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journal or publication title	Science reports of the Tohoku University. Ser. 5, Geophysics
volume	1
number	2
page range	91-95
year	1949-11
URL	http://hdl.handle.net/10097/44427

ON VISCO-ELASTIC MEDIUM
(PART I)

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(Received May 28, 1949)

Introduction.

On the visco-elastic medium Hosali studied the wave propagation in it, assuming the stress components are the linear functions of the strain components and their time differentials.

After his assumption, however, if the time differentials of the strain components are very large comparing to the strain components themselves, the stress components are to be proportional to the time differentials of the strain components or the medium may be regarded to be viscous fluid, and if, on the contrary, the time differentials of the strain components are negligibly small, the stress components are to be proportional to the strain components, or the medium may be regarded to be perfect elastic.

These results are quite contrary to our fundamental idea of the visco-elastic medium, which is practically viscous if the strain varies very gradually and is perfectly elastic if the strain varies very quickly.

The present writer intended to improve this contradiction by assuming the relatively simple stress-strain relations. For the sake of the mathematical facilities of further developments in practical purpose, the formula must be linear function of stress and strain components and their time differentials.

In Hosali's assumption the time differential of the strain components only are introduced, but the time differentials of the stress components are neglected.

The writer introduced the latter in Hosali's formula. This very simple and quite natural improvement of the stress-strain relation practically removed the contradictions in the conclusions of the fundamental properties of the visco-elastic medium.

Chapter I.
Fundamantal Stress-strain Relation.

1. Henceforth, uniform isotropic medium only is treated, otherwise noticed. Or the coefficients represent the elastic and viscous properties of the medium are all independent of the space co-ordinates. The fundamental stress-strain relation is given by

$$\left.\begin{aligned}
\left(a+\frac{\partial}{\partial t}\right)X_x&=\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)\theta+2\left(\mu'+\mu\frac{\partial}{\partial t}\right)e_{xx}\\
\left(a+\frac{\partial}{\partial t}\right)Y_y&=\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)\theta+2\left(\mu'+\mu\frac{\partial}{\partial t}\right)e_{yy}\\
\left(a+\frac{\partial}{\partial t}\right)Z_z&=\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)\theta+2\left(\mu'+\mu\frac{\partial}{\partial t}\right)e_{zz}\\
\left(a+\frac{\partial}{\partial t}\right)Y_z&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)e_{yz}\\
\left(a+\frac{\partial}{\partial t}\right)Z_x&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)e_{zx}\\
\left(a+\frac{\partial}{\partial t}\right)X_y&=\left(\mu'+\mu\frac{\partial}{\partial t}\right)e_{xy}.
\end{aligned}\right\}\qquad\cdots\cdots(1.1)$$

2. When $a\xi\ll\frac{\partial}{\partial t}\xi$, $\lambda'\xi\ll\lambda\frac{\partial}{\partial t}\xi$ and $\mu'\xi\ll\mu\frac{\partial}{\partial t}\xi$, (1.1) becomes

$$\frac{\partial}{\partial t}X_x=\lambda\frac{\partial}{\partial t}\theta+2\mu\frac{\partial}{\partial t}e_{xx},\ \text{etc.}\cdots(1.2)$$

Integrating this with respect to t, we get

$$X_x = \lambda\theta + 2\mu e_{xx},\ \text{etc.}, \qquad \cdots\cdots(1.3)$$

which is the Hooke's law for elastic medium. In (1.3) the integrating constants are dropped, but it does no changes in the physical meaning.

3. When, on the contrary, it is assumed that $a\xi \gg \frac{\partial}{\partial t}\xi$, (1.1) becomes

$$aX_x = \left(\lambda' + \lambda\frac{\partial}{\partial t}\right)\theta + 2\left(\mu' + \mu\frac{\partial}{\partial t}\right)e_{xx},\ \text{etc.} \qquad \cdots\cdots(1.4)$$

If in this formula, $\lambda'\xi \ll \lambda\frac{\partial}{\partial t}\xi$ and $\mu'\xi \ll \mu\frac{\partial}{\partial t}\xi$, we get

$$X_x = \frac{\lambda}{a}\frac{\partial}{\partial t}\theta + 2\frac{\mu}{a}\frac{\partial}{\partial t}e_{xx},\ \text{etc.}, \cdots(1.5)$$

which is the stress-strain relation quite analogous to that in viscous medium. But there is one important differece. As seen in (1.2) λ and μ are Lamé's constants. If, however, the ordinary assumption in viscous medium that the mean of three normal stresses on mutually orthogonal planes at a point is equal to the statical pressure, which is assumed here to be zero, is adopted, (1.5) gives

$$3\lambda = -2\mu, \qquad \cdots\cdots(1.6)$$

which cannot be fulfiled.

As there is, of course, no theoretical foundation of this ordinary assumption in viscous fluid, in visco-elastic medium this assumption on the normal stresses may be safely abandoned.

4. Most of amorphous mediums have practically elastic properties at low temperature, and are viscous at high temperature. This transition can be explained by that λ' and μ' quickly diminish with increasing temperature and λ and μ so vary to gain at last the relation (1.6).

Chapter II.
The Case of Slow Displacement.

5. There is no strict statical case in the dynamics of the visco-elastic medium. Any stress applied to visco-elastic medium must set it in motion however slow it may be. We will deal at first the case only in which the square of velocity is negligibly small or the term including the inertia of particle can be neglected.

6. A body of any form subjected to the action of a constant pressure P will be the simplest state of stress. In this case the stress will be given by the equations

$$\left.\begin{array}{l} X_x = Y_y = Z_z = -P, \\ \text{and} \\ Y_z = Z_x = X_y = 0, \end{array}\right\} \qquad \cdots\cdots(2.1)$$

which gives

$$\left.\begin{array}{l} -aP = \lambda'\theta + 2\mu' e_{xx} + \frac{\partial}{\partial t}(\lambda\theta + 2\mu e_{xx}) \\ -aP = \lambda'\theta + 2\mu' e_{yy} + \frac{\partial}{\partial t}(\lambda\theta + 2\mu e_{yy}) \\ -aP = \lambda'\theta + 2\mu' e_{zz} + \frac{\partial}{\partial t}(\lambda\theta + 2\mu e_{zz}), \end{array}\right\} \qquad \cdots\cdots(2.2)$$

and

$$e_{yz} = e_{zx} = e_{xy} = 0.$$

From (2.2) we get

$$-3aP = (3\lambda' + 2\mu')\theta + \frac{\partial}{\partial t}(3\lambda + 2\mu)\theta. \qquad \cdots\cdots(2.3)$$

Integrating (2.3), we get

$$\theta = A\exp\left(-\frac{3\lambda' + 2\mu'}{3\lambda + 2\mu}t\right) - \frac{3aP}{3\lambda' + 2\mu'}. \qquad \cdots\cdots(2.4)$$

If when $t=0$, $\theta=\theta_0$, then

$$\theta = \left(\theta_0 + \frac{3aP}{3\lambda' + 2\mu}\right)\exp\left(-\frac{3\lambda' + 2\mu'}{3\lambda + 2\mu}t\right) - \frac{3aP}{3\lambda' + 3\mu'}. \qquad \cdots\cdots(2.5)$$

7. When the pressure increases linearly from $t=0$ to $t=\tau$, put

$$X_x = Y_y = Z_z = -pt, \qquad \cdots\cdots(2.6)$$

then from (1.1) we get

$$\left.\begin{array}{l} -(1+at)p = \lambda'\theta + 2\mu' e_{xx} \\ \qquad + \frac{\partial}{\partial t}(\lambda\theta + 2\mu e_{xx}) \\ -(1+at)p = \lambda'\theta + 2\mu' e_{yy} \\ \qquad + \frac{\partial}{\partial t}(\lambda\theta + 2\mu e_{yy}) \\ -(1+at)p = \lambda'\theta + 2\mu' e_{zz} \\ \qquad + \frac{\partial}{\partial t}(\lambda\theta + 2\mu e_{zz}) \\ e_{yz} = e_{zx} = e_{xy} = 0. \end{array}\right\} \qquad \cdots\cdots(2.7)$$

Adding the first three of (2.7), we get

$$-3p(1+at) = (3\lambda' + 2\mu')\theta + (3\lambda + 2\mu)\frac{\partial\theta}{\partial t}. \qquad \cdots\cdots(2.8)$$

Integrating this and taking in account that when $t=0$, $\theta=0$, we get easily

$$\theta=\frac{3p}{3\lambda+2\mu}\left\{\frac{3\lambda+2\mu}{3\lambda'+2\mu'}-a\left(\frac{3\lambda+2\mu}{3\lambda'+2\mu'}\right)^2\right\}\times\left\{\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)-1\right\}-\frac{3pt}{3\lambda'+2\mu'}. \quad \cdots\cdots(2.9)$$

For $t=\tau$, θ is given by

$$\theta_\tau=\frac{3P}{3\lambda'+2\mu'}\left(1-a\frac{3\lambda+2\mu}{3\lambda'+2\mu'}\right)\times\left\{\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}\tau\right)-1\right\}\frac{1}{\tau}-\frac{3P}{3\lambda'+2\mu'},$$

where $P=p\tau$.

If τ is very small, we get

$$\theta_0=-\frac{3P}{3\lambda+2\mu}. \quad \cdots\cdots(2.10)$$

Put this value of θ_0 in (2.5) in Article 6, θ is given by

$$\theta=-\frac{3P}{3\lambda+2\mu}\left[\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)+\frac{3\lambda+2\mu}{3\lambda'+2\mu'}a\left\{1-\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)\right\}\right]. \quad \cdots\cdots(2.11)$$

8. A cylinder subjects to a tension at its ends, but with no force acting on the lateral surface is another simple case. In this case the stress components in the cylinder is given by

$$X_x\neq 0,\ Y_y=Z_z=Y_z=Z_x=X_y=0,$$

if the x-axis is taken parallel to the axis of the cylinder.

If X_x is a function of time we get

$$\left.\begin{aligned}\left(a+\frac{\partial}{\partial t}\right)X_x&=\lambda'\theta+2\mu' e_{xx}+\frac{\partial}{\partial t}(\lambda\theta+2\mu e_{xx})\\ 0&=\lambda'\theta+2\mu' e_{yy}+\frac{\partial}{\partial t}(\lambda\theta+2\mu e_{yy})\\ 0&=\lambda'\theta+2\mu' e_{zz}+\frac{\partial}{\partial t}(\lambda\theta+2\mu e_{zz})\\ 0&=\mu' e_{yz}+\mu\frac{\partial}{\partial t}e_{yz}\\ 0&=\mu' e_{zx}+\mu\frac{\partial}{\partial t}e_{zx}\\ 0&=\mu' e_{xy}+\mu\frac{\partial}{\partial t}e_{xy}.\end{aligned}\right\} \quad \cdots\cdots(2.12)$$

From this equation we get

$$\left.\begin{aligned}e_{yz}&=Fe^{-\frac{\mu'}{\mu}t},\\ e_{zx}&=Ge^{-\frac{\mu'}{\mu}t},\\ e_{xy}&=He^{-\frac{\mu'}{\mu}t},\end{aligned}\right\} \quad \cdots\cdots(2.13)$$

$$e_{yy}-e_{zz}=Ke^{-\frac{\mu'}{\mu}t}, \quad \cdots\cdots(2.14)$$

and

$$\left.\begin{aligned}\frac{\partial^2}{\partial t^2}e_{xx}&=m\frac{\partial}{\partial t}e_{xx}+le_{xx}+\frac{F_1(t)}{\mu(3\lambda+2\mu)},\\ \frac{\partial^2}{\partial t^2}e_{yy}&=m\frac{\partial}{\partial t}e_{yy}+le_{yy}+\frac{F_2(t)}{\mu(3\lambda+2\mu)},\end{aligned}\right\} \quad \cdots\cdots(2.15)$$

where

$$\left.\begin{aligned}m&=-\frac{3(\lambda'\mu+\lambda\mu')+4\mu\mu'}{\mu(3\lambda+2\mu)},\\ l&=-\frac{(3\lambda'+2\mu')\mu'}{(3\lambda+2\mu)\mu},\\ F_1(t)&=\left(a+\frac{\partial}{\partial t}\right)\left\{(\lambda'+\mu')+(\lambda+\mu)\frac{\partial}{\partial t}\right\}X_x,\\ F_2(t)&=-\frac{1}{2}\left(a+\frac{\partial}{\partial t}\right)\left(\lambda'+\lambda\frac{\partial}{\partial t}\right)X_x.\end{aligned}\right\} \quad \cdots\cdots(2.16)$$

(2.15) can be easily solved and we get

$$\left.\begin{aligned}\frac{3(\lambda'\mu-\lambda\mu')}{\mu(3\lambda'+2\mu')}e_{xx}&=C_1\frac{\mu'}{\mu}e^{-\frac{\mu'}{\mu}t}-C_2\frac{3\lambda+2\mu}{3\lambda'+2\mu'}\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)\\ &+e^{-\frac{\mu'}{\mu}t}\int\frac{F_1(t)}{\mu'(3\lambda'+2\mu')}e^{\frac{\mu'}{\mu}t}\,dt\\ &-\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)\int\frac{F_1(t)}{\mu'(3\lambda'+2\mu')}\times\exp\left(\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)dt,\\ \frac{3(\lambda'\mu-\lambda\mu')}{\mu(3\lambda'+2\mu')}e_{yy}&=C_1'\frac{\mu}{\mu}e^{-\frac{\mu'}{\mu}t}-C_2'\frac{3\lambda+2\mu}{3\lambda'+2\mu'}\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)\\ &+e^{-\frac{\mu'}{\mu}t}\int\frac{F_2(t)}{\mu'(3\lambda'+2\mu')}e^{\frac{\mu'}{\mu}t}\,dt\\ &-\exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)\int\frac{F_2(t)}{\mu'(3\lambda'+2\mu')}\times\exp\left(\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right)dt.\end{aligned}\right\} \quad \cdots\cdots(2.17)$$

where C_1, C_2, C_1', and C_2' are integration constants.

Case 1. If $X_x=Tt$, and at $t=0$ $e_{xx}=e_{yy}=0$, then $\frac{\partial X_x}{\partial t}=T$ and $\frac{\partial^2 X_x}{\partial t^2}=0$. In this case we get from (2.16)

$$\left.\begin{aligned}F_1(t)&=a(\lambda'+\mu')Tt+\left\{(\lambda'+\mu')+a(\lambda+\mu)\right\}T,\\ F_2(t)&=\frac{1}{2}\left\{a\lambda' Tt+(\lambda'+a\lambda)T\right\}.\end{aligned}\right\} \quad \cdots\cdots(2.18)$$

If there is no strain at $t=0$, then we get

$$\left.\begin{aligned} e_{xx}=&\frac{T}{\mu'(3\lambda'+2\mu')}\Big[\Big\{(\lambda'+\mu')\\ &-\Big((\lambda'+\mu')\frac{\mu}{\mu'}-\frac{\lambda'\mu-\lambda\mu'}{3\lambda'+2\mu'}\Big)a\Big\}\\ &\times\Big\{1-\frac{\mu(3\lambda'+2\mu')}{3(\lambda'\mu-\lambda\mu')}e^{-\frac{\mu'}{\mu}t}\\ &+\frac{\mu'(3\lambda+2\mu)}{3(\lambda'\mu-\lambda\mu')}\exp\Big(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\Big)\Big\}\\ &+(\lambda'+\mu')at\Big],\\ e_{yy}=&\frac{T}{2\mu'(3\lambda'+2\mu')}\Big[\Big\{\lambda'-\Big(\lambda'\frac{\mu}{\mu'}\\ &-\frac{2(\lambda'\mu-\lambda\mu')}{3\lambda'+2\mu'}\Big)a\Big\}\\ &\times\Big\{\frac{\mu(3\lambda'+2\mu')}{3(\lambda'\mu-\lambda\mu')}e^{-\frac{\mu'}{\mu}t}\\ &-\frac{\mu'(3\lambda+2\mu)}{3(\lambda'\mu-\lambda\mu')}\exp\Big(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\Big)-1\Big\}\\ &-\lambda'at\Big]. \end{aligned}\right\} \quad \cdots\cdots(2.19)$$

If t is small but $Tt=X_r$ is finite, (2.19) can be written as

$$\left.\begin{aligned} e_{xx}&=\frac{\lambda'+\mu'}{\mu'(3\lambda'+2\mu')}aX_x,\\ e_{yy}&=-\frac{\lambda'}{2\mu'(3\lambda'+2\mu')}aX_r. \end{aligned}\right\}\cdots\cdots(3.20)$$

Case 2. $X_x=T_0$, and $e_{xx}=E_{xx}$, $e_{yy}=E_{yy}$, $\frac{\partial e_{xx}}{\partial t}=\dot{E}_{xx}$, and $\frac{\partial e_{yy}}{\partial t}=\dot{E}_{yy}$ when $t=0$, then we get

$$F_1(t)=a(\lambda'+\mu')T,$$

and

$$F_2(t)=-\frac{1}{2}a\lambda'T.$$

Put these values in (2.16), we get

$$\left.\begin{aligned} \frac{3(\lambda'\mu-\lambda\mu')}{\mu(3\lambda'+2\mu')}e_{xx}=&C_1\frac{\mu}{\mu'}e^{-\frac{\mu'}{\mu}t}\\ &-C_2\frac{3\lambda+2\mu}{3\lambda'+2\mu'}\exp\Big(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\Big)\\ &+\frac{a(\lambda'+\mu')T_0}{\{\mu'(3\lambda'+2\mu')\}^2}3(\lambda'\mu-\lambda\mu'),\\ \frac{3(\lambda'\mu-\lambda\mu')}{\mu(3\lambda'+2\mu')}e_{yy}=&C_1'\frac{\mu}{\mu'}e^{-\frac{\mu'}{\mu}t}\\ &-C_2'\frac{3\lambda+2\mu}{3\lambda'+2\mu'}\exp\Big(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\Big)\\ &-\frac{1}{2}\frac{a\lambda'T_0}{\{\mu'(3\lambda'+2\mu')\}^2}3(\lambda'\mu-\lambda\mu'). \end{aligned}\right\} \quad \cdots\cdots(2.21)$$

Constants in this eqation are to be determined by initial conditions.

9. Circular cylinder subjects to torsional couple at its ends, but with no force on the lateral surface is another simplest uniform deformation. Similar to the well known case of pure elastic deformation of circular cylinder, the couple at ends N and the torsional angle τ is given by

$$\Big(a+\frac{\partial}{\partial t}\Big)N=\Big(\mu'+\mu\frac{\partial}{\partial t}\Big)\tau\frac{\pi r^4}{2}, \quad \cdots\cdot(2.22)$$

where r is the radius of the cylinder.

This case is easily solved when N is given as a function of t. (2.22) can be written as

$$\frac{\partial}{\partial t}\tau+P\tau=Q(t), \quad \cdots\cdots(2.23)$$

where

$$P(t)=\frac{\mu'}{\mu}\tau,$$

and

$$Q(t)=\frac{2}{\pi r^4\mu}\Big(a+\frac{\partial}{\partial t}\Big)N.$$

(2.23) has the solution

$$\tau=Ce^{-\int P(t)dt}+e^{-\int P(t)dt}\int Qe^{\int P(t)dt}dt, \quad \cdots\cdots(2.24)$$

or

$$\tau=Ce^{-\frac{\mu'}{\mu}t}+\frac{2}{\pi r^4\mu}e^{-\frac{\mu'}{\mu}t}\int\Big(a+\frac{\partial}{\partial t}\Big)Ne^{\frac{\mu'}{\mu}t}dt. \quad \cdots\cdot(2.25)$$

Case 1. Put $N=nt+N_0$, we get

$$\tau=Ce^{-\frac{\mu'}{\mu}t}+\frac{2}{\pi r^4\mu'}\Big\{\Big(aN_0+n-an\frac{\mu}{\mu'}\Big)+ant\Big\}. \quad \cdots\cdots(2.26)$$

If at $t=0$, $\tau=\tau_0$, then we get

$$\tau=\tau_0e^{-\frac{\mu'}{\mu}t}+\frac{2}{\pi r^4\mu'}\Big\{\Big(aN_0+n-an\frac{\mu}{\mu'}\Big)\times\Big(1-e^{-\frac{\mu'}{\mu}t}\Big)+ant\Big\}. \quad \cdots\cdot(2.27)$$

Case 2. If $N_0=0$ or $N=nt$, then we get

$$\tau=\tau_0e^{-\frac{\mu'}{\mu}t}+\frac{2}{\pi r^4\mu'}\Big\{\Big(n-an\frac{\mu}{\mu'}\Big)\times\Big(1-e^{-\frac{\mu'}{\mu}t}\Big)+ant\Big\}, \quad \cdots\cdots(2.28)$$

and if $\tau_0=0$, then we get

$$\tau=\frac{2}{\pi r^4\mu'}\Big\{n\Big(1-a\frac{\mu}{\mu'}\Big)\Big(1-e^{-\frac{\mu'}{\mu}t}+ant\Big\}. \quad \cdots\cdots(2.29)$$

If further t is small, but $N=nt$ is finite, then we get

$$\tau=\frac{2N}{\pi r^4\mu}. \qquad \cdots\cdots(2.30)$$

Case 3. If $n=0$, or $N=N_0$, then

$$\left(a+\frac{\partial}{\partial t}\right)N=aN_0.$$

In this case we get

$$\tau=\tau_0 e^{-\frac{\mu'}{\mu}t}+\frac{2a}{\pi r^4\mu'}N_0\left(1-e^{-\frac{\mu'}{\mu}t}\right). \qquad \cdots\cdots(2.31)$$

Case 4. If $N=0$, then we get

$$\tau=\tau_0 e^{-\frac{\mu'}{\mu}t}. \qquad \cdots\cdots(2.32)$$

10. From equation (2.32) in Article 9, we get

$$\mu'>0, \qquad \cdots\cdots(2.33)$$

because τ/τ_0 must be decreasing with t.

From equation (2.31) we get

$$a>0, \qquad \cdots\cdots(2.34)$$

because τ must be always positive if $\tau_0>0$ and $N_0>0$.

Put $P=0$ in (2.5) in Article 6, we get

$$\theta=\theta_0 \exp\left(-\frac{3\lambda'+2\mu'}{3\lambda+2\mu}t\right).$$

In this case θ cannot be greater than θ_0, therefore

$$\left.\begin{aligned} &3\lambda'+2\mu'>0,\\ \text{or}\quad &\lambda'>-\frac{2}{3}\mu'. \end{aligned}\right\} \qquad \cdots\cdots(2.35)$$