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THE VERTICAL TRANSFER OF HEAT AND THE CHANGE OF AIR TEMPERATURE BY TURBULENCE

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§ 1. Introduction.

In 1915 G. I. TAYLOR obtained a formula expressing the rate of the vertical transfer of heat by turbulence. In obtaining this he assumed that there is no resultant transfer of mass across any horizontal plane. However when the heat is transferred vertically and the vertical distribution of temperature changes, the change of the vertical distribution of pressure will be followed. This means that the mass is transferred vertically and so TAYLOR's assumption is not correct. We obtained an equation which expresses the rate of the vertical transfer of heat and the change of the temperature of the air taking the transfer of mass into consideration and compared this with results obtained by observations.

§ 2. The Vertical Transfer of Heat and the Change of Air Temperature.

Let us first obtain the rate of the vertical flow of heat across any horizontal plane. Let an eddy start from z_0 at time t_0, having the potential temperature $\theta\ (z_0, t_0)$ appropriate to that point, and let it reach a level z, at time t. The absolute temperature of the eddy at its new level will be $\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}} \times \theta\ (z_0, t_0)$. At each stage the eddy takes up the pressure of its surroundings, and in mixing it shares with its surroundings its excess or defect of thermal energy. The motion of the eddy across the horizontal plane at z is therefore equivalent to a flow of heat

$$I = \Sigma\, \rho c_p w \left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}} \theta\,(z_0, t_0), \quad \cdots\cdots(1)$$

where w is the vertical component of the turbulent velocity of the air and Σ is the summation over all eddies which reach the new level in unit time. Provided $z_0 - z$ and $t_0 - t$ are both small

$$\theta\,(z_0, t_0) = \theta\,(z, t) + (z_0 - z)\frac{\partial\theta}{\partial z} + (t_0 - t)\frac{\partial\theta}{\partial t}\ , \quad \cdots\cdots\cdots\cdots(2)$$

and so

$$I = \rho c_p \left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}} \left\{\theta\,(z, t)\,\Sigma w + \frac{\partial\theta}{\partial z}\Sigma w\,(z_0 - z) + \frac{\partial\theta}{\partial t}\Sigma w(t_0 - t)\right\}. \quad \cdots\cdots\cdots\cdots(3)$$

In the above equation $\rho\,\Sigma w$ is the flow of mass across horizontal plane at level z and will be written as

$$g\rho\,\Sigma w = -\frac{\partial}{\partial t}(p_0 - p), \quad \cdots\cdots\cdots\cdots(4)$$

where p_0 is the pressure at the ground. In the second term if we put $K = -\Sigma w(z_0 - z)$, then K is the so-called eddy conductivity. Again if we assume the distribution of $t_0 - t$ as same for both eddies from upper and lower level than z, which will not be an unreasonable assumption, then

$$\Sigma w\,(t_0 - t) \fallingdotseq \overline{t_0 - \ }\ \Sigma w,$$

hence

$$\frac{\partial\theta}{\partial t}\,\Sigma\, w\,(t_0-t)=\overline{t_0-t}\,\frac{\partial\theta}{\partial t}\,\Sigma\, w \ll \theta\,\Sigma\, w.$$

Thus

$$I=-\rho c_p\left(\frac{p}{P}\right)^{\frac{\gamma-1}{\gamma}}\left\{K\frac{\partial\theta}{\partial z}+\theta\frac{1}{g\rho}\frac{\partial}{\partial t}(p_0-p)\right\}$$
$$=-\rho c_p\left\{K\left(\frac{\partial T}{\partial z}+\Gamma\right)+\frac{RT^2}{g\,p}\frac{\partial}{\partial t}(p_0-p)\right\}, \quad \cdots\cdots\cdots\cdots(5)$$

where Γ is the dry adiabatic lapse rate of the air temperature.

The net gain of heat in the layer between z and $z+dz$ level is equal to the product of ρc_p and the change of the temperature in the layer. Therefore the change of the air temperature at z is given by

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left\{K\left(\frac{\partial T}{\partial z}+\Gamma\right)+\frac{RT^2}{g\,p}\frac{\partial}{\partial t}(p_0-p)\right\}. \quad \cdots\cdots\cdots (6)$$

When the total mass of the air undergoes no change p_0 must be constant. Substituting the expression $p=p_0\exp\left(-\frac{g}{R}\int_0^z\frac{dz}{T}\right)$ into (6) we get

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left\{K\left(\frac{\partial T}{\partial z}+\Gamma\right)+T^2\frac{\partial}{\partial t}\int_0^z\frac{dz}{T}\right\}. \quad \cdots\cdots\cdots\cdots\cdots (7)$$

This is the equation which gives for the change of temperature by turbulence.

§ 3. Application of the Result to the Atmosphere.

(1) The diurnal variation of temperature.

As an example of the application of the above result to the atmosphere, let us find the change of temperature with height in the form of the curve of diurnal variation of temperature for the case when K is constant with height. As it is not easy to get a general solution of the equation (7), we shall make an approximate calculation assuming the second term of the right-hand side of the equation to be smaller than other terms.

From TAYLOR's equation

$$\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial z^2}, \quad \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

we have $T=Be^{az+ct}$, where $c=Ka^2$ and a is generally a complex number. Introducing this into the second term of the right-hand side of (7) we have

$$\frac{\partial}{\partial z}\left(T^2\frac{\partial}{\partial t}\int_0^z\frac{dz}{T}\right)=Bc\,(1-2e^{az})\,e^{az+ct}.$$

Putting this expression into (7) and solving the equation

$$\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial z^2}+Bc\,(1-2e^{az})\,e^{az+ct},$$

we have

$$T=C\left(1-\frac{a}{2}z+\frac{2}{3}e^{az}\right)e^{az+ct}+E+Fz. \quad \cdots\cdots\cdots\cdots\cdots (9)$$

Let the temperature at the ground be given by

$$T=T_0+A\sin qt,$$

then $T_0+A\sin qt=\frac{5}{3}Ce^{ct}+E,$

from which

$$C=-i\frac{3}{5}A,\quad c=iq,\quad E=T_0.$$

Thus we have

$$T=T_0-\beta z-\frac{3}{5}A\Re\left[i\exp\left(-\sqrt{\frac{q}{2K}}(1+i)z+iqt\right)\times\left\{1+\frac{1}{2}\sqrt{\frac{q}{2K}}(1+i)z+\frac{2}{3}\exp\left(-\sqrt{\frac{q}{2K}}(1+i)z\right)\right\}\right]$$
$$=T_0-\beta z+\frac{3}{5}Ae^{-bz}\left\{\sin(qt-bz)+\frac{bz}{\sqrt{2}}\sin\left(qt-bz+\frac{\pi}{4}\right)+\frac{2}{3}e^{-bz}\sin(qt-2bz)\right\}, \quad \cdots\cdots(10)$$

where β is the mean lapse-rate during the period and $\Re$ denotes the symbol to take the real part of the following expression and

b is a constant defined by $b^2 = q/2K$.

Euqation (10) will be written as

$$T = T_0 - \beta z + \frac{3}{5} A\sqrt{G^2+H^2}\, e^{-bz} \sin(qt - bz + \alpha) \cdots (11)$$

where $G = 1 + \frac{bz}{2} + \frac{2}{3} e^{-bz}\cos bz$,

$$H = \frac{bz}{2} - \frac{2}{3} e^{-bz}\sin bz,$$

and $\alpha = \tan^{-1}\frac{H}{G}$.

Then the amplitude of the diurnal variation of temperature at any height z will be

$$\frac{6}{5} A\sqrt{G^2 + H^2} \equiv 2A_z'.$$

Solving the equation (8) we have

$$T = T_0 - \beta z + Ae^{-bz}\sin(qt - bz), \cdots (12)$$

so the amplitude by TAYLOR's expression will be given by $2Ae^{-bz} \equiv 2A_z$. TAYLOR compared the above result with observations made on the Eiffel Tower and evaluated the mean value of K for the whole year as 10^5 in C. G. S. units.

Now the diurnal variation of temperature at the ground can be represented with reasonable accuracy by a single sine-term of period 24 hours. Thus $q = 2\pi / (24 \times 60 \times 60) = 7.3 \times 10^{-5}$ and $b = 2 \times 10^{-5}$ for $K = 10^5$. Then at $z = 500$ m the diurnal variation falls to 0.37 of the surface value and 0.05 at 1500 m. The ratio of the diurnal variation deduced from (11) and (12) or A_z'/A_z is equal to $\frac{3}{5}\sqrt{G^2 + H^2}$, which is the function of bz only. Values of A_z'/A_z were illustrated in Fig. 1 with reference to bz, and were tabulated below for a range of height below 2000 m assuming as $K = 10^5$.

z	0	200	500	1000	1500	2000 m
A_z'/A_z	1.0	0.96	0.99	1.26	1.66	2.12

Let us now compare these results with observations made at Lindenberg.

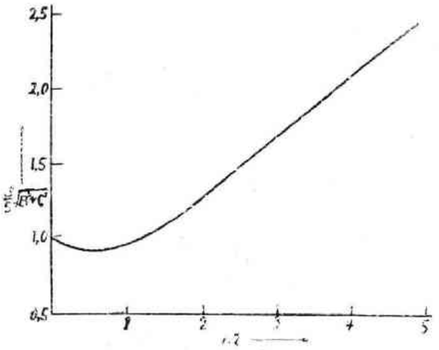

Fig. 1 Ratio of Amplitudes Referred to bz.

Ampl \ z	0	500	1000	1500	2000 m
Obs.	3.0	1.1	0.7	0.5	0.2 °C
Cal. ($2A_z$)	3.0	1.1	0.43	0.15	0.05°C
Cal. ($2A_z'$)	3.0	1.1	0.54	0.25	0.12°C

From the above table we can find that TAYLOR's result holds only below 500 m from the ground and the discrepancy increases with height, which means that K must increase with height in TAYLOR's expression. On the contrary our result holds fairly good up to higher levels even if K is assumed to be constant.

The lag in the occurrence of maximum temperature from the ground to the height z is bz in TAYLOR's expression and $bz - \alpha$ in our expression. Therefore if we assume as $K = 10^5$, the latter is more retarded up to about 125 m from the ground and beyond this level it is gained than the former.

However in practice the method to compare our result with TAYLOR's one from the lag in the time of maximum temperature between any two levels is unreliable on account of the difficulty in estimating the extent of the lag.

(2) Change in the distribution of temperature within a current of air.

When a current of air, after being heated during the passage over warm land, passes over a cold sea, heat is transferred vertically downward. Suppose the air on reaching the coast had a constant lapse-rate β, its surface temperature being T_0. Let the surface temperature of the sea be T_1. Then the solution of equation (8) which was given by TAYLOR will be written down as

$$T = T_0 - \beta z + (T_1 - T_0)\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{4Kt}} e^{-x^2}dx\right\}, \quad \cdots(13)$$

whereas the temperature from (7) is given by

$$T = T_0 - \beta z + (T_1 - T_0) \times \left\{1 - \frac{2}{\sqrt{\pi}}\left(\int_0^{z/\sqrt{4Kt}} e^{-x^2}dx + \frac{1}{2}\frac{z}{\sqrt{4Kt}}e^{-\frac{z^2}{4Kt}}\right)\right\} \quad \cdots\cdots(14)$$

approximately. TAYLOR has deduced values of K of the order of 10^3 from the vertical distribution of temperature above the Great Banks of Newfoundland. He obtained this value from the height at which the inversion ceased, but it is impossible by this method to decide which result is in closer agreement with observations.

Reference.

(1) G. I. TAYLOR; *Phil. Trans. Roy. Soc. A.* **215**, 1 (1915).