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# ON THE FORMATION OF RAIN DROPS IN CLOUDS WHICH ARE ENTIRELY COMPOSED OF WATER (STUDIES ON THE FORMATION OF CLOUD AND RAIN II)

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## § 1. Introduction.

There are two processes by which water cloud particles grow to rain drops, the one is the condensation and the other is the coagulation. By condensation only, however, cloud particles cannot produce the drops of which normal rain is composed, only the small drops of about one hundred microns of diameter at the largest. Water drops coagulate by turbulence of the air, relative rates of fall of drops of different sizes etc. Wigand and Frankenberger [1] have shown that the coagulation by turbulence is exceedingly hindered when the drops are charged with electricity of the same sign. But this conclusion is not yet fully justified by experiments or observations. The opinion that cloud particles can coagulate by the relative rates of fall of drops of different sizes has also been rejected by Köhler etc. Experiments performed by Findeisen [2], however, shown that this opinion is correct and cloud particls can coagulate by the relative rates of fall.

In this paper we made some calculations to obtain the rate of growth of rain drops by coagulation due to relative rates of fall, disregarding the effect of turbulence of the air. Water drops do not necessarily capture all cloud particles which encountered during the fall in the air, as small particles *en route* will be deflected by the stream around the drop which falls in the air. So we calculated first the proportion of captured cloud particles by falling rain drops and then estimated the rate of growth of drops.

## § 2. The Proportion of Captured Particles by Falling Rain Drop.

With reference to ice formation on aircraft the problem of capture of small particles by a cylindrical body is investigated by many authors, the present problem, however, is equivalent to that of capture by a spherical one.

Let the radius and the rate of fall of the rain drop be $R$ and $V$ respectively, and the radius of the cloud particle be $a$. For convenience we consider that the rain drop is at rest in the air and the air moves upwards with the velocity $V$ neglecting the rate of fall of the cloud particle which is so small as compared with that of the rain drop.

As shown in Fig. 1 we introduce the polar coordinates $r,\theta$ in any vertical plane through the centre of the drop.

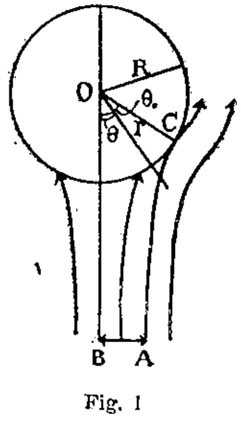


Fig. 1

First we assume the potential flow around the drop and Stokes' resistance on the cloud by the air flow. Then equations of motion of the cloud particle are written down as

$$m\left(\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right)=6\pi\mu a\left(\frac{\partial\varphi}{\partial r}-\frac{dr}{dt}\right),\quad \cdots\cdots(1)$$

$$m\left(r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right)=6\pi\mu a\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}-r\frac{d\theta}{dt}\right),\quad \cdots\cdots(2)$$

where $t$, $m$ and $\mu$ are the time, mass of the cloud particle and the coefficient of viscosity of the air respectively. The velocity potential $\varphi$ is given by

$$\varphi=Vr\cos\theta\left(1+\frac{R^3}{2r^3}\right).$$

Putting

$$K=\frac{6\pi\mu a}{m}=\frac{9}{2}\frac{\mu}{a^2}$$

and introducing dimensionless quantities $r/R$, $Vt/R$ and $\varphi/VR$ instead of $r,t$ and $\varphi$ respectively, (1) and (2) become

$$\alpha\left(\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right)+\frac{dr}{dt}=\cos\theta\left(1-\frac{1}{r^3}\right)\quad \cdots\cdots\cdots(1)'$$

$$\alpha\left(r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right)+r\frac{d\theta}{dt}=-\sin\theta\left(1+\frac{1}{2r^3}\right)\quad \cdots\cdots(2)'$$

where $\alpha=V/KR$.

By dimensional analysis [3] the proportion of captured particles is reduced to a function of $\alpha$ only and is equal to nearly unity when $\alpha$ is large and tends to zero as it decreases. As will be seen later $\alpha$ is much larger than unity in our case, we assume that the velocity of the particle can be expressed as the inverse power series of $\alpha$.

Namely we put

$$\frac{dr}{dt}=P_0+\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}+\cdots,\quad \cdots\cdots(3)$$

$$r\frac{d\theta}{dt}=Q_0+\frac{Q_1}{\alpha}+\frac{Q_2}{\alpha^2}+\cdots,\quad \cdots(4)$$

where $P_0$, $P_1$, $P_2,\cdots$ and $Q_0$, $Q_1$, $Q_2\cdots$ are functions of $r$ and $\theta$ and are independent of $\alpha$. Introducing (3) and (4) into (1)′ and (2)′ we can get $P_0, P_1, P_2,\cdots$; $Q_0, Q_1, Q_2,\cdots$ by comparing the same power of $\alpha$ on the both sides of (1)′ and (2)′. In these circumstances we must be borne in mind that $\frac{dr}{dt}=\cos\theta$ and $r\frac{d\theta}{dt}=-\sin\theta$, for $r=\infty$ and $\alpha=\infty$ respectively.

As

$$\frac{d^2r}{dt^2}=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{dr}{dt}\frac{\partial}{\partial t}\left(\frac{dr}{dt}\right)+\frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left(\frac{dr}{dt}\right)$$

and

$$2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{1}{r}\left\{\frac{dr}{dt}\frac{\partial}{\partial r}\left(r^2\frac{d\theta}{dt}\right)+\frac{d\theta}{dt}\frac{\partial}{\partial\theta}\left(r^2\frac{d\theta}{dt}\right)\right\}$$

(1)′ and (2)′ become

$$\alpha\Big\{\left(P_0+\frac{P_1}{\alpha}+\cdots\right)\left(\frac{\partial P_0}{\partial r}+\frac{1}{\alpha}\frac{\partial P_1}{\partial r}+\cdots\right)+\frac{1}{r}\left(Q_0+\frac{Q_1}{\alpha}+\cdots\right)\left(\frac{\partial P_0}{\partial\theta}+\frac{1}{\alpha}\frac{\partial P_1}{\partial\theta}+\cdots\right)-\frac{1}{r}\left(Q_0+\frac{Q_1}{\alpha}+\cdots\right)^2\Big\}+P_0+\frac{P_1}{\alpha}+\cdots-\cos\theta\left(1-\frac{1}{r^3}\right)=0\quad \cdots\cdots\cdots(1)''$$

and

$$\alpha\Big\{\left(P_0+\frac{P_1}{\alpha}+\cdots\right)\left(Q_0+\frac{1}{\alpha}Q_1+\cdots\right)+r\left(P_0+\frac{P_1}{\alpha}+\cdots\right)\left(\frac{\partial Q_0}{\partial r}+\frac{1}{\alpha}\frac{\partial Q_1}{\partial r}+\cdots\right)+\left(Q_0+\frac{Q_1}{\alpha}+\cdots\right)\left(\frac{\partial Q_0}{\partial\theta}+\frac{1}{\alpha}\frac{\partial Q_1}{\partial\theta}+\cdots\right)\Big\}+r\left(Q_0+\frac{Q_1}{\alpha}+\cdots\right)+r\sin\theta\left(1+\frac{1}{2r^3}\right)=0\quad \cdots\cdots(2)''$$

respectively.

Comparing the terms which involve $\alpha^1$ on the both sides of (1)″ and (2)″ we have

$$\begin{cases} rP_0\frac{\partial P_0}{\partial r}+Q_0\frac{\partial P_0}{\partial\theta}-Q_0^2=0 \\ P_0Q_0+rP_0\frac{\partial Q_0}{\partial r}+Q_0\frac{\partial Q_0}{\partial\theta}=0 \end{cases}$$

from which we get

$$P_0=\cos\theta,\qquad Q_0=-\sin\theta.$$

Next comparing the terms of $\alpha^0$ we have

$$P_1=\frac{1}{2r^2},\qquad Q_1=0.$$

Similarly,

$$P_2=\frac{1}{2r}-\frac{\cos\theta}{r^4}\Big(\frac{1}{8}+\frac{1}{10}\sin^2\theta+\frac{3}{35}\sin^4\theta+\frac{4}{55}\sin^6\theta+\cdots\Big)$$

$$Q_2=\frac{\cos\theta-1}{2r\sin\theta}+\frac{\sin\theta}{r^4}\Big(\frac{1}{40}+\frac{1}{70}\sin^2\theta+\frac{1}{105}\sin^4\theta+\frac{8}{1155}\sin^6\theta+\cdots\Big).$$

$P_3, P_4, \cdots; Q_3, Q_4, \cdots$ can be obtained by the same way and if we neglect these small quantities the velocity of the particle will be written down as follows

$$\frac{dr}{dt}=V\Big[\cos\theta+\frac{1}{2\alpha}\Big(\frac{R}{r}\Big)^2+\frac{1}{\alpha^2}\Big\{\frac{1}{2}\frac{R}{r}-\Big(\frac{R}{r}\Big)^4\cos\theta\Big(\frac{1}{8}+\frac{1}{10}\sin^2\theta+\frac{3}{35}\sin^4\theta+\frac{4}{55}\sin^6\theta\Big)\Big\}\Big]. \quad\cdots\cdots\cdots(5)$$

$$r\frac{d\theta}{dt}=V\Big[-\sin\theta+\frac{1}{\alpha^2}\Big\{\frac{1}{2}\frac{R}{r}\frac{\cos\theta-1}{\sin\theta}+\Big(\frac{R}{r}\Big)^4\sin\theta\Big(\frac{1}{40}+\frac{1}{70}\sin^2\theta+\frac{1}{105}\sin^4\theta+\frac{8}{1155}\sin^6\theta\Big)\Big\}\Big]. \quad\cdots\cdots(6)$$

In these formulae, however, $r$, $t$ and $\varphi$ are not $r/R$, $Vt/R$, and $\varphi/VR$, but are original expressions.

The cloud particle which moves tangentially to the drop makes the angle $\theta_0$ on the drop with the axis of stream. This angle is determined from (5) with the condition that

$$\frac{dr}{dt}=0 \text{ at } r=R$$

and is tabulated below with reference to $\alpha$.

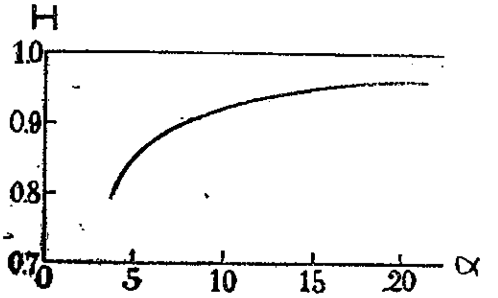


Fig. 2

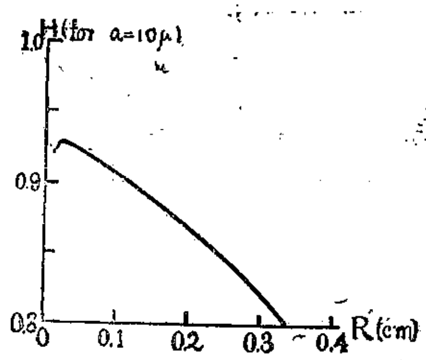


Fig. 3

| $\alpha$ | 5 | 6 | 8 | 10 | 15 | 20 | 30 |
|---|---|---|---|---|---|---|---|
| $\theta_0$ | 85°25′ | 86°00′ | 86°50′ | 87°25′ | 88°15′ | 88°35′ | 89°05′ |

Thus the path of the cloud particle which moves tangentially to the drop will be obtained from (5) and (6) with the condition that $\theta=\theta_0$ at $r=R$. This was done by the aid of Runge-Kutta's method for the numerical solution and the proportion of captured particles $H$, which is expressed as

$H=\frac{1}{\pi R^2}\lim_{r\to\infty}\pi(r\sin\theta)^2$, was estimated for various values of $\alpha$. In the above expression $\lim_{r\to\infty}\pi(r\sin\theta)^2$ refers to the cloud particle which moves tangentially to the drop and so is equal to $\pi\overline{AB}^2$ in Fig. 1.

Since $\alpha=V/KR=1.29\times10^3a^2V/R$, $\alpha$ and $H$ are the functions of $a$ and $R$. As normal size of the cloud particle is about $10\,\mu$ in radius the relation between $H$ and $R$ for $a=10\,\mu$ was illustrated in Fig. 3.

## § 3. Growth of Rain Drops by Capture of Cloud Particles.

We consider that rain drop of which radius is $R_0$ falls in the cloud which is composed of $N_1/\text{cm}^3$ particles of radius $a_1$, $N_2/\text{cm}^3$ particles of $a_2$, $N_3$ of $a_3$ and so on. Then the increment of the volume of rain drop while it falls $dZ$ is

$$\frac{4}{3}\pi\Big\{(R+dR)^3-R^3\Big\}=\frac{\pi}{3}\Big(\sum_{n=1}N_nH_na_n^3\Big)dZ$$

where $H_n$ is the proportion of captured particles of which radius is $a_n$.

As special case when the particle is uniform and its radius is $a$,

$$dR=\frac{\pi}{3}\ NHa^3dZ$$

or $$Z=\frac{3}{\pi Na^3}\int_{R_0}^{R}\frac{dR}{H}.$$

As an example when $a=10\,\mu$ and $N=100$ and consequently the water content is 0.42 g/m³ and the visibility is 140 m by Trabert's formula, the rain drop of $200\,\mu$ grows to $310\,\mu$ after it falls 1 km and $419\,\mu$ for 2 km, $638\,\mu$ or 0.638 mm for 4 km.

Another example in which water content is especially, large, say 2 g/m³ or 30 m of visibility, they are 0.71, 1.22, and 2.24 mm respectively.

## § 4. On The Formation of Rain from Water Clouds.

From above examples we can suggest that rain drops formed in the cloud of the normal water content and thickness are fairly small and large drops are formed only in the clouds which are thick and contain much water.

Bergeron-Findeisen's theory on the formation of rain pronounces that almost every real rain drops and all snowflakes originate around an ice crystal, but it is recognized that rain frequently occurs without the tops of clouds reaching the freezing layer, especially in the middle and low latitudes. In these circumstances it will be supposed that the rain drops are formed by coagulation, and we can examine the above result to some extent by observations. Our observations in which we are engaged are that of the drop size of rain, the thickness of cloud and the temperature in the cloud. Unfortunately we cannot measure the drop size of cloud particles directly so we are obliged to assume it adequately. However by these observations we shall be able at least to decide whether Bergeron-Findeisen's theory is true or not.

## References


(1) A.Wigand and E.Frankenberger, Phys. ZS. 1930.

(2) W.Findeisen, Met. ZS. 1939.

(3) I.Imai, Journ. Met. Soc. Jap. 1941 and 1943.