Effect of Viscosity on the Model Experiment of Tsunami.

EFFECT OF VISCOSITY ON THE MODEL EXPERIMENT OF TSUNAMI

Sekiji Ogiwara

Institute of Geophysics, Faculty of Science, Tohoku University

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§ 1. Law of Similitude and the Effect of Viscosity.

Model experiment of Tsunami or seismic sea waves demands to satisfy the law of similitude. However it is generally impossible when the effect of viscosity is taken into consideration and it has been considered this effect as considerablly serious on the model experiment.(1)(2) This erroneous consideration comes from the fact that we have been concerned only with the molecular viscosity of water of which coefficient is nearly same both for the sea and the model. In these circumstances, however, turbulent viscosity predominates the molecular viscosity and its coefficient of the sea water is considerably larger than that of the water in the model. Therefore we shall examine the effect of viscosity again from the standpoint of the law of similitude.

The equation of motion of the sea water is

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \cdots\cdots (1)$$

and the elevation of water from the undisturbed level is given by

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x}. \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

In the case of the model these equations are

$$\frac{du'}{dt'} = -\frac{1}{\rho'}\frac{\partial p'}{\partial x'} + \frac{\mu'}{\rho'}\frac{\partial^2 u'}{\partial y'^2}, \cdots (1)'$$

and

$$\frac{\partial p'}{\partial x'} = \rho' g \frac{\partial \eta'}{\partial x'} \cdots\cdots\cdots\cdots\cdots\cdots (2)'$$

respectively. Here x- and x'-axis are chosen in the direction of the propagation of the wave and y- and y'-axis vertically upwards.

If we put

$$x = L_0 x',\ y = D_0 y',\ t = T_0 t',$$

$$\eta = D_0 \eta',\ \frac{\mu}{\rho} = \Phi_0 \frac{\mu'}{\rho'}$$

and introduce these into (1) and (2) we get

$$\frac{L_0}{T_0^2}\frac{du'}{dt'} = -\frac{gD_0}{L_0}\frac{\partial \eta'}{\partial x'} + \frac{\Phi_0 L_0}{T_0 D_0^2}\frac{\mu'}{\rho'}\frac{\partial^2 u'}{\partial y'^2}.$$

Then from the law of similitude

$$T_0 = \frac{L_0}{\sqrt{D_0}},\quad \Phi_0 = \frac{D_0^{\frac{5}{2}}}{L_0}.$$

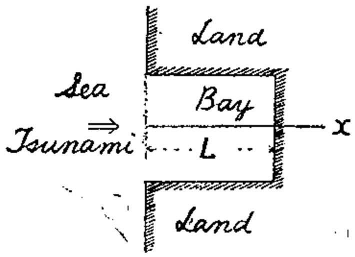

Fig. 1

When we are concerned only with molecular viscosity Φ_0 must be equal to unity, therefore $L_0 = D_0^{\frac{5}{2}}$, and $T_0 = D_0^2$. In usual cases D_0 should be about 100 and so the time must be reduced to 1/10000 and the horizontal dimension to 10^{-5}. In such circumstances no experiment can be performed. However the value of the turbulent viscosity of the sea water is about 100 in the bay. On the contrary it is much smaller in the model. Observations in the Japan Sea show that turbulent viscosity depends on the velocity of the sea water and is about 4 when the velocity is very small. If we adopt this value for the model $\Phi_0=100/4=25$ and if we assume $D_0=100$, then $L_0=4000$ and $T_0=400$ which is a suitable reduction for experiment. As the value of the coefficient of turbulent viscosity is not definite, especially that of the water in the model, we cannot perfectly satisfy the law of similitude, but the effect of viscosity can be fairly reduced by adopting the adequate dimensions of the model.

§ 2. Effect of Viscosity on the Height of Waves in the Bay.

As a simple case we consider a rectangular bay whose length is L and depth is D which is uniform throughout the bay. Tsunami is supposed to invade the bay in the direction of its length. We shall obtain the height of wave in the bay giving the changes in the elevation at the mouth of the bay.

Introducing the expression $\xi=\int u dt$ into (1) and (2) we get

$$\frac{\partial^2\xi}{\partial t^2}=-g\frac{\partial\eta}{\partial x}+\nu\frac{\partial^3\xi}{\partial t\partial y^2}\cdots\cdots\cdots(3)$$

As the horizontal velocity u depends on y the equation of continuity is given by

$$\eta=-\frac{\partial}{\partial x}\int_0^D \xi dy. \qquad \cdots\cdots(4)$$

From (3) and (4) we get

$$\frac{\partial^2\xi}{\partial t^2}=g\frac{\partial^2}{\partial x^2}\int_0^D \xi dy+\nu\frac{\partial^3\xi}{\partial t\partial y^2}\cdots(5)$$

If we take the origin of the axes of coordinates at the bottom of the open end which communicates with the sea, boundary and initial conditions are expressed as follows

$$\begin{cases} u=0 \quad \text{at} \quad y=0, \quad \dfrac{\partial u}{\partial y}=0 \quad \text{at} \quad y=D, \\ \eta=f(t) \quad \text{at} \quad x=0, \quad \dfrac{\partial \eta}{\partial x}=0 \quad \text{at} \quad x=L, \end{cases}$$

$$\eta=0 \text{ and } \frac{\partial\eta}{\partial t}=0 \text{ when } t=0.$$

The condition at $y=0$ means that the water is at rest at the bottom of the sea, and that at $y=D$ is given by the stress condition at the surface. Again the condition at $x=L$ means that the closed end of the bay coincides with the loop of the wave. To solve the equation (5) we shall first obtain the solution for $\eta=1$ at $x=0$ and then the solution for $\eta=f(t)$ at $x=0$.

Putting

$$\xi=X(x)T(t)\left\{Y(y)+1\right\}-\frac{x}{D}$$

and introducing it into (5) we have

$$XT''(Y+1)=gX''T\left(\int_0^D Ydy+D\right)+\nu XT'Y''.$$

If we separate the terms which contain y from those not contain,

$$\left.\begin{aligned} XT''Y&=\nu XT'Y'', \\ XT''&=gX''T\left(\int_0^D Ydy+D\right) \end{aligned}\right\}\cdots\cdots(6)$$

From (6)

$$\frac{T''}{T'}=\nu\frac{Y''}{Y}=b \qquad \cdots\cdots(7)$$

where b is in general a complex number, and so

$$T=e^{bt},\ y=A\exp\left(\sqrt{\frac{b}{\nu}}y\right)+B\exp\left(-\sqrt{\frac{b}{\nu}}y\right).$$

Therefore by (7)

$$X=E\sin cx+F\cos cx$$

where

$$\frac{1}{c^2}=-\frac{g}{b^2}\left[\sqrt{\frac{\nu}{b}}\left\{A\left(\exp(\sqrt{\frac{b}{\nu}}D)-1\right)\right.\right.$$
$$\left.\left.-B\left(\exp(-\sqrt{\frac{b}{\nu}}D)-1\right)\right\}+D\right]\cdots\cdots(8)$$

Thus we get

$$\xi=\Re\,(E\sin cx+F\cos cx)$$
$$\times e^{bt}\left(A\exp(\sqrt{\frac{b}{\nu}}y)\right.$$
$$\left.+B\exp(-\sqrt{\frac{b}{\nu}}y)+1\right)-\frac{x}{D}.$$

$\Re$ denotes the symbol to take the real part of the above expression.

From conditions at the surface and the bottom of the sea

$$A+B+1=0,$$
$$\sqrt{\frac{b}{\nu}}\left(A\exp(\sqrt{\frac{b}{\nu}}D)-B\exp(-\sqrt{\frac{b}{\nu}}D)\right)=0$$

or

$$A=-\frac{1}{1+\exp\left(2\sqrt{\frac{b}{\nu}}D\right)},$$

$$B=-\frac{\exp\left(2\sqrt{\frac{b}{\nu}}D\right)}{1+\exp\left(2\sqrt{\frac{b}{\nu}}D\right)}.$$

Thus the height of wave is expressed as

$$\eta=-\Re\frac{\partial}{\partial x}\int_0^D \xi dy$$
$$=\Re\frac{b^2}{cg}e^{bt}(E\cos cx-F\sin cx)+1.$$

Using the conditions at the open end (or $\eta=1$ at $x=0$) and the closed end

$$E=0,\ c=\frac{s+\frac{1}{2}}{L}\pi,\quad s=0,1,2,\cdots.$$

Consequently

$$\eta=-\Re\frac{1}{g}\sum_{s=0}^{\infty}\frac{Lb_s^2\ e^{b_s t}}{(s+\frac{1}{2})\pi}F_s\sin\frac{s+\frac{1}{2}}{L}\pi x+1$$

$$=\Re\sum_{s=0}^{\infty}H_s\ e^{b_s t}\sin\frac{s+\frac{1}{2}}{L}\pi x+1.\quad\cdots\cdots(9)$$

From the initial conditior, or $\eta=0$ when $t=0$

$$\Re H_s=\frac{-2}{(s+\frac{1}{2})\pi},$$

and from $\frac{\partial\eta}{\partial t}=0$ when $t=0$

$$\Re H_s b_s=0.$$

Thus we can get the height of wave with the condition that $\eta=1$ at $x=0$. The solution for $\eta=f(t)$ at $x=0$ is given by Duhamel's theorem as

$$\eta=\Re\sum_{s=0}^{\infty}H_s b_s\sin\frac{s+\frac{1}{2}}{L}\pi x\int_0^t f(\lambda)e^{b_s(t-\lambda)}d\lambda,$$
$$\cdots\cdots\cdots(10)$$

or putting

$$H_s=P_s+iQ_s,\quad b_s=p_s+iq_s,$$

then

$$P_s=-\frac{2}{(s+\frac{1}{2})\pi},\ Q_s=-\frac{2}{(s+\frac{1}{2})\pi}\ \frac{p_s}{q_s},$$

so we have

$$\eta=\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{1}{s+\frac{1}{2}}\ \frac{1}{q_s}(p_s^2+q_s^2)\sin\frac{s+\frac{1}{2}}{L}\pi x$$
$$\times\int_0^t f(t)e^{p_s(t-\lambda)}\sin q_s(t-\lambda)d\lambda.\quad\cdots\cdots\cdots(11)$$

When the viscosity is neglected

$$p_s=0,\quad q_s=\sqrt{gD}\ \frac{s+\frac{1}{2}}{L}\pi,$$

so introducing these into (11) we get the height of wave directly and this is in accordance with the result obtained by G. Nishimura & K. Kanai [3] using the Stokes' method.

As an example when $f(t)=H\sin\alpha t$,

$$\eta=2\frac{H\sqrt{gD}}{L}\sum_{s=0}^{\infty}\frac{1}{B_s^2-\alpha^2}\sin\frac{s+\frac{1}{2}}{L}\pi x$$
$$\times(B_s\sin\ \alpha t-\alpha\sin\ B_s t)$$

for non-viscous fluid, where

$$B_s=\sqrt{gD}\ \frac{s+\frac{1}{2}}{L}\pi.$$

For viscous fluid, from (8)

$$b_s^2+B_s^2\left(1-\sqrt{\frac{\nu}{b_s}}\frac{1}{D}\tanh\sqrt{\frac{b_s}{L}D}\right)=0,$$

but it is difficult to get the value of b_s from the above equation. However, as in general

$$\sqrt{\frac{\sqrt{gD}(s+\frac{1}{2})\pi}{\nu L}}D>0,$$

so
$$b_s=-\frac{B_s}{2D}\sqrt{\frac{\nu L}{\sqrt{gD}(2s+1)\pi}}$$
$$+iB_s\left(1-\frac{1}{2D}\sqrt{\frac{\nu L}{\sqrt{gD}(2s+1)\pi}}\right)$$

approximately.

If we put

$$f_s=\frac{1}{2D}\sqrt{\frac{\nu L}{\sqrt{gD}(2s+1)\pi}}\ (0<f_s<1),$$

then

$$b_s=B_s\left\{-f_s+i(1-f_s)\right\},$$

thus

$$\eta=\frac{2H\sqrt{gD}}{L}\sum_{s=0}^{\infty}\frac{1}{B_s^2-\alpha^2}\sin\frac{s+\frac{1}{2}}{L}\pi x$$
$$\times\left\{B_s\left(1+f_s\frac{B_s^2+\alpha^2}{B_s^2-\alpha^2}\right)\sin\left(\alpha t-\frac{2B_s\alpha f_s}{B_s^2-\alpha^2}\right)\right.$$
$$\times e^{-B_s f_s t}\left(1+\frac{2B_s^2 f_s}{B_s^2-\alpha^2}\right)$$
$$\left.\times\sin\left(B_s t-B_s f_s(t+\frac{2B_s}{B_s^2-\alpha^2})\right)\right\}.$$

When we compare this with that of non-viscous fluid, it is easily seen that for forced oscillation the amplitude is larger and the phase is more retarded than the latter and for free oscilllation the ampitude is larger at first but it decreases exponentially and the phase is also retarded. After a long time free oscillation diminishes and so the difference between them increases as the time elapses.

In the near future we intend to make a model experiment of Tsunami in Shizugawa Bay, Miyagi Ken, Japan. In this case $L \fallingdotseq 1.5$ km, $D \fallingdotseq 6$ m and if we assume $\mu/\rho = 100$, then $f_0 = 0.066$. The longest period of the free oscillation of this bay is 13.1 min.

In Fig. 2 the relative height of waves at the open end with the closed end or $\eta_{x=L}/H$ is illustrated, assuming $\alpha = 2\pi/30\times60$ or the period of forced oscillation as 30 min. This ratio depends not only on the coefficient of viscosity but on the period of free and forced oscillation. In the case of the above example the ratio $\eta_{x=L}/H$ of viscous fluid is at first about 10% smaller than that of the non-viscous fluid, but its difference increases as the time elapses. In the same way the relative height of the wave in the bay and in the model in which turbulent viscosity is not same will be obtained with ease.

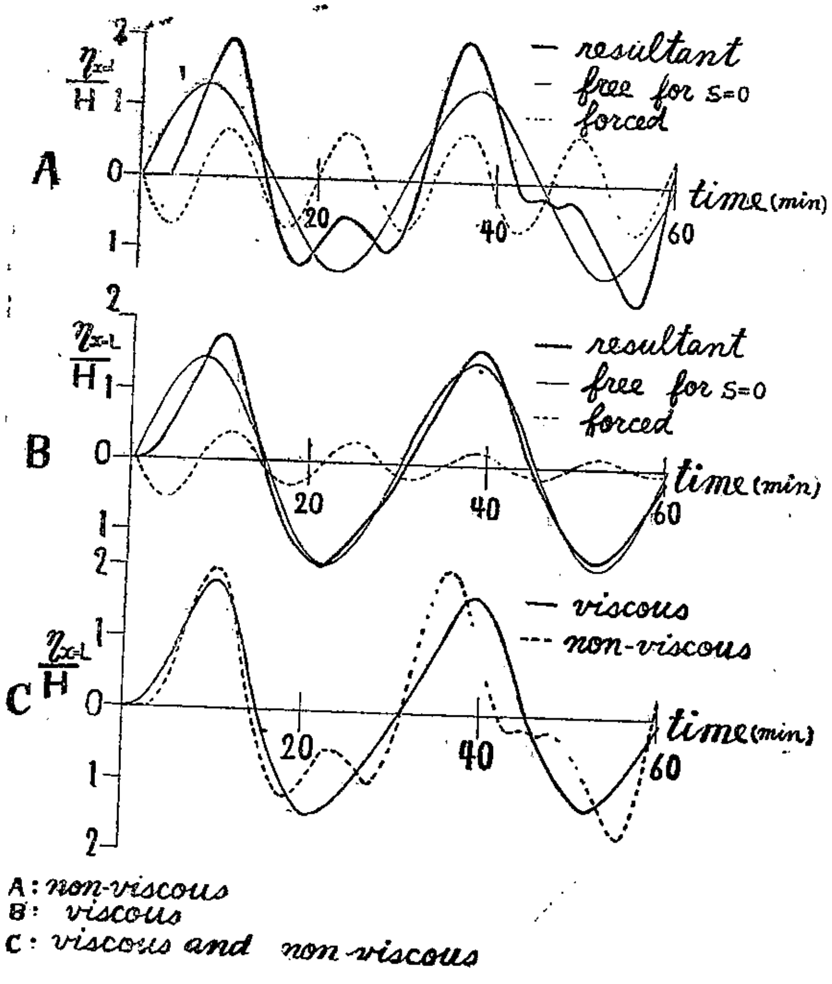

Fig. 2

References

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(3) G. Nishimura & K. Kanai. Bull. Earthq. Res. Inst. Suppl. Vol. 1.