On the Relation between Transmission Function of Column and that of Slab for Infra-Red Absorption Band.

ON THE RELATION BETWEEN TRANSMISSION FUNCTION OF COLUMN AND THAT OF SLAB FOR INFRA-RED ABSORPTION BAND

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1 Introduction

ROBERTS (4) and PHILIPPS (3) found that for a continuous spectrum the transmission of a slab of thickness u is practically equivalent to that of a linear column of length 1.5 u. While, according to ELSASSER (2) the transmission through a slab of thickness u is approximately equal to that of a column of length 1.66 u for the idealized absorption band of equal and equi-distant lines. However, it will be shown in the present paper that the result for the continuous spectrum is also available for the absorption band.

2 Transmission functions for the idealized band

According to ELSASSER (2) the transmission function of a column, Γ_I, for the idealized band is given by

$$\Gamma_I = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-k(s)u}\, ds = \sinh\beta \int_{\frac{Su}{d\sinh\beta}}^{\infty} e^{-y\cosh\beta} J_0(iy)dy \tag{1}$$

$$= \sinh\beta\left(\frac{Su}{d\sinh\beta}\right)\int_1^{\infty} e^{-\frac{Suy\cosh\beta}{d\sinh\beta}} J_0\left(i\frac{Suy}{d\sinh\beta}\right)dy, \tag{2}$$

where $k(s) = \dfrac{S}{d}\,\dfrac{\sinh\beta}{\cosh\beta\text{-}\cos s}$, $\beta = \dfrac{2\pi\alpha}{d}$, $s = \dfrac{2\pi\nu}{d}$, k is the absorption coefficient, ν the frequency in cm^{-1}, S the total intensity of a line, α the half-width, d the line distance, J_0 the zeroth order Bessel function of a imaginary argument. The corresponding transmission function of a slab, Γ_f, is defined by

$$\Gamma_f = \frac{1}{\pi}\int_1^{\infty}\frac{dt}{t^3}\int_{-\pi}^{\pi} e^{-k(s)ut}\, ds. \tag{3}$$

Introducing (1) in (3) and integrating by parts, we have

$$\Gamma_f = \sinh\beta\int_{\frac{Su}{d\sinh\beta}}^{\infty} e^{-y\cosh\beta} J_0(iy)dy - \sinh\beta\left(\frac{Su}{d\sinh\beta}\right)^2\int_{\frac{Su}{d\sinh\beta}}^{\infty}\frac{1}{y^2}e^{-y\cosh\beta} J_0(iy)dy. \tag{4}$$

Or introducing (2) in (3), we have

$$\Gamma_f = 2\sinh\beta\,\frac{Su}{d\sinh\beta}\int_1^{\infty}\frac{dt}{t^2}\int_1^{\infty} e^{-\frac{Suty\cosh\beta}{d\sinh\beta}} J_0\left(i\frac{Suty}{d\sinh\beta}\right)dy. \tag{5}$$

If β is small and u is large, we can put $\cosh\beta = 1 + \dfrac{\beta^2}{2}$, $\sinh\beta = \beta$, $J_0(i\eta) = \dfrac{e^{\eta}}{\sqrt{2\pi\eta}}$ in (5).

Then we have

$$\Gamma_f = \frac{2}{3}\frac{1}{\sqrt{\pi}}\sqrt{\frac{lu}{2}}\left(2\frac{lu}{2}-1\right)e^{-\frac{lu}{2}} - \left\{\frac{4}{3}\left(\frac{lu}{2}\right)^2-1\right\}\left\{1-\phi\left(\sqrt{\frac{lu}{2}}\right)\right\}, \qquad (6)$$

where l is ELSASSER's generalized absorption coefficient, given by $l = \frac{S\beta}{d}$, and $\phi(\eta)$ is the error integral, given by

$$\phi(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\, d\eta.$$

Equation (6) was already obtained by ELSASSER.

3 The Relation between transmission functions of column and slab

At first we shall summarize the known result for a continuous spectrum. In this case the transmission of a column, τ_I, is given by $\tau_I = e^{-x}$ and that of a slab, τ_f, is given by $\tau_f = 2\int_1^{\infty}\frac{e^{-xt}}{t^3}\,dt$, where $x = ku$. Then by the mean value theorem of integral, we have $\tau_f(x) = \tau_I(\xi x)$ where the value of ξ lies between 1 and ∞ and depends on x. For small x, expanding τ_f in series by the well known formula for exponential integral, we have $\tau_f = 1 - 2x$, while $\tau_I = 1 - x$ for this case. Hence $\xi = 2$ for small x. With increase of x, ξ decreases rapidly and tends to 1 for large x. The mean value of ξ was defined by ROBERTS as the value of ξ which satisfies the following condition:

$$\int_0^{\infty}\tau_I(\xi x)\,dx = \int_0^{\infty}\tau_f(x)\,dx. \qquad (7)$$

It is easily shown that $\xi = 1.5$ satisfies this condition.

Now quite similar argument holds in the case of absorption band too. By the mean value theorem of integral, $\Gamma_f(x)$ of (5) for the band can be expressed as $\Gamma_I(\xi' x)$ of (2) with $1 < \xi' < \infty$, where $x = \frac{Su}{d\sinh\beta}$. For small x, we have from (1)

$$\Gamma_I(x) = 1 - \sinh\beta\int_0^{x} e^{-y\cosh\beta} J_0(iy)\,dy = 1 - x\sinh\beta. \qquad (8)$$

And referring to the following relation, *i. e.*,

$$\lim_{x\to 0} x\int_x^{\infty}\frac{1}{y^2}e^{-y\cosh\beta} J_0(iy)\,dy = 1, \qquad (9)$$

we have from (4)

$$\Gamma_f(x) = 1 - 2x\sin\beta. \qquad (10)$$

Hence we have $\xi' = 2$ for small x in the case of a band too. With increase of x, ξ' also decreases rapidly and tends to 1 for large x. Now it will be reasonable to define the mean value of ξ' so as to satisfy the condition,

$$\int_0^{\infty}\Gamma_I(\xi' x)\,dx = \int_0^{\infty}\Gamma_f(x)\,dx. \qquad (11)$$

Then from (2) and (5), by the aid of the known integral*

$$\int_0^{\infty} e^{-at}J_\nu(bt)t^{\nu+1}\,dt = \frac{2a(2b)^\nu\Gamma(\nu+\frac{3}{2})}{(a^2+b^2)^{\nu+\frac{3}{2}}\sqrt{\pi}}, \qquad (12)$$

where $R(\nu) > -1$ and Γ is the Gamma function, we again arrive at the result that $\xi' = 1.5$. The only difference between the values of ξ and ξ' is due to the difference of respective transmission functions for a continuous spectrum and for a band. In the following discussion $x = \frac{lu}{2}$ will be used for the band instead of $x = \frac{Su}{d\sinh\beta}$ in the above discussion. This change of variable is for the sake of practical convenience in numerical calculation, because transmission functions of a slab are generally expressed as function of $\frac{lu}{2}$. Now in formulae which determine ξ and ξ', *i. e.*,

$$\tau_f(x) = 2\int_1^{\infty}\tau_I(xt)\frac{dt}{t^3} = \tau_I(\xi x), \qquad (13)$$

$$\Gamma_f(x) = 2\int_1^{\infty}\Gamma_I(xt)\frac{dt}{t^3} = \Gamma_I(\xi' x), \qquad (14)$$

the weight function $\frac{1}{t^3}$ decreases rapidly with

*G. N. WATSON, Theory of Bessel Function, 1922, p. 386

increase of t, and as shown by ELSASSER's numerical calculation, both τ_I and Γ_I are monotone decreasing functions and Γ_I decreases more rapidly with x than τ_I. So that we can say that $\xi > \xi'$ for a given small x. While, as was shown above, mean values of both ξ and ξ' are equal, hence it will be said that $\xi < \xi'$ for large value of x.

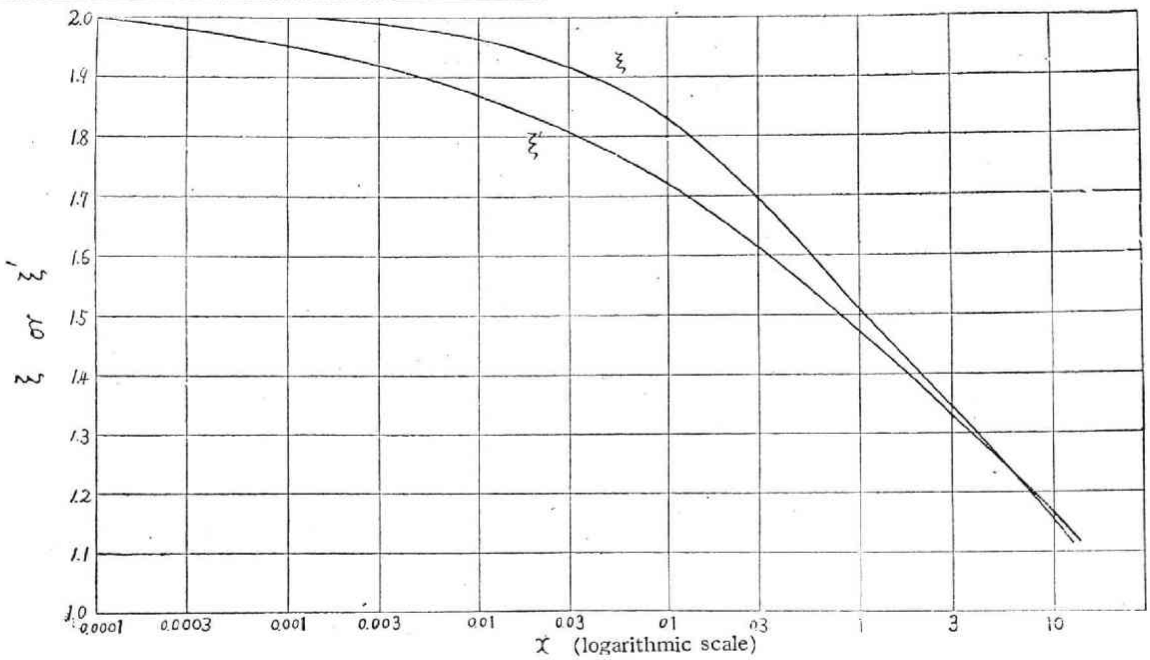

Fig. 1. $x = ku$ for ξ curve.
$x = \dfrac{lu}{2}$ for ξ' curve.

Actual values of ξ and ξ' as function of x are shown in Fig. 1. These curves are obtained by calculating numerically τ_I, τ_f, Γ_I and Γ_f as function of x. In this case we assumed that $\beta = 0.2$, *i. e.*, $\alpha = 0$ and $d = 3$, which will be appropriate values for the water vapour band at normal pressure. Then we have $\dfrac{Su}{d \sinh \beta} = 50 \dfrac{lu}{2}$, and for small x, Γ_I and Γ_f were calculated from (1) and (4) by numerical integration, and for moderate and large x, ELSASSER's approximate formulae, $\Gamma_I(x) = 1 - \phi(\sqrt{x})$ and (6) were used. It is shown in Fig. 1 that both ξ and ξ' decrease rapidly at first with increase of x and that $\xi' < \xi$ for small x as was predicted qualitatively.

Recently COWLING (1) has shown that ELSASSER's transmission functions for the idealized band are somewhat smaller than actual ones for large value of u. The result obtained in this paper ought to be modified accordingly. The required modification will have the tendency to decrease ξ' for small x. However the necessary modification will probably be very small, because the weight function will be $\dfrac{1}{t^3}$ in the case of actual band too, so that the change of the transmission function for large xt will not influence much effect on ξ'.

References

(1) T. G. COWLING, *Phil. Mag.*, 41, 109, (1950)
(2) W. M. ELSASSER, Heat Transfer by Infra-red Radiation in the Atmosphere, *Harvard Meteor. Studies*, No. 6, (1942)
(3) H. PHILIPPS, *Gerl. Beitr. z. Geophys.*, 56, 229, (1940)
(4) O. F. T. ROBERTS, *Proc. Roy. Soc. Edingburgh*, 50, 225, (1929/30)