}{2\sin\theta}} \quad \text{from (27) there,}$$

$$|PcP| = \sqrt{\frac{m+1}{2}}\,\frac{1}{a}\sqrt{\frac{\tan\tau_4\cos\tau_2}{\sin\theta}}\,\frac{E}{\left\{\left(\frac{a}{c}\right)^{m+1}\cos\tau_4 - \cos\tau_2\right\}^{\frac{1}{2}}} \quad \text{from the above (46).}$$

$$|S| = \sqrt{\frac{m+1}{2}}\,\frac{n}{2a}\sqrt{\frac{\tan\tau_3}{\sin\theta}} \quad \text{from (21) in the previous paper,}$$

$$|SS| = \sqrt{\frac{m+1}{2}}\,\frac{n}{2a}\,\frac{\sin 2\tau_2 \sin 2\tau_3 - \frac{\xi^2}{k^2}\cos^2 2\tau_3}{\sin 2\tau_2 \sin 2\tau_3 + \frac{\xi^2}{k^2}\cos^2 2\tau_3}\sqrt{\frac{\tan\tau_3}{2\sin\theta}} \quad \text{from (35) there,}$$

$$|ScS| = \sqrt{\frac{m+1}{2}}\,\frac{n}{2a}\sqrt{\frac{\tan\tau_4\cos\tau_2}{\sin\theta}}\,\frac{W}{\left\{\left(\frac{a}{c}\right)^{m+1}\cos\tau_4 - \cos\tau_2\right\}^{\frac{1}{2}}} \quad \text{from the above (51).}$$

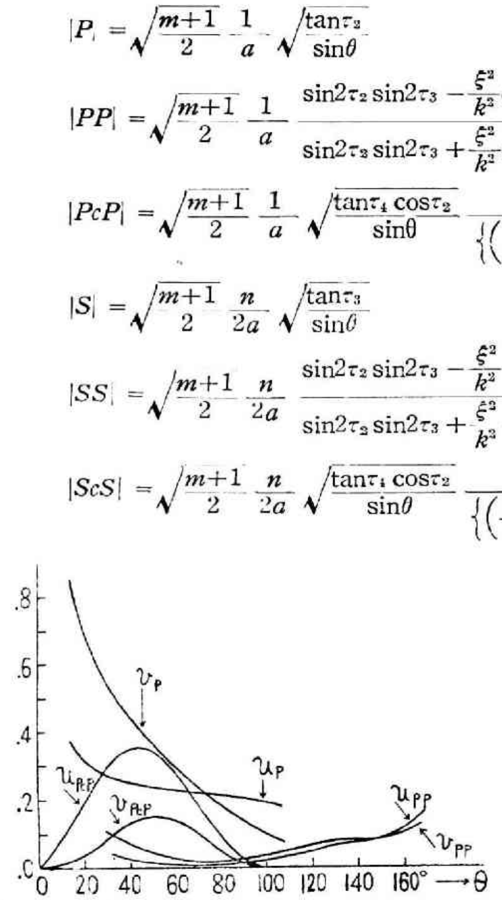

Fig. 6 Displacements of the free surface (u; vertical, v; horizontal). and the central angle θ.

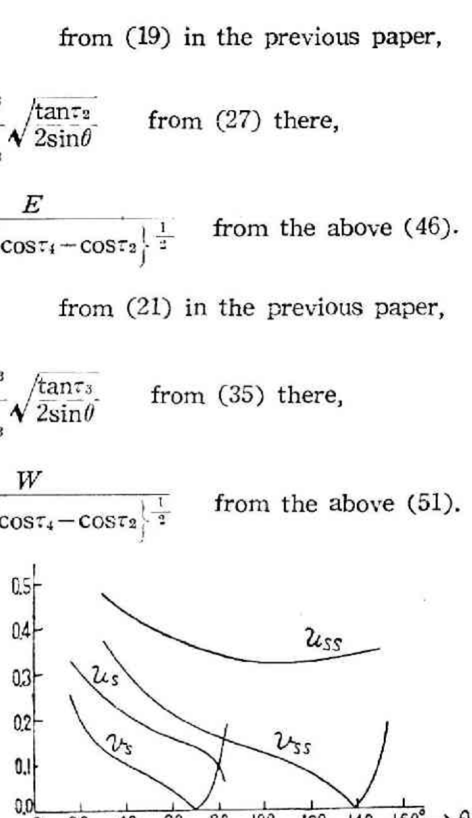

Fig. 7 Displacements of the free surface (u; vertical, v; horizontal) and the central angle θ.

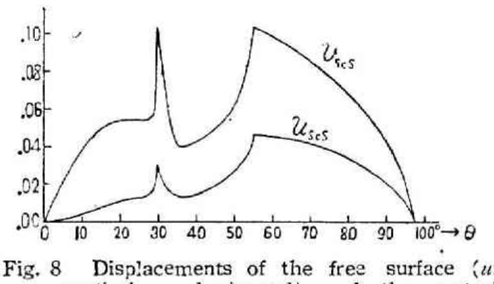

Fig. 8 Displacements of the free surface (u: vertical, v. horizontal) and the central angle θ.

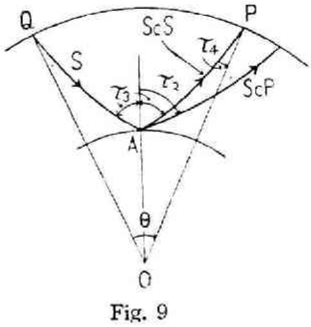

Fig. 9

The displacements of the surface at the point of emergence of these waves are shown in Figs. 6, 7 and 8. The angles τ_2, τ_3, τ_4 and θ in (46), (51) have the meaning shown in Fig. 9.

6. Summary

An elastic sphere of large radius is assumed to have a concentric fluid spherical core whose radius is also large. Both the media of the elastic shell and fluid core are supposed to be inhomogeneous and the velocities of the elastic waves propagating in them are proportional to r^{-m} and r^{-s} respectively, where m and s are arbitrary real number and both are larger than -1. Then the wave equations have rigorous solutions expressed in terms of the Bessel functions. They are superposed so as to construct the wave transmitted from a point source, which coincides asymptotically $R^{-1}\exp(ikR)$ when the medium is homogeneous. The distribution of the external force in the vicinity of the source makes clear how the source concerned be constructed mechanically to send out such waves as $R^{-1}\exp(ikR)$ or $\frac{\partial}{\partial\theta}[R^{-1}\exp(i\xi R)]$ approximately.

The most part of our attention has been confined to the waves which are reflected and refracted at the inner fluid boundary. The results of general discussions have been applied to the seismic phenomena. In the previous paper we took the value $m=3$, and calculated numerically. This is valid as far as our attention is confined to the shallow part of the earth. It seems better, however, to take the value $m = 0.3$, when the waves at a large distance from the source are taken into account. Thus we have calculated numerically with the latter value of m in this paper, and obtained consideradly different numerical results as compared to those of the previous paper.

Owing to the fact that the elastic properties of the earth crust are much complicated, rather tedious calculation will be needed for the more precise explanation of the actual seismic waves, if one wishes to start with the rigorous solutions of the wave equations.

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