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A GLOBAL CORRESPONDENCE BETWEEN CMC-SURFACES IN S^3 AND PAIRS OF NON-CONFORMAL HARMONIC MAPS INTO S^2

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ABSTRACT. We show there is a global correspondence between branched *constant mean curvature* (i.e. CMC-) immersions in $S^3/\{\pm 1\}$ and pairs of non-conformal harmonic maps into S^2 in the same associated family. Furthermore, we give two applications.

Let x be a conformal immersion of a Riemann surface M into the Euclidean 3-sphere $S^3(c^2)$ of radius $1/c$, or the real projective space $P^3(c^2)$. The *generalized Gauss map* $\mathcal{G} : M \to G_{2,2}$ is defined by $\mathcal{G}(z) = [x_u \wedge x_v]$, where $G_{2,2}$ is the Grassmann manifold of oriented 2-planes in $\mathbf{R}^4$ and $z = u + iv$ is a complex coordinate on M. Since $G_{2,2}$ is isometric to the product of two unit spheres, $\mathcal{G}$ splits into $\mathcal{G} = (g_1, g_2) : M \to S^2 \times S^2$. When x is a non-totally umbilic CMC-H immersion, g_1 and g_2 are both non-conformal harmonic maps in the same associated S^1-family (see [1], [5], [8] and [9]). More precisely, the holomorphic quadratic differential φ_j $(j = 1, 2)$ of g_j satisfies the relation $\varphi_2 = e^{2i\alpha}\varphi_1$ $(\alpha := \arg(H + ic) \in (0, \pi/2])$. In this case, we express $g_2 = g_1^\alpha$. Conversely, we prove the following:

Theorem. *Let $g : M \to S^2$ be a non-conformal harmonic map. Suppose that there exists a real number $\alpha \in (0, \pi/2]$ such that the associated harmonic map g^α is single-valued on M. Then there exists a branched conformal CMC-H immersion x of M into $P^3(c^2)$ with $\mathcal{G} = (g, g^\alpha)$, where $H = c \cot \alpha$.*

Proof. There exists a CMC-H immersion $x : \tilde{M} \to S^3(c^2)$ having the generalized Gauss map $\tilde{\mathcal{G}} = (g \circ \pi, g^\alpha \circ \pi)$ (see, for instance, [1] and [8, §4 (replace H by H_1)]), where $\pi : \tilde{M} \to M$ is the universal cover. Let U_1 and U_2 be domains in $\tilde{M}$ such that $\pi(U_1) = \pi(U_2)$. Then $x(U_1)$ is congruent with $x(U_2)$ by an isometry Φ of $S^3(c^2)$. Since the tangent planes at the corresponding points are parallel to each other because $\tilde{\mathcal{G}}$ has the same value, we must have $\Phi = \pm id$. □

Remark. The theorem also follows from Bobenko [2, Theorem 14.1]. The parallel branched CMC-immersion arising as the Bonnet pair has the reversed orientation. Taking a finite cover, we get a branched CMC-H immersion into $S^3(c^2)$. However, x is non-branched when M is a torus [5, (4.35)].

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For a non-conformal harmonic map $g : M \to S^2$, there exists uniquely the branched CMC-1 immersion $x_g : \tilde{M} \to \mathbf{R}^3$ with the Gauss map g (see [6]). Let $\rho_\alpha : \pi_1(M) \to SO(3)$ be the monodromy representation of g^α.

Corollary 1 ([3, §5])**.** *x_g is single-valued on M if and only if* $\left.\dfrac{d\rho_\alpha}{d\alpha}\right|_{\alpha=0} = 0.$

Proof. For any α, there exists a branched CMC-immersion $x^\alpha : \tilde{M} \to S^3(c^2)$ ($c = \sin\alpha$) with $\mathcal{G} = (g \circ \pi, g^\alpha)$. By the stereographic projection, x^α is considered as a map into $\mathbf{R}^3$. Using the deformation of the Lie group $SO(4)$ into $SO(3) \ltimes \mathbf{R}^3$ as in [10], we can obtain $\lim_{\alpha\to 0} x^\alpha = x_g$. Then the monodromy of x_g can be seen as a limit of that of x^α when $\alpha \to 0$, which yields the assertion. □

There are at most two distinct isometric immersions of a closed surface with the same *non-constant* mean curvature function H (see [7]). In the CMC case, we obtain:

Corollary 2. *Let (M, ds^2) be a closed Riemannian 2-manifold and $x : M \to S^3(c^2)$ or $\mathbf{R}^3$ an isometric immersion of constant mean curvature H. Then the number N_x of congruent classes of isometric CMC-H immersions is finite. In particular, there exist no global non-trivial isometric deformations of x preserving the mean curvature.*

Proof. Suppose $N_x = \infty$. Then for countably many $e^{i\theta} \in S^1$, there exist isometric CMC-H immersions $x^\theta : M \to S^3(c^2)$ (resp. $\mathbf{R}^3$) with the generalized Gauss map $\mathcal{G} = (g, g^\theta)$ (resp. Gauss map g^θ). Since such $e^{i\theta}$ has accumulation points, g^θ is single-valued on M for all $e^{i\theta} \in S^1$, which contradicts the fact that there exist no non-conformal harmonic maps from M to S^2 with single-valued associated S^1-families (see, for instance, [4, Proposition 2.3]). □

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