On Visco-Elastic Medium. (Part III)

ON VISCO-ELASTIC MEDIUM.

(PART III)

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Chapter V. Bodily Waves.

§ 1. From equation (3.9) in Chapter III, taking divergence of each term, we get

$$\left(a+\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}\theta=\frac{1}{\rho}\left\{(\lambda'+2\mu')+(\lambda+2\mu)\frac{\partial}{\partial t}\right\}\nabla^2\theta, \qquad \cdots\cdots(5.\ 1)$$

and putting $\theta=0$, we get

$$\left(a+\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}(u,\ v,\ w)=\frac{1}{\rho}\left(\mu'+\mu\frac{\partial}{\partial t}\right)\nabla^2(u, v, w). \qquad \cdots\cdots(5.\ 2)$$

If u, v, w are proportional to $e^{i\kappa ct}$, these equations become to

$$\left.\begin{aligned}&(a+i\kappa c)\kappa^2c^2\theta+\frac{1}{\rho}\{(\lambda'+2\mu')+i\kappa c(\lambda+2\mu)\}\nabla^2\theta=0,\\ \text{and in the case of}\quad&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\\ &(a+i\kappa c)\kappa^2c^2(u, v, w)+\frac{1}{\rho}\{\mu'+i\kappa c\mu\}\nabla^2(u, v, w)=0.\end{aligned}\right\} \cdots\cdots(5.\ 3)$$

These equations can be regarded to represent wave motions.

§ 2. Taking the simplest case in which u, v, w are all independent of y and z, we get

$$\nabla^2\theta=\frac{\partial^2\theta}{\partial x^2}\ \text{and}\ \nabla^2(u, v, w)=\frac{\partial^2}{\partial x^2}(u, v, w).$$

Assuming

$$\theta=\theta_0e^{(\alpha+i\beta)x}, \qquad \cdots\cdots(5.\ 4)$$

we get

$$\nabla^2\theta=\theta_0(\alpha^2-\beta^2+2i\alpha\beta)e^{(\alpha+i\beta)x}. \qquad \cdots\cdots(5.\ 5)$$

Putting this in (5. 3), we get

$$\rho\kappa^2c^2(a+i\kappa c)+\{(\lambda'+2\mu')+(\lambda+2\mu)i\kappa c\}(\alpha^2-\beta^2+2i\alpha\beta)=0. \qquad \cdots\cdots(5.\ 6)$$

Putting the real and imaginary parts of this equation independently to be zero, following equations are obtained:

$$\left.\begin{aligned}&\rho\kappa^2c^2a+(\lambda'+2\mu')(\alpha^2-\beta^2)-2\alpha\beta\kappa c(\lambda+2\mu)=0,\\ \text{and}\quad&\rho\kappa^3c^3+(\lambda+2\mu)\kappa c(\alpha^2-\beta^2)+2\alpha\beta(\lambda'+2\mu')=0,\end{aligned}\right\} \qquad \cdots\cdots(5.\ 7)$$

which gives

$$\left.\begin{aligned} \alpha^2-\beta^2&=-\frac{\rho\kappa^2c^2\{\kappa^2c^2(\lambda+2\mu)+a(\lambda'+2\mu')\}}{(\lambda+2\mu)^2\kappa^2c^2+(\lambda'+2\mu')^2}, \\ \text{and} \qquad \alpha\beta&=\frac{\rho\kappa^3c^3\{(\lambda+2\mu)a-(\lambda'+2\mu')\}}{2\{\kappa^2c^2(\lambda+2\mu\}^2+(\lambda'+2\mu')^2}. \end{aligned}\right\} \quad \cdots\cdots(5.\ 8)$$

If we denote the positive roots of (5. 8) by α and β, then the solution of (5. 1) is

$$\theta=\theta_1 e^{(\alpha+i\beta)x+i\kappa ct}+\theta_2 e^{-(\alpha+i\beta)x+i\kappa ct}, \quad \cdots\cdots(5.\ 9)$$

because the wave must be damped as it propagates and α and β must have the same sign.

By this evidence, from the second equation of (5. 8), the following relation is obtained:

$$a(\lambda+2\mu)>\lambda'+2\mu'. \quad \cdots\cdots(5.\ 10)$$

Similarly the solution of (5. 2) can be obtained by replacing μ and μ' to $\lambda+2\mu$ and $\lambda'+2\mu'$ respectively in (5. 9). We get thus

$$(u, v, w)=(0, v_1, w_1)e^{(\alpha'+i\beta')x+i\ ct}+(0, v_2, w_2)e^{-(\alpha'+i\beta')x+i\kappa ct}, \quad \cdots\cdots(5.\ 11)$$

where α' and β' are the positive roots of

$$\left.\begin{aligned} \alpha'^2-\beta'^2&=-\frac{\rho\kappa^2c^2(\kappa^2c^2\mu+a\mu')}{\mu^2\kappa^2c^2+\mu'^2}, \\ \text{and} \qquad \alpha'\beta'&=\frac{\rho\kappa^3c^3(\mu a-\mu')}{2(\kappa^2c^2\mu^2+\mu'^2)}. \end{aligned}\right\} \quad \cdots\cdots(5.\ 12)$$

As α' and β' are both positive, we get from the second equation of (5. 12)

$$\mu a>\mu'. \quad \cdots\cdots(5.\ 13)$$

If specially

$$a(\lambda+2\mu)=(\lambda'+2\mu'), \quad \cdots\cdots(5.\ 14)$$

then we get

$$\alpha=0 \text{ and } \beta=\sqrt{\frac{\rho}{\lambda+2\mu}}\,\kappa c \quad \cdots\cdots(5.\ 15)$$

or the velocity of propagation of the wave is $\sqrt{\frac{\lambda+2\mu}{\rho}}$. The case of perfect elastic body or $a=\lambda'+2\mu'=0$ is included in this case, but this is more general.

If $\lambda'+2\mu'=0$ but not $a=0$, then we get

$$\left.\begin{aligned} \alpha^2&=\frac{\rho\kappa c}{2(\lambda+2\mu)}\{\sqrt{\kappa^2c^2+a^2}-\kappa c\}, \\ \beta^2&=\frac{\rho\kappa c}{2(\lambda+2\mu)}\{\sqrt{\kappa^2c^2+a^2}+\kappa c\}. \end{aligned}\right\} \quad \cdots\cdots(5.\ 16)$$

If specially $a<\kappa c$, then we get

$$\left.\begin{aligned} \alpha^2&=\frac{\rho\kappa^2c^2}{2(\lambda+2\mu)}\left\{\frac{1}{2}\left(\frac{a}{\kappa c}\right)^2-\frac{1}{8}\left(\frac{a}{\kappa c}\right)^4+\frac{1}{16}\left(\frac{a}{\kappa c}\right)^6\cdots\cdots\right\}, \\ \beta^2&=\frac{\rho\kappa c}{2(\lambda+2\mu)}\left\{2+\frac{1}{2}\left(\frac{a}{\kappa c}\right)^2-\frac{1}{8}\left(\frac{a}{\kappa c}\right)^4+\frac{1}{16}\left(\frac{a}{\kappa c}\right)^6\cdots\cdots\right\}. \end{aligned}\right\} \quad \cdots\cdots(5.\ 17)$$

If $a>\kappa c$, then we get

$$\left.\begin{aligned}\alpha^2&=\frac{\rho\kappa c}{2(\lambda+2\mu)}\left[\left\{1+\frac{1}{2}\left(\frac{\kappa c}{a}\right)^2-\frac{1}{8}\left(\frac{\kappa c}{a}\right)^4+\frac{1}{16}\left(\frac{\kappa c}{a}\right)^6-\frac{1}{128}\left(\frac{\kappa c}{a}\right)^8+\cdots\cdots\right\}a-\kappa c\right],\\ \beta^2&=\frac{\rho\kappa c}{2(\lambda+2\mu)}\left[\left\{1+\frac{1}{2}\left(\frac{\kappa c}{a}\right)^2-\frac{1}{8}\left(\frac{\kappa c}{a}\right)^4+\frac{1}{16}\left(\frac{\kappa c}{a}\right)^6-\frac{1}{128}\left(\frac{\kappa c}{a}\right)^8+\cdots\cdots\right\}a+\kappa c\right].\end{aligned}\right\}\cdots(5.\ 18)$$

If $\frac{\kappa c}{a}=0$, then we get

$$\alpha^2=\frac{\rho\kappa ca}{2(\lambda+2\mu)},\qquad \beta^2=\frac{\rho\kappa ca}{2(\lambda+2\mu)}.\qquad\cdots\cdots(5.\ 19)$$

The velocity of the propagation of wave is given by $\kappa c/\beta$, which is $\sqrt{\frac{2(\lambda+2\mu)}{\rho a}}$ in the case κc is very small or the period of oscillation is infinitely long.

The general solutions of (5. 8) and (5. 12) are

$$\left.\begin{aligned}\alpha^2&=\frac{\rho\kappa^2c^2}{2(\lambda+2\mu)}\,\frac{1}{\kappa^2c^2+m^2}\{\sqrt{(\kappa^2c^2+am)^2+\kappa^2c^2(a-m)^2}-(\kappa^2c^2+am)\},\\ \beta^2&=\frac{\rho\kappa^2c^2}{2(\lambda+2\mu)}\,\frac{1}{\kappa^2c^2+m^2}\{\sqrt{(\kappa^2c^2+am)^2+\kappa^2c^2(a-m)^2}+(\kappa^2c^2+am)\},\\ \alpha'^2&=\frac{\rho\kappa^2c^2}{2\mu}\,\frac{1}{\kappa^2c^2+l^2}\{\sqrt{(\kappa^2c^2+al)^2+\kappa^2c^2(a-l)^2}-(\kappa^2c^2+al)\},\\ \text{and}\quad \beta'^2&=\frac{\rho\kappa^2c^2}{2\mu}\,\frac{1}{\kappa^2c^2+l^2}\{\sqrt{(\kappa^2c^2+al)^2+\kappa^2c^2(a-l)^2}+(\kappa^2c^2+al)\},\end{aligned}\right\}\cdots\cdots(5.\ 20)$$

respectively,
where

$$a>m=\frac{\lambda'+2\mu'}{\lambda+2\mu'}\ \text{ and }\ a>l=\frac{\mu'}{\mu}.\qquad\cdots\cdots(5.\ 21)$$