Studies on Tsunami on the Pacific Coasts of Northern Honshu.(I.The Model Experiment of Tsunami in Shizukawa Harbour)

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雑誌名	Science reports of the Tohoku University. Ser. 5, Geophysics
巻	2
号	1
ページ	66-78
発行年	1950-03
URL	http://hdl.handle.net/10097/44437

ON VISCO-ELASTIC MEDIUM (PART II)

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(Received October 22, 1949)

Chapter III Equations of Motion and Energy.

§ 1. The equations of motion when the velocity is not very large are

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \\ \rho\frac{\partial^2 v}{\partial t^2} &= \rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}, \\ \rho\frac{\partial^2 w}{\partial t^2} &= \rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}, \end{aligned}\right\} \qquad \cdots\cdots(3.1)$$

and the total work done at the boundary of the elementary volume (W) is

$$\frac{\partial W}{\partial t} = \frac{\partial T}{\partial t} + X_x\frac{\partial e_{xx}}{\partial t} + Y_y\frac{\partial e_{yy}}{\partial t} + Z_z\frac{\partial e_{zz}}{\partial t} + Y_z\frac{\partial e_{yz}}{\partial t} + Z_x\frac{\partial e_{zx}}{\partial t} + X_y\frac{\partial e_{xy}}{\partial t}, \cdots\cdots(3.2)$$

where T is the kinetic energy.

Replace the stress components in the last part of this expression, which expresses the elastic internal energy and the dissipation of energy by strain components, we get

$$\begin{aligned} \left(a + \frac{\partial}{\partial t}\right)E &= \left(a + \frac{\partial}{\partial t}\right)\left(X_x\frac{\partial e_{xx}}{\partial t} + Y_y\frac{\partial e_{yy}}{\partial t} + Z_z\frac{\partial e_{zz}}{\partial t} + Y_z\frac{\partial e_{yz}}{\partial t} + Z_x\frac{\partial e_{zx}}{\partial t} + X_y\frac{\partial e_{xy}}{\partial t}\right) \\ &= \left(\lambda' + \lambda\frac{\partial}{\partial t}\right)(e_{xx}+e_{yy}+e_{zz})^2 + \left(\mu' + \mu\frac{\partial}{\partial t}\right)(2e_{xx}^2+2e_{yy}^2+2e_{zz}^2 + e_{yz}^2 + e_{zx}^2+e_{xy}^2). \cdots\cdots(3.3) \end{aligned}$$

If the strain components are given as functions of t, this equation has the form

$$\left(a + \frac{\partial}{\partial t}\right)E = \left(\lambda' + \lambda\frac{\partial}{\partial t}\right)\theta^2 + \left(\mu' + \mu\frac{\partial}{\partial t}\right)\phi^2, \qquad \cdots\cdots(3.4)$$

where

$$\phi^2 = 2(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + e_{yz}^2 + e_{zx}^2 + e_{xy}^2. \qquad \cdots\cdots(3.5)$$

Integrating this equation by t, reminding that if $\theta = \phi = 0$, E must be zero, we get

$$E = \lambda\theta^2 + \mu\phi^2 + e^{-at}\int\left\{(\lambda'\theta^2 + \mu'\phi^2) - a(\lambda\theta^2 + \mu\phi^2)\right\}e^{at}dt. \qquad \cdots\cdots(3.6)$$

The first two terms of this equation express the elastic internal energy, the last term is, therefore, the dissipation of energy.

§ 2. In equation (3.1), the right hand side is transformed, by replacing the stress components by strain components, in the form

$$\left.\begin{aligned} \left(a + \frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial t^2} &= \left(a + \frac{\partial}{\partial t}\right)X + \frac{1}{\rho}\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}\frac{\partial\theta}{\partial x} + \frac{1}{\rho}\left(\mu' + \mu\frac{\partial}{\partial t}\right)\nabla^2 u, \\ \left(a + \frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial t^2} &= \left(a + \frac{\partial}{\partial t}\right)Y + \frac{1}{\rho}\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}\frac{\partial\theta}{\partial y} + \frac{1}{\rho}\left(\mu' + \mu\frac{\partial}{\partial t}\right)\nabla^2 v, \\ \left(a + \frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial t^2} &= \left(a + \frac{\partial}{\partial t}\right)Z + \frac{1}{\rho}\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}\frac{\partial\theta}{\partial z} + \frac{1}{\rho}\left(\mu' + \mu\frac{\partial}{\partial t}\right)\nabla^2 w. \end{aligned}\right\} (3.7)$$

If the external force is negligible, (3.7) becomes

$$\left(a+\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}(u,v,w) = \frac{1}{\rho}\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\theta+\frac{1}{\rho}\left(\mu'+\mu\frac{\partial}{\partial t}\right)\nabla^2(u,v,w). \quad \cdots\cdots (3.8)$$

If the acceleration is negligible in (3.7) we get

$$\rho\left(a+\frac{\partial}{\partial t}\right)\mathfrak{K}+\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}\operatorname{grad}\theta+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\nabla^2\mathfrak{v}=0, \quad \cdots\cdots (3.9)$$

where $\mathfrak{K}$ is the external force and $\mathfrak{v}$ the displacement.

In equation (3.8), if $\theta=0$, then we get

$$\left(a+\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}(u,\ v,\ w)=\frac{1}{\rho}\left(\mu'+\mu\frac{\partial}{\partial t}\right)\nabla^2(u,\ v,\ w). \quad \cdots\cdots (3.10)$$

Take the divergence of displacement in all terms of (3.8), then we get

$$\left(a+\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}\theta=\frac{1}{\rho}\left\{(\lambda'+2\mu')+(\lambda+2\mu)\frac{\partial}{\partial t}\right\}\nabla^2\theta. \quad \cdots\cdots (3.11)$$

These equations (3.10) and (3.11) are important as the fundamental equations for wave motion.

If $\theta=0$ in equation (3.9), we get

$$\rho\left(a+\frac{\partial}{\partial x}\right)\mathfrak{K}+\nabla^2\mathfrak{v}=0. \quad \cdots\cdots (3.12)$$

Chapter IV Transmission of Force.

§ 1. Put

$$\mathfrak{v}=\operatorname{grad}\phi+\operatorname{rot}\mathbf{A}, \quad \cdots\cdots (4.1)$$

where $\mathfrak{v}$ is the displacement potential, ϕ the scalar quantity (scalar potential) and $\operatorname{div}\mathbf{A}=0$ ($\mathbf{A}$ the vector potential), then we get

$$\theta=\nabla^2\phi,\quad \operatorname{rot}\mathfrak{v}=-\nabla^2\mathbf{A}. \quad \cdots\cdots (4.2)$$

In like manner, we express the extenal force $\mathfrak{K}$ by

$$\mathfrak{K}=\operatorname{grad}\Phi+\operatorname{rot}\mathbf{B}. \quad \cdots\cdots (4.3)$$

Then equation (3.9) is transformed in

$$\left(a+\frac{\partial}{\partial t}\right)\rho(\operatorname{grad}\Phi+\operatorname{rot}\mathbf{B})+\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}\operatorname{grad}\nabla^2\phi$$
$$+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\nabla^2(\operatorname{grad}\phi+\operatorname{rot}\mathbf{A})=0,$$

or

$$\left(a+\frac{\partial}{\partial t}\right)\rho(\operatorname{grad}\Phi+\operatorname{rot}\mathbf{B})+\left\{(\lambda'+2\mu')+(\lambda+2\mu)\frac{\partial}{\partial t}\right\}\operatorname{grad}\nabla^2\phi$$
$$+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\operatorname{rot}\nabla^2\mathbf{A}=0. \quad \cdots\cdots (4.4)$$

The particular solutions of this equation can be obtained by writing down the particular solutions of equations

$$\left.\begin{aligned}&\left(a+\frac{\partial}{\partial t}\right)\rho\Phi+\left\{(\lambda'+2\mu')+(\lambda+2\mu)\frac{\partial}{\partial t}\right\}\nabla^2\phi=0,\\&\left(a+\frac{\partial}{\partial t}\right)\rho\mathbf{B}+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\nabla^2\mathbf{A}=0.\end{aligned}\right\} \quad \cdots\cdots (4.5)$$

Now $\mathfrak{K}$ can be expressed in forms of type (4.3) by putting

$$\left.\begin{aligned}
&\Phi = -\frac{1}{4\pi}\iiint\left\{X'\frac{\partial r^{-1}}{\partial x} + Y'\frac{\partial r^{-1}}{\partial y} + Z'\frac{\partial r^{-1}}{\partial z}\right\}dx'dy'dz' \\
\text{or}\quad &\Phi = -\frac{1}{4\pi}\iiint\left(\mathfrak{K}'\,\mathrm{grad}\frac{1}{r}\right)dv, \\
&L = \frac{1}{4\pi}\iiint\left(Z'\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - Y'\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right)dx'dy'dz', \\
&M = \frac{1}{4\pi}\iiint\left(X'\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - Z'\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right)dx'dy'dz', \\
&N = \frac{1}{4\pi}\iiint\left(Y'\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - X'\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right)dx'dy'dz', \\
\text{or}\quad &\mathbf{A} = \frac{1}{4\pi}\iiint\left(\mathfrak{K}'\,\mathrm{grad}\frac{1}{r}\right)dv,
\end{aligned}\right\} \quad \cdots\cdots (4.6)$$

where $\mathfrak{K}'(X', Y', Z')$ denotes the external force at any point x', y', z' within a volume T, r is the distance of this point from x, y, z and the integration extends through T.

We now pass to a limit by diminishing all the linear dimensions of T indefinitely, but supposing that $\iiint X'\,dx'dy'dz'$ has a finite value.

Put

$$\rho\iiint X'dx'dy'dz' = X_0, \quad \cdots\cdots (4.7)$$

then we have

$$\Phi = -\frac{1}{4\pi\rho}X_0\frac{\partial}{\partial x}\left(\frac{1}{r}\right),\quad L = 0,\quad M = \frac{1}{4\pi\rho}X_0\frac{\partial}{\partial z}\left(\frac{1}{r}\right), N = -\frac{1}{4\pi\rho}X_0\frac{\partial}{\partial y}\left(\frac{1}{r}\right). \cdots (4.8)$$

Putting these in eqaution (4.5), we get

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\nabla^2\phi + \frac{\lambda' + 2\mu'}{\lambda + 2\mu}\nabla^2\phi = \frac{\nabla^2\left(\frac{\partial r}{\partial x}\right)}{8\pi(\lambda + 2\mu)}\left(a + \frac{\partial}{\partial t}\right)X_0, \\
&\frac{\partial}{\partial t}\nabla^2 F + \frac{\mu'}{\mu}\nabla^2 F = 0, \\
&\frac{\partial}{\partial t}\nabla^2 G + \frac{\mu'}{\mu}\nabla^2 G = -\frac{1}{8\pi\mu}\nabla^2\left(\frac{\partial r}{\partial z}\right)\left(a + \frac{\partial}{\partial t}\right)X_0, \\
&\frac{\partial}{\partial t}\nabla^2 H + \frac{\mu'}{\mu}\nabla^2 H = \frac{1}{8\pi\mu}\nabla^2\left(\frac{\partial r}{\partial y}\right)\left(a + \frac{\partial}{\partial t}\right)X_0,
\end{aligned}\right\} \quad \cdots\cdots (4.9)$$

where F, G, H are the components of the vector potential $\mathbf{A}$.

The solutions of these equations are

$$\left.\begin{aligned}
&\nabla^2\phi = C_1 e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} + e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int\left\{\frac{\nabla^2\left(\frac{\partial r}{\partial x}\right)}{8\pi(\lambda + 2\mu)}\left(a + \frac{\partial}{\partial t}\right)X_0 e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\right\}dt, \\
&\nabla^2 F = C_2 e^{-\frac{\mu'}{\mu}t}, \\
&\nabla^2 G = C_3 e^{-\frac{\mu'}{\mu}t} - e^{-\frac{\mu'}{\mu}t}\int\frac{\nabla^2\left(\frac{\partial r}{\partial z}\right)}{8\pi\mu}\left(a + \frac{\partial}{\partial t}\right)X_0 e^{\frac{\mu'}{\mu}t}\,dt, \\
&\nabla^2 H = C_4 e^{-\frac{\mu'}{\mu}t} + e^{-\frac{\mu'}{\mu}t}\int\frac{\nabla^2\left(\frac{\partial r}{\partial z}\right)}{8\pi\mu}\left(a + \frac{\partial}{\partial t}\right)X_0 e^{\frac{\mu'}{\mu}t}\,dt.
\end{aligned}\right\} \quad \cdots\cdots (4.10)$$

Put $C_1 = C_2 = C_3 = C_4 = 0$, then we get

$$\left.\begin{aligned}
\phi &= \frac{\frac{\partial r}{\partial x}}{8\pi(\lambda+2\mu)} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\, dt\,, \\
F &= 0\,, \\
G &= -\frac{\frac{\partial r}{\partial z}}{8\pi\mu} e^{-\frac{\mu'}{\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\mu'}{\mu}t}\, dt\,, \\
H &= -\frac{\frac{\partial r}{\partial y}}{4\pi\mu} e^{-\frac{\mu'}{\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\mu'}{\mu}t}\, dt\,.
\end{aligned}\right\} \quad \cdots\cdots(4.11)$$

The corresponding forms of u, v, w, the components of displacement are

$$\left.\begin{aligned}
u &= \frac{\partial^2 r}{\partial x^2}\left\{\frac{1}{8\pi(\lambda+2\mu)} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\, dt \right.\\
&\qquad \left. -\frac{1}{8\pi\mu} e^{-\frac{\mu'}{\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\mu'}{\mu}t}\, dt\right\} \\
&\qquad\qquad + \frac{1}{4\pi\mu}\,\frac{1}{r} e^{-\frac{\mu'}{\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\mu'}{\mu}t}\, dt\,, \\
v &= \frac{1}{8\pi}\,\frac{\partial^2 r}{\partial x \partial y}\left\{\frac{1}{\lambda+2\mu} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t} dt \right.\\
&\qquad\qquad \left. -\frac{1}{\mu} e^{-\frac{\mu'}{\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\mu'}{\mu}t} dt\right\}, \\
w &= \frac{1}{8\pi}\,\frac{\partial^2 r}{\partial x \partial z}\left\{\frac{1}{\lambda+2\mu} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t} dt \right.\\
&\qquad\qquad \left. -\frac{1}{\mu} e^{-\frac{\mu'}{\mu}t} \int \left(a+\frac{\partial}{\partial t}\right) X_0 e^{\frac{\mu'}{\mu}t} dt\right\}.
\end{aligned}\right\} \quad \cdots\cdots(4.12)$$

Hence we get

$$\left.\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial^3 r}{\partial x^3}\left\{\frac{1}{8\pi(\lambda+2\mu)} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt \right.\\
&\qquad \left. -\frac{1}{8\pi\mu} e^{-\frac{\mu'}{\mu}t} \int e^{\frac{\mu'}{\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt\right\} \\
&\qquad\qquad + \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\frac{1}{4\pi\mu} e^{-\frac{\mu'}{\mu}t} \int e^{\frac{\mu'}{\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt\,, \\
e_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial^3 r}{\partial x \partial y^2}\left\{\frac{1}{8\pi(\lambda+2\mu)} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt \right.\\
&\qquad\qquad \left. -\frac{1}{8\pi\mu} e^{-\frac{\mu'}{\mu}t} \int e^{\frac{\mu'}{\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt\right.\,, \\
e_{zz} &= \frac{\partial w}{\partial z} = \frac{\partial^3 r}{\partial x \partial z^2}\left\{\frac{1}{8\pi(\lambda+2\mu)} e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt \right.\\
&\qquad\qquad \left. -\frac{1}{8\pi\mu} e^{\frac{\mu'}{\mu}} \int e^{\frac{\mu'}{\mu}t} \left(a+\frac{\partial}{\partial t}\right) X_0 dt\right\},
\end{aligned}\right\} \quad \cdots\cdots(4.13)$$

$$
\left.\begin{aligned}
e_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{1}{4\pi}\frac{\partial^3 r}{\partial x\partial y\partial z}\left\{\frac{1}{\lambda+2\mu}e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt\right.\\
&\qquad\left. -\frac{1}{\mu}e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt\right\},\\
e_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{4\pi}\frac{\partial^3 r}{\partial z\partial x^2}\left\{\frac{1}{\lambda+2\mu}e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt\right.\\
&\quad\left. -\frac{1}{\mu}e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt\right\} + \frac{1}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt,\\
e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{4\pi}\frac{\partial^3 r}{\partial y\partial x^2}\left\{\frac{1}{\lambda+2\mu}e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt\right.\\
&\quad\left. -\frac{1}{\mu}e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt\right\} + \frac{1}{4\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt.
\end{aligned}\right\}
$$

And

$$
\left.\begin{aligned}
\theta &= \frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt,\\
2\widetilde{\omega}_x &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0,\\
2\widetilde{\omega}_y &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{1}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt,\\
2\widetilde{\omega}_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{1}{4\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)X_0dt.
\end{aligned}\right\} \quad\cdots\cdots\cdots(4.14)
$$

The stress components corresponding to these strain components are

$$
\left.\begin{aligned}
\left(a+\frac{\partial}{\partial t}\right)X_x &= \left\{\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)+\left(\mu+\mu'\frac{\partial}{\partial t}\right)\right\}\Phi(m)\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\\
&\quad -\left(\mu+\mu'\frac{\partial}{\partial t}\right)\Phi(m)\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\left(\frac{x^2}{r^3}\right)+\frac{1}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{r}+\frac{x^2}{r^3}\right)\left(\mu+\mu'\frac{\partial}{\partial t}\right)\Phi(l),\\
\left(a+\frac{\partial}{\partial t}\right)Y_y &= \left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\Phi(m)\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\left(\mu+\mu'\frac{\partial}{\partial t}\right)\left\{\frac{\Phi(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi(l)}{4\pi\mu}\right\}\frac{\partial^3 r}{\partial x\partial y^2},\\
\left(a+\frac{\partial}{\partial t}\right)Z_z &= \left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\Phi(m)\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\left(\mu+\mu'\frac{\partial}{\partial t}\right)\left\{\frac{\Phi(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi(l)}{4\pi\mu}\right\}\frac{\partial^3 r}{\partial x\partial z^2},\\
\left(a+\frac{\partial}{\partial t}\right)Y_z &= \left(\mu+\mu'\frac{\partial}{\partial t}\right)\left(\frac{\Phi(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi(l)}{4\pi\mu}\right)\frac{\partial^3 r}{\partial x\partial y\partial z},\\
\left(a+\frac{\partial}{\partial t}\right)Z_x &= \left(\mu+\mu'\frac{\partial}{\partial t}\right)\left\{\left(\frac{\Phi(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi(l)}{4\pi\mu}\right)\frac{\partial^3 r}{\partial z\partial x^2}+\frac{\Phi(l)}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\},\\
\left(a+\frac{\partial}{\partial t}\right)X_y &= \left(\mu+\mu'\frac{\partial}{\partial t}\right)\left\{\left(\frac{\Phi(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi(l)}{4\pi\mu}\right)\frac{\partial^3 r}{\partial y\partial x^2}+\frac{\Phi(l)}{4\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right\},
\end{aligned}\right\}
$$

$$\cdots\cdots\cdots(4.15)$$

where

$$
\left.\begin{aligned}
\Phi(\xi) &= e^{-\xi t}\int e^{\xi t}\left(a+\frac{\partial}{\partial t}\right)X_0dt,\\
m &= \frac{\lambda'+2\mu'}{\lambda+2\mu}, \quad l = \frac{\mu'}{\mu}.
\end{aligned}\right\} \quad\cdots\cdots\cdots(4.16)
$$

and

§ 2. Let a force P/h be applied at the origin in the direction of the axis of x and let an equal and opposite force be applied at $(h, 0, 0)$ and let be h diminished indefinitely while P remains constant. The displacement is

$$\left(P\frac{\partial u}{\partial x},\quad P\frac{\partial v}{\partial x},\quad P\frac{\partial w}{\partial x}\right). \qquad \cdots\cdots(4.17)$$

Such a force are described a double force without moment.

If three double forcs without moment be applied at origin with their axes parallel to the axes of co-ordinates and specified by the same quantity P, the resultant displacement is

$$P\left\{\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right),\quad \left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right),\quad \left(\frac{\partial w_1}{\partial x}+\frac{\partial w_2}{\partial y}+\frac{\partial w_3}{\partial z}\right)\right\}.$$

In (4. 18), (4. 19) and (4. 20)

$$\begin{matrix} Pu_1, & Pv_1, & Pw_1, \\ Pu_2, & Pv_2, & Pw_2, \\ Pu_3, & Pv_3, & Pw_3, \end{matrix}$$

are the displaements due to the forces in the directions of x, y and z.

$$\left.\begin{aligned} Pu_1 &= \frac{\partial^2 r}{\partial x^2}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}+\frac{1}{r}\,\frac{1}{4\pi\mu}\Phi_p(l), \\ Pv_1 &= \frac{\partial^2 r}{\partial x\partial y}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}, \\ Pw_1 &= \frac{\partial^2 r}{\partial z\partial x}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}, \end{aligned}\right\} \cdots\cdots(4.18)$$

$$\left.\begin{aligned} Pu_2 &= \frac{\partial^2 r}{\partial x\partial y}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}, \\ Pv_2 &= \frac{\partial^2 r}{\partial y^2}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}+\frac{1}{r}\,\frac{1}{4\pi\mu}\Phi_p(l), \\ Pw_2 &= \frac{\partial^2 r}{\partial y\partial z}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}, \end{aligned}\right\} \cdots\cdots(4.19)$$

$$\left.\begin{aligned} Pu_3 &= \frac{\partial^2 r}{\partial z\partial x}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}, \\ Pv_3 &= \frac{\partial^2 r}{\partial y\partial z}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}, \\ Pw_3 &= \frac{\partial^2 r}{\partial z^2}\left\{\frac{1}{8\pi(\lambda+2\mu)}\Phi_p(m)-\frac{1}{8\pi\mu}\ \Phi_p(l)\right\}-\frac{1}{r}\,\frac{1}{4\pi\mu}\Phi_p(l), \end{aligned}\right\} \cdots\cdots(4.20)$$

where

$$\Phi_p(l)=e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)Pdt,$$

and

$$\Phi_p(m)=e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\left(a+\frac{\partial}{\partial t}\right)Pdt. \qquad \cdots\cdots(4.21)$$

Thus the resultant displacement is

$$u=\frac{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m),\ v=\frac{\frac{\partial}{\partial y}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m),\ w=\frac{\frac{\partial}{\partial z}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m), \qquad \cdots\cdots(4.22)$$

and the strain components are

$$\left.\begin{aligned} &\theta=\frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\nabla^2\frac{1}{r}=0, \\ &e_{xx}=\frac{\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m),\quad e_{yy}=\frac{\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m),\quad e_{zz}=\frac{\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m), \end{aligned}\right\} \cdots(4.23)$$

$$e_{yz}=\frac{\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right)}{2\pi(\lambda+2\mu)}\Phi_p(m),\ e_{zx}=\frac{\frac{\partial^2}{\partial z\partial x}\left(\frac{1}{r}\right)}{2\pi(\lambda+2\mu)}\Phi_p(m),\ e_{xy}=\frac{\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)}{2\pi(\lambda+2\mu)}\Phi_p(m),$$
$$2\widetilde{\omega}_x=2\widetilde{\omega}_y=2\omega_z=0.$$

Hence the stress components are given by

$$\left.\begin{aligned}
\left(a+\frac{\partial}{\partial t}\right)X_x&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{1}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\Phi_p(m),\\
\left(a+\frac{\partial}{\partial t}\right)Y_y&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{1}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right)\Phi_p(m),\\
\left(a+\frac{\partial}{\partial t}\right)Z_z&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{1}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\Phi_p(m),\\
\left(a+\frac{\partial}{\partial t}\right)Y_z&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{1}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right)\Phi_p(m),\\
\left(a+\frac{\partial}{\partial t}\right)Z_x&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{1}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial z\partial x}\left(\frac{1}{r}\right)\Phi_p(m),\\
\left(a+\frac{\partial}{\partial t}\right)X_y&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{1}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)\Phi_p(m).
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(4.24)$$

The stress components on a spherical surface with its centre at origin are, thus, given by

$$\left(a+\frac{\partial}{\partial t}\right)X_r=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\left\{\frac{\partial r}{\partial x}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial r}{\partial y}\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)+\frac{\partial r}{\partial z}\frac{\partial^2}{\partial z\partial x}\left(\frac{1}{r}\right)\right\},$$

$$\left(a+\frac{\partial}{\partial t}\right)Y_r=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\left\{\frac{\partial r}{\partial x}\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)+\frac{\partial r}{\partial y}\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right)+\frac{\partial r}{\partial z}\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right)\right\},$$

$$\left(a+\frac{\partial}{\partial t}\right)Z_r=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\left\{\frac{\partial r}{\partial x}\frac{\partial^2}{\partial z\partial x}\left(\frac{1}{r}\right)+\frac{\partial r}{\partial y}\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right)+\frac{\partial r}{\partial z}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\right\},$$

or

$$\left.\begin{aligned}
\left(a+\frac{\partial}{\partial t}\right)X_r&=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{\pi(\lambda+2\mu)}\frac{x}{r^4},\\
\left(a+\frac{\partial}{\partial t}\right)Y_r&=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{\pi(\lambda+2\mu)}\frac{y}{r^4},\\
\left(a+\frac{\partial}{\partial t}\right)Z_r&=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{\pi(\lambda+2\mu)}\frac{z}{r^4}.
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(4.25)$$

Therefore the stress component is perpendicular to the spherical surface with its centre at origin and its magnitude is given by

$$\left(a+\frac{\partial}{\partial t}\right)R_r=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{\pi(\lambda+2\mu)r^3}.\quad\cdots\cdots\cdots\cdots\cdots\cdots(4.26)$$

§ 3. We suppose the centres of dilatation as discussed in § 2 are distributed along the axis of z from origin to $z=-\infty$. The displacement is given by integrating the equation (4.22) from $z=0$ to $z=-\infty$. The result is

$$u = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\, x\int_0^\infty \frac{dz'}{R^3} = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\, \frac{x}{r(z+r)},$$

$$v = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\, y\int_0^\infty \frac{dz'}{R^3} = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\, \frac{y}{r(z+r)},$$

$$w = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)} \int_0^\infty \frac{z+z'}{R^3}dz' = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\, \frac{1}{r},$$

where $R^2 = x^2+y^2+(z+z')^2$, or

$$\left.\begin{aligned} u &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\log(z+r),\\ v &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial y}\log(z+r),\\ w &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial z}\log(z+r). \end{aligned}\right\} \qquad \cdots\cdots\cdots\cdots(4.27)$$

For this displacement, the components of strain are as follows:

$$\left.\begin{aligned} e_{xx} &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial^2}{\partial x^2}\log(z+r) = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\left\{\frac{y^2+z^2}{r^3(z+r)} - \frac{x^2}{r^2(z+r)^2}\right\},\\ e_{yy} &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial^2}{\partial y^2}\log(z+r) = \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\left\{\frac{z^2+x^2}{r^3(z+r)} - \frac{y^2}{r^2(z+r)^2}\right\},\\ e_{zz} &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial^2}{\partial z^2}\log(z+r) = -\frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{z}{r^3},\\ e_{yz} &= \frac{\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial y\partial z}\log(z+r) = -\frac{\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{y}{r^3},\\ e_{zx} &= \frac{\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial z\partial x}\log(z+r) = -\frac{\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{x}{r^3},\\ e_{xy} &= \frac{\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial x\partial y}\log(z+r) = -\frac{\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{xy(z+2r)}{r^3(z+r)^2},\\ \theta &= 0,\\ 2\widetilde{\omega}_x &= 2\widetilde{\omega}_y = 2\widetilde{\omega}_z = 0. \end{aligned}\right\} \qquad \cdots(4.28)$$

And the stress components are given by

$$\begin{aligned} \left(a+\frac{\partial}{\partial t}\right)X_x &= \frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial x^2}\log(z+r) = \frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\left\{\frac{y^2+z^2}{r^3(z+r)} - \frac{x^2}{r^2(z+r)^2}\right\}\\ \left(a+\frac{\partial}{\partial t}\right)Y_y &= \frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial y^2}\log(z+r) = \frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\left\{\frac{z^2+x^2}{r^3(z+r)} - \frac{y^2}{r^2(z+r)^2}\right\}\\ \left(a+\frac{\partial}{\partial t}\right)Z_z &= \frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial z^2}\log(z+r) = -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{z}{r^3},\\ \left(a+\frac{\partial}{\partial t}\right)Y_z &= -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial y\partial z}\log(z+r) = -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{y}{r^3},\\ \left(a+\frac{\partial}{\partial t}\right)Z_x &= -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial z\partial x}\log(z+r) = -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{x}{r^3}, \end{aligned}$$

$$\left(a+\frac{\partial}{\partial t}\right)X_y=-\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial x\partial y}\log(z+r)=-\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{xy(z+2r)}{r^3(z+r)^2}. \qquad \cdots\cdots(4.29)$$

At a surface of a hemisphere, for which r is constant and z is positive, these give rise to tractions

$$\left.\begin{aligned}\left(a+\frac{\partial}{\partial t}\right)X_r&=-\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{x}{r^2(z+r)},\\ \left(a+\frac{\partial}{\partial t}\right)Y_r&=-\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{y}{r^2(z+r)},\\ \left(a+\frac{\partial}{\partial t}\right)Z_r&=-\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{1}{r^2}.\end{aligned}\right\} \qquad \cdots\cdots(4.30)$$

Or the spherical surface is subject to a traction whose magnitude is

$$\left(a+\frac{\partial}{\partial t}\right)R_r=-\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{1}{r(z+r)}. \qquad \cdots\cdots(4.31)$$

§ 4. We consider a visco-elastic body to which forces are applied in the neighbourhood of a single point on the surface.

We take the origin to be the point at which the load is applied, the plane $z=0$ to be the boundary surface of the body, and the positive direction of the axis of z to be that which goes into the interior of the body.

The local effect of force applied at the origin being very great, we suppose the origin to be excluded by a hemispherical surface.

The displacement in visco-elastic body subjected by a force applied at the origin parallel to the axis of z can be easily written down from (4.13),

$$\left.\begin{aligned}u&=\frac{1}{8\pi}\frac{\partial^2 r}{\partial z\partial x}\left\{\frac{1}{\lambda+2\mu}\psi(m)-\frac{1}{\mu}\psi(l)\right\},\\ v&=\frac{1}{8\pi}\frac{\partial^2 r}{\partial y\partial z}\left\{\frac{1}{\lambda+2\mu}\psi(m)-\frac{1}{\mu}\psi(l)\right\},\\ w&=\frac{1}{8\pi}\frac{\partial^2 r}{\partial z^2}\left\{\frac{1}{\lambda+2\mu}\psi(m)-\frac{1}{\mu}\psi(l)\right\}+\frac{1}{4\pi\mu}\frac{1}{r}\psi(l),\end{aligned}\right\} \qquad \cdots\cdots(4.32)$$

where

$$\psi(m)=e^{-\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\int e^{\frac{\lambda'+2\mu'}{\lambda+2\mu}t}\left(a+\frac{\partial}{\partial t}\right)Z_0dt,$$

and

$$\psi(l)=e^{-\frac{\mu'}{\mu}t}\int e^{\frac{\mu'}{\mu}t}\left(a+\frac{\partial}{\partial t}\right)Z_0dt. \qquad \cdots\cdots(4.33)$$

The corresponding stress in the body being given by, as (4.15)

$$\left(a+\frac{\partial}{\partial t}\right)X_x=\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)\psi(m)\frac{\frac{\partial}{\partial z}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\left\{\frac{\psi(m)}{4\pi(\lambda+2\mu)}-\frac{\psi(l)}{4\pi\mu}\right\}\frac{\partial^3 r}{\partial z\partial x^2},$$

$$\left(a+\frac{\partial}{\partial t}\right)Y_y=\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)\psi(m)\frac{\frac{\partial}{\partial z}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)}+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\left\{\frac{\psi(m)}{4\pi(\lambda+2\mu)}-\frac{\psi(l)}{4\pi\mu}\right\}\frac{\partial^3 r}{\partial z\partial y^2},$$

$$
\left.\begin{aligned}
\left(a+\frac{\partial}{\partial t}\right)Z_z &= \left\{\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)+\left(\mu'+\mu\frac{\partial}{\partial t}\right)\right\}\psi(m)\frac{\frac{\partial}{\partial z}\left(\frac{1}{r}\right)}{4\pi(\lambda+2\mu)} \\
&\quad -\left(\mu'+\mu\frac{\partial}{\partial t}\right)\psi(m)\frac{\frac{\partial}{\partial z}\left(\frac{z^2}{r^3}\right)}{4\pi(\lambda+2\mu)}+\frac{1}{4\pi\mu}\frac{\partial}{\partial z}\left(\frac{1}{r}+\frac{z^2}{r^3}\right)\left(\mu'+\mu\frac{\partial}{\partial t}\right)\psi(l), \\
\left(a+\frac{\partial}{\partial t}\right)Y_z &= \left(\mu'+\mu\frac{\partial}{\partial t}\right)\left\{\left(\frac{\psi(m)}{4\pi(\lambda+2\mu)}-\frac{\psi(l)}{4\pi\mu}\right)\frac{\partial^3 r}{\partial y\partial z^2}+\frac{\psi(l)}{4\pi\mu}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right\}, \\
\left(a+\frac{\partial}{\partial t}\right)Z_x &= \left(\mu'+\mu\frac{\partial}{\partial t}\right)\left\{\left(\frac{\psi(m)}{4\pi(\lambda+2\mu)}-\frac{\psi(l)}{4\pi\mu}\right)\frac{\partial^3 r}{\partial x\partial z^2}+\frac{\psi(l)}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right\}, \\
\left(a+\frac{\partial}{\partial t}\right)X_y &= \left(\mu'+\mu\frac{\partial}{\partial t}\right)\left(\frac{\psi(m)}{4\pi(\lambda+2\mu)}-\frac{\psi(l)}{4\pi\mu}\right)\frac{\partial^3 r}{\partial x\partial y\partial z}.
\end{aligned}\right\}
\qquad \cdots\cdots(4.34)
$$

The displacement given by (4.32) can be maintained in the body by tractions over the plane boudary, which are expressed by the equations

$$
\left.\begin{aligned}
\left[\left(a+\frac{\partial}{\partial t}\right)X_z\right]_{z=0} &= -\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{\psi(m)}{4\pi(\lambda+2\mu)}\frac{x}{r^3}, \\
\left[\left(a+\frac{\partial}{\partial t}\right)Y_z\right]_{z=0} &= -\left(\mu'+\mu\frac{\partial}{\partial t}\right)\frac{\psi(m)}{4\pi(\lambda+2\mu)}\frac{y}{r^3}, \\
\left[\left(a+\frac{\partial}{\partial t}\right)Z_z\right]_{z=0} &= 0.
\end{aligned}\right\}
\qquad \cdots\cdots(4.35)
$$

The displacement expressed by (4.27) could be maitained in the body by the tractions over the plane boundary, which are expressed by

$$
\left.\begin{aligned}
\left(a+\frac{\partial}{\partial t}\right)X_z &= -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{x}{r^3}, \\
\left(a+\frac{\partial}{\partial t}\right)Y_z &= -\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{2\pi(\lambda+2\mu)}\frac{y}{r^3}, \\
\left(a+\frac{\partial}{\partial t}\right)Z_z &= 0,
\end{aligned}\right\}
\qquad \cdots\cdots(4.36)
$$

and by traction over the hemispherical boundary, which are expressed by equations(4.30).

The resultant of the latter is a force

$$
F=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{\lambda+2\mu} \qquad \cdots\cdots(4.37)
$$

in the direction of axis of z.

If put $\psi(m)=-2\Phi_p(m)$, then the state of displacement expressed by the sum of the displacements (4.27) and (4.32) will be maintained by forces applied to the hemispherical surface only and the displacement is given by the equations

$$
\left.\begin{aligned}
u &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\log(z+r)-\frac{\partial^2 r}{\partial z\partial x}\left(\frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi_p(l)}{4\pi\mu}\right), \\
v &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial y}\log(z+r)-\frac{\partial^2 r}{\partial y\partial z}\left(\frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi_p(l)}{4\pi\mu}\right), \\
w &= \frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial z}\log(z+r)-\frac{\partial^2 r}{\partial z^2}\left(\frac{\Phi_p(m)}{4\pi(\lambda+2\mu)}-\frac{\Phi_p(l)}{4\pi\mu}\right)-\frac{1}{2\pi r}\Phi_p(l).
\end{aligned}\right\}
\qquad \cdots(4.38)
$$

Or the displacement expressed by the quations

$$\left.\begin{aligned} u &= \frac{\frac{\partial}{\partial x}\left(\log(z+r)-\frac{\partial r}{\partial z}\right)}{4\pi(\gamma+2\mu)}\Phi_p(m)+\frac{\frac{\partial^2 r}{\partial z\partial x}}{4\pi\mu}\Phi_p(l)\,,\\ v &= \frac{\frac{\partial}{\partial y}\left(\log(z+r)-\frac{\partial r}{\partial z}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m)+\frac{\frac{\partial^2 r}{\partial z\partial x}}{4\pi\mu}\Phi_p(l)\,,\\ w &= \frac{\frac{\partial}{\partial z}\left(\log(z+r)-\frac{\partial r}{\partial z}\right)}{4\pi(\lambda+2\mu)}\Phi_p(m)+\frac{\frac{\partial^2 r}{\partial z^2}-\frac{2}{r}}{4\pi\mu}\Phi_p(l)\,, \end{aligned}\right\} \quad\cdots\cdots(4.39)$$

are maintained by a force

$$F=\frac{\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)}{\lambda+2\mu}. \quad\cdots\cdots(4.40)$$

These are the generalized forms of BOUSSINESQ's equations.

§ 5. As an example, we put

$$P=C+Dt \quad\text{and}\quad F=A+Bt, \quad\cdots\cdots(4.41)$$

then we get

$$\left.\begin{aligned} &\left(a+\frac{\partial}{\partial t}\right)P=(aC+D)+aDt\,,\\ &\Phi_p(m)=\left(\frac{aC+D}{m}-\frac{aD}{m^2}\right)+\frac{aD}{m}t\,,\\ &\left(\mu'+\mu\frac{\partial}{\partial t}\right)\Phi_p(m)=\mu'\left(\frac{aC+D}{m}-\frac{aD}{m^2}\right)+\mu'\frac{aD}{m}t+\mu\frac{aD}{m}\,. \end{aligned}\right\} \quad\cdots\cdots(4.42)$$

Therefore we get the relations

$$\left.\begin{aligned} (\lambda+2\mu)A &= \mu'\left(\frac{aC+D}{m}-\frac{aD}{m^2}\right)+\mu\frac{aD}{m}\,,\\ C &= (\lambda+2\mu)A\left\{\frac{\mu'(a-1)-\mu a}{\mu'^2a^2}m+\frac{1}{\mu' a}\right\}\\ &= \left\{(\lambda'+2\mu')\frac{\mu'(a-1)-\mu a}{\mu'^2a^2}+\frac{\lambda+2\mu}{\mu' a}\right\}A, \end{aligned}\right\} \quad\cdots\cdots(4.43)$$

which give

$$\left.\begin{aligned} D &= \frac{\lambda+2\mu}{\mu' a}Bm=\frac{\lambda'+2\mu'}{\mu' a}B\,.\\ C &= (\lambda+2\mu)A\left\{\frac{\mu'(a-1)-\mu a}{\mu'^2a^2}m+\frac{1}{\mu' a}\right\}\\ &= \left\{(\lambda'+2\mu')\frac{\mu'(a-1)-\mu a}{\mu'^2a^2}+\frac{\lambda+2\mu}{\mu' a}\right\}A\,,\\ \Phi_p(m) &= \frac{\lambda+2\mu}{\mu' a}\left[\left\{(a-1)-\frac{\mu}{\mu'}a\right\}A+B\right]+\frac{(\lambda+2\mu)^2}{\mu'\mu(\lambda'+2\mu')}(A-B)+\frac{\lambda+2\mu}{\mu'}Bt. \end{aligned}\right\} \cdots(4.44)$$

Similarly putting l instead of m in $\Phi_p(m)$, we get

$$\Phi_p(l)=\frac{\mu}{\mu'^2}\left\{(\lambda'+2\mu')\frac{a-1}{a}+(\lambda+2\mu)\right\}A-\frac{(\lambda'+2'\mu)\mu^2}{\mu'^2}(A+B)+\frac{\lambda'+2\mu'}{\mu'^2}\mu Bt. \cdots(4.45)$$

§ 6. Most important case of (4.39) in geophysical problems is to estimate the value of w on the plane boundary or at $z=0$. We get

$$w_{z=0}=-\frac{1}{4\pi\mu r}\Phi_p(l)\,.$$

If the body subject to a distributed tractions, P is a function of x', y' and t and we have

$$w_{z=0}=-\frac{1}{4\pi r}\iint\frac{\Phi_p(l)}{r}\,dx'dy',\qquad\cdots\cdots(4.46)$$

where the integration is extented over the all area in which the tractions are distributed.

If, for example, the body subjects to constant traction over a circular area of radius R with its centre at origin, F and accordingly P are constant over the area and zero outside of it. We get, therefore, in this case

$$w_{z=0}=-\frac{\Phi_p(l)}{4\pi\mu}\int_0^R\int_0^{2\pi}\frac{\rho d\theta d\rho}{(r^2+\rho^2-2r\rho\cos\theta)^{\frac{1}{2}}}.\qquad\cdots\cdots(4.47)$$

This can be evaluated by using zonal harmonics, and we get

$$w_{z=0}=-\frac{\Phi_p(l)}{2\pi\mu R}\left\{\frac{1}{2}\frac{R}{r}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\frac{R^3}{r^3}+\frac{3}{8}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\frac{R^5}{r^5}+\frac{5}{16}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}\frac{R^7}{r^7}+\cdots\cdots\right\}\quad\text{for } r>R,$$

and

$$w_{z=0}=-\frac{\Phi_p(l)}{2\pi\mu R}\left\{1-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{r^2}{R^2}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{8}\frac{r^4}{R^4}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{16}\frac{r^6}{R^6}-\cdots\cdots\right\}\quad\text{for } r<R.$$