Effect of chaotic noise on multistable systems

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In a recent Letter [Phys. Rev. Lett. **30**, 3269 (1995)], we reported that a macroscopic chaotic determinism emerges in a multistable system: the unidirectional motion of a dissipative particle subject to an apparently symmetric chaotic noise occurs even if the particle is in a spatially symmetric potential. In this paper, we study the global dynamics of a dissipative particle by investigating the barrier crossing probability of the particle between two basins of the multistable potential. We derive analytically an expression of the barrier crossing probability of the particle subject to a chaotic noise generated by a general piecewise linear map. We also show that the obtained analytical barrier crossing probability is applicable to a chaotic noise generated not only by a piecewise linear map with a uniform invariant density but also by a nonpiecewise linear map with nonuniform invariant density. We claim, from the viewpoint of the noise induced motion in a multistable system, that chaotic noise is a first realization of the effect of *dynamical asymmetry* of general noise which induces the symmetry breaking dynamics. [S1063-651X(96)01209-3]

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I. INTRODUCTION

Chaotic systems show several unexpected and complex dynamics. ''Chaotic itinerancy'' [1–4] and ''evolution to edge of chaos'' [5–8] are good examples. The mysterious role of chaos in neural networks has also been studied extensively [9–16]. However, the origin of such interesting behaviors has not been clarified sufficiently because an important feature of complex systems, multistability, has not been discussed explicitly in regard to chaos.

The studies of multistable systems subject to probabilistic noise have been extensively carried out in the field of reaction-rate theory, which is analyzed as stochastic processes [17]. The theory makes it possible to calculate a barrier crossing probability in multistable systems, in which the noise may have a simple time correlation. However, this theory also has difficulty treating dynamical noise (perturbation), especially for chaotic noise, because the theory is based on stochastic processes, in which a simple structure of the time correlation of the noise is necessary for its integrability.

In addition to these backgrounds, some chaotic time series have been assumed to be too random to retrieve its deterministic nature in physical systems, because they may have the same randomness even as the coin tosses [18]. Therefore the effect of chaotic noise has not been recognized as an important property even for a macroscopic physical system, whereas the chaotic noise has dynamical asymmetry [19].

In a recent Letter [20], we reported that the short-time correlation of chaotic noise caused by its determinism is unexpectedly important in understanding the dynamics of mul-

tistable systems with chaotic structures [21]. In this paper, we detail the analytical derivation of the barrier crossing probability of the dissipative particle in multistable systems subject to chaotic noise and show that the analytical result is applicable to wider classes of chaotic noise.

We also emphasize in this paper that chaotic noise is a first realization of the effect of dynamical asymmetry of any noise that induces unidirectional motion of a dissipative particle in a *symmetric* potential. This, to our knowledge, is a new insight in regard to the discussions on the possible mechanisms of protein motors by ratchet models [22–25].

In Sec. II, we describe the system in which we will discuss the effect of chaotic noise on the dissipative particle in a periodic potential. In Sec. III we will derive the barrier crossing probability of the dissipative particle over the potential barrier, where we use two kinds of generalized chaotic maps for wider application. In Sec. IV, we show that the present analytical result is applicable both to the chaotic noise, which is a nonpiecewise linear map, and to the chaotic noise, which has nonuniform invariant density, by using a logistic map chaos as an example. In Sec. VI, we summarize our discussions, where we remark the relation of our result to ratchet models of protein motors.

II. SYSTEM

In this section, we describe a system where we argue the effect of chaotic noise on multistable systems. We discuss a dynamics of a dissipative particle in a periodic potential subject to chaotic noise. We believe that the present system is a minimal one, which shows an effect of chaotic noise on a multistable system clearly.

A dissipative particle in a potential V and noise η obeys the equation

$$\frac{dx}{dt} = -\frac{\partial V}{\partial x} + \eta(t), \qquad (1)$$

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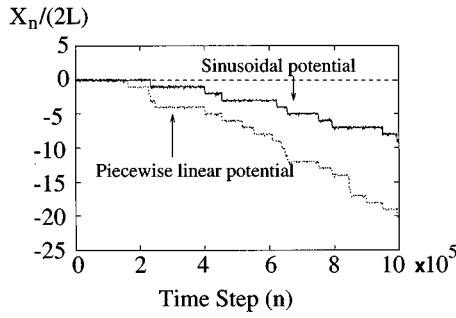
FIG. 1. Typical time evolution of the two kinds of systems under a tent map chaos: (1) a smooth periodic potential, $V_s(x) = h/2\sin[2\pi x/(2L) - \pi/2]$, and (2) the piecewise linear potential, where the same amplitude and the period of the potentials are used: $L=5$ and $h=0.5$.

where $\eta(t)$ is additive noise. We introduce chaotic noise: $\eta(t) = \sum_{j=-\infty}^{\infty} \eta_j \delta(t-j)$, where η_j is a chaotic time series generated by

$$\eta_{n+1} = f(\eta_n), \qquad (2)$$

where f is a chaotic map. The potential $V(x)$ is any periodic potential. In this paper, we report mainly the results of our study using a piecewise linear potential with parity symmetry: $V(x) = h - (h/L)|x[\mathrm{mod}(2L)] - L|$ for $x \geq 0$, $V(-x) \equiv V(x)$, where L is the half-width of the period of the potential and h is the height of the potential barrier. We consider a system that satisfies the following condition:

$$0 < h/L < |\eta|_{max} \ll L. \qquad (3)$$

In this condition, the dissipative particle can move against the gradient of the potential in both direction and the particle staying near the bottom of the potential needs to be driven by chaotic noise many times to cross the potential barrier. In the following we study a discretized equation

$$x_{n+1} = x_n - \left.\frac{\partial V}{\partial x}\right|_{x=x_n} + \eta_n \quad (n=0,1,2,\dots), \qquad (4)$$

which is approximately obtained by integrating Eq. (1) from t_n to $t_{n+1} = t_n + 1$. The choice of the finite $\Delta t \equiv t_{n+1} - t_n(=1)$ does not alter the central result as shown in the following.

We show here that the present system is a sufficient one, which exhibits an unexpected dynamics under chaotic noise. We also show that the central result is not altered if the potential is not piecewise linear. For this purpose, although this paper is intended to discuss the effect of general chaotic noise, we briefly summarize the qualitative result of the symmetry breaking dynamics by using the tent map chaos [26,27], $\eta_{n+1} = f(\eta_n) = 1/2 - 2|\eta_n|$. The tent map chaos has a uniform invariant density with parity symmetry of $\rho(\eta) = 1$ [for $-0.5 \leq x < 0.5$, otherwise $\rho(\eta) = 0$] and δ correlated [28]; these properties are the same as the uniform random number, r_n, $|r_n| < 0.5$. The tent map chaos is one of the most random chaotic sequences and has the same randomness as the coin tosses. Therefore the macroscopically broken parity dynamics induced by this apparently symmet-
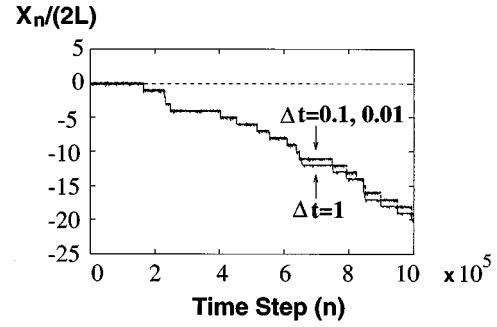

FIG. 2. Dependence of a difference interval Δt, which is used to derive Eq. (4) from Eq. (1) on the evolution of the system, where $L=5$ and $h=0.5$. Data for $\Delta t = 1$, 0.1, and 0.01 are shown.

ric chaotic noise had never been realized explicitly before our discovery, to our knowledge.

As found in Ref. [20], the chaotic noise generated by this tent map can induce a broken symmetry dynamics of a dissipative particle even in a symmetric multistable system. It is easily verified that the qualitative results, namely, the broken symmetry dynamics, are the same both for a smooth periodic potential and for a piecewise linear potential (Fig. 1). If we replace the piecewise linear potential $V(x)$ with the sinusoidal potential having the same amplitude and the period, $V_s(x) = (h/2)\sin[2\pi x/(2L) - \pi/2]$, the direction in which the particle moves does not change. The quantitative difference of the velocity as shown in Fig. 1 can be attributed to the difference of the absolute maximum gradients of the potentials: $|\partial V_s/\partial x|_{max} = (\pi/2)|\partial V_0/\partial x|_{max}$. It can also be verified that the choice of $\Delta t \equiv t_{n+1} - t_n$ does not essentially alter the time evolution of the dissipative particle (Fig. 2).

III. ANALYTICAL DERIVATION OF BARRIER CROSSING PROBABILITY

In this section, we argue a barrier crossing probability of a particle in a periodic potential subject to chaotic noise [29]. An average velocity of the particle is expressed in terms of the barrier crossing probabilities:

$$\langle v \rangle = \lim_{n \to \infty} \frac{x_n - x_0}{n} = 2L\left\{\sum_i p_i^+ - \sum_j p_j^-\right\}, \qquad (5)$$

where p_i^+ is a barrier crossing probability in a positive direction caused by a process i and p_i^- is that in a negative direction. As is found later, the average velocity is often dominated by one barrier crossing probability p: $|\langle v \rangle| \sim 2Lp$.

When the slope of the potential is large enough, a particle is found mostly in the neighborhood of one of the basins of the potential. Therefore, the particle needs to be forced continuously by the noise having the coherent values to cross the barrier. Chaotic noise works effectively for the barrier crossing when the noise stays in the neighborhood of an unstable fixed point η^*. There are two types of chaotic sequences staying near an unstable fixed point: One is the chaotic sequence leaving the unstable fixed point monotonically in its
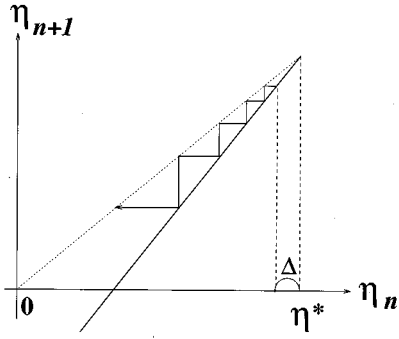
FIG. 3. Emergence of short-time correlation (or clustering) of the chaotic noise near an unstable fixed point (case A): The chaotic sequence η_n leaves an unstable fixed point η^* of the piecewise linear map (solid line) monotonically and the distance between η^* and η_n increases exponentially, where the dotted line shows the line, $\eta_{n+1} = \eta_n$. The strong correlation starts when the noise is injected in the Δ neighborhood of the unstable fixed point. The less the slope of the map, Λ, the more the sequence stays near the unstable fixed point.

stay and the other is the sequence leaving it with oscillation. First, we discuss the former case, which is relatively simple to treat.

In this paper, we have restricted ourselves to chaotic noise in which the two successive events of clustering around an unstable fixed point are not strongly correlated; in other words, successive clustering does not occur without a sufficient intermission. However, there exists a case where the two successive events can be strongly correlated. Bernoulli shift chaos is the case [20]. We will not discuss the complex case in this paper. The study is under way.

A. Chaotic sequence monotonically leaving an unstable fixed point

As shown in Fig. 3, the nearer the injected chaotic noise η is to the unstable fixed point η^*, the longer η stays in the neighborhood of η^*. Therefore, we have to calculate first how near the chaotic noise needs to be injected in the neighborhood of an unstable fixed point for the particle to cross the barrier.

In the following, we calculate the maximum distance Δ_c between an injected chaotic noise and an unstable fixed point for the barrier crossing. The maximum distance Δ_c is necessary to obtain the barrier crossing probability for the particle under chaotic noise. To make the following discussion applicable to wider classes of chaotic maps, we investigate the effect of chaotic noise generated by a generalized piecewise linear map, which is characterized by the absolute value of the slope of the map Λ and an unstable fixed point η^*.

In a system that satisfies Eq. (3), the particle is mostly found near one of the bottoms of the potential. Therefore we assume that the particle is at an origin of the x coordinate at discrete time, $n=1$, namely, $x_1 = 0$, when the chaotic noise starts to drive the particle to cross the potential barrier. We also assume that the particle crosses the barrier by N ($\gg 1$) time steps. The nearer to an unstable fixed point the chaotic sequence η_n is injected, the longer the particle continues to climb the potential. If the particle moves over a half-width of

the potential, L, within N time steps, we judge that the particle crosses the potential barrier. In this consideration, we calculate the maximum distance $\Delta_c = |\eta^* - \eta_1|$ for the barrier crossing due to the effect of an unstable fixed point, as a function of η^*, Λ, h, and L. Note that the slope of the potential $|\Lambda| > 1$ because the fixed point η^* is unstable.

From these conditions, we write the following inequality in which the particle can cross the barrier:

$$\eta_N \geq h/L, \tag{6}$$

$$x_{N+1} \geq L, \tag{7}$$

where we assume that the sign of η is plus for simplicity; but the following result is also valid for negative η by replacing it with an absolute value of η. In such a case, the direction of the velocity of the particle is reversed.

Let us set

$$\eta_1 = \eta^* - \Delta \quad (\Delta \ll 1), \tag{8}$$

then one finds

$$\eta_N = \eta^* - \Lambda^{N-1} \Delta. \tag{9}$$

We obtain through Eq. (4)

$$x_{N+1} = \sum_{k=1}^{N} \eta_k - \frac{h}{L}(N-1), \tag{10}$$

where we use an approximation that $x_1 = 0$. Inserting Eq. (9) into Eq. (10) and using Eq. (7), one obtains

$$x_{N+1} = N\eta^* - \frac{\Delta(1-\Lambda^N)}{1-\Lambda} - \frac{h}{L}(N-1) \geq L. \tag{11}$$

Inserting Eq. (9) into Eq. (6), we get

$$\Delta \leq \left(\eta^* - \frac{h}{L} \right) \Big/ \Lambda^{N-1}. \tag{12}$$

Equation (12) gives the maximum N for which the noise keeps driving the particle upward against the potential as a function of η^*, Δ, and h/L:

$$N \leq N_0 \equiv \log_\Lambda \left[\left(\eta^* - \frac{h}{L} \right) \Big/ \Delta \right] + 1. \tag{13}$$

Replacing N of Eq. (11) by N_0 and approximating $1-\Lambda^N$ of Eq. (11) by $-\Lambda^N$ for $N \gg 1$, we obtain the maximum value of Δ for which Eq. (7) is satisfied, that the particle crosses the peak and makes the transition:

$$\Delta \leq \left(\eta^* - \frac{h}{L} \right) \Lambda^{-[(\Lambda-1)L^2 + L\eta^* - \Lambda h]/[(\Lambda-1)(L\eta^* - h)]}$$

$$\equiv \Delta_c(L, h, \eta^*, \Lambda). \tag{14}$$

This expression contains the previous result calculated only for the tent map [29], which is directly shown by setting $\Lambda = 2$ and $\eta^* = 1/2$.

In the following, we clarify in what limit the simple expression of the maximum distance Δ_c, which is comparable
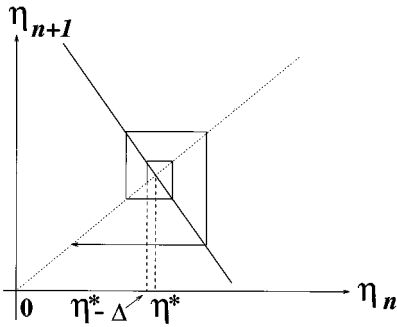
FIG. 4. Emergence of short-time correlation of the chaotic noise (case B): The chaotic sequence η_n leaves an unstable fixed point η^* with oscillating around the unstable fixed point. The strong correlation starts when the noise is injected in the Δ neighbor of the unstable fixed point.

to the scaling of the barrier crossing probability P shown in Ref. [20], is obtained. First, we rewrite the maximum distance Δ_c, obtained in Eq. (14):

$$\Delta_c = \left(\eta^* - \frac{h}{L} \right) \Lambda^{-\alpha}, \qquad (15)$$

where

$$\alpha = \frac{L(1 + (L\eta^* - \Lambda h)/[(\Lambda - 1)L^2])}{\eta^* - h/L}. \qquad (16)$$

By the following limits, we get

$$\alpha \overset{(L\eta^* - \Lambda h)/[(\Lambda-1)L^2] \ll 1}{\longrightarrow} \frac{L}{\eta^* - h/L}, \qquad (17)$$

$$\overset{h/(\eta^* L) \ll 1}{\longrightarrow} \frac{L}{\eta^*}. \qquad (18)$$

This yields the same scaling expression as the barrier crossing probability in Ref. [20]:

$$P(L) \sim (1/\Lambda)^{L/|\eta^*|} \qquad (19)$$

if $\Delta_c \propto P$. The first limit in Eq. (17) is valid if the width of the potential barrier is sufficiently large. The second in Eq. (18) is valid if the effect of the potential gradient is sufficiently small.

B. Chaotic sequence leaving an unstable fixed point with oscillation

In this subsection, we discuss the maximum distance Δ_c for the case where the chaotic noise leaves an unstable fixed point with oscillation (Fig. 4). In a similar way as Eq. (6) and Eq. (7), we write the inequalities in which the particle can cross the potential barrier:

$$|\eta_N - \eta^*| \leqslant \eta^* - h/L, \qquad (20)$$

$$x_{N+1} \geqslant L. \qquad (21)$$

The chaotic noise works to drive the particle against the gradient of the potential up to $n = N$ if Eq. (20) is satisfied. One

finds that the noise may drive the particle against the potential, even if Eq. (20) is not satisfied: This happens when $\eta_N - \eta^* > \eta^* - h/L$. However, this ambiguity in the inequality [Eq. (20)] does not change the following result of Δ_c without a prefactor as shown later [Eqs. (31) and (32)].

Let us set

$$\eta_1 = \eta^* \pm \Delta \quad (\Delta \ll 1), \qquad (22)$$

then one finds

$$\eta_N = \eta^* \pm (-1)^{N-1} \Lambda^{N-1} \Delta. \qquad (23)$$

We obtain through Eq. (4)

$$x_{N+1} = \sum_{k=1}^{N} \eta_k - \frac{h}{L}(N-1) \qquad (24)$$

by setting $x_1 = 0$. One obtains from Eq. (23)

$$\left\langle \sum_{k=1}^{N} \eta_k \right\rangle_{\pm} = N\eta^*, \qquad (25)$$

where $\langle \rangle_{\pm}$ means an average over the sign, $\pm$, of η_1. By use of this averaging procedure, Eq. (24) is replaced by

$$x_{N+1} = N\eta^* - \frac{h}{L}(N-1). \qquad (26)$$

Inserting Eq. (26) into Eq. (21), one obtains

$$N\eta^* - \frac{h}{L}(N-1) \geqslant L. \qquad (27)$$

Inserting Eq. (23) into Eq. (20), we get

$$N \leqslant N_c \equiv \log_\Lambda \frac{\eta^* - h/L}{\Delta} + 1. \qquad (28)$$

Equation (28) gives the maximum N for which the noise keeps driving the particle upward against the potential as a function of η^*, Δ, and h/L. Replacing N of Eq. (27) by N_c, we obtain the maximum value of Δ for which Eq. (21) is satisfied, that the particle crosses the peak and makes the transition:

$$\Delta \leqslant \Delta_c \equiv \left(\eta^* - \frac{h}{L} \right) \Lambda^{-(L-\eta^*)/(\eta^* - h/L)}, \qquad (29)$$

where we used an approximation for $\Delta \ll 1$.

By the following limits, we get simpler expressions of Δ_c such as Eqs. (17) and (18):

$$\Delta_c \propto \Lambda^{-(L-\eta^*)/(\eta^* - h/L)} \overset{\eta^*/L \ll 1}{\longrightarrow} \Lambda^{-L/(\eta^* - h/L)} \overset{h/(\eta^* L) \ll 1}{\longrightarrow} \Lambda^{-L/\eta^*}. \qquad (30)$$

The last expression is the same as that obtained for the chaotic noise, which monotonically leaves an unstable fixed point [Eq. (18)].

We note that the main result is not altered even if the right-hand side of Eq. (20) is replaced by an arbitrary constant A, namely,
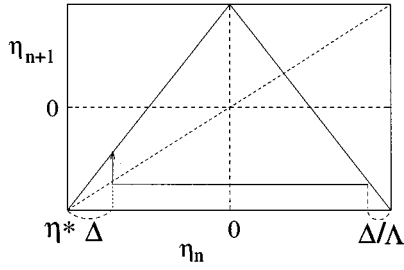
FIG. 5. Injection mechanism of chaotic sequence near to the Δ neighbor of the unstable fixed point η^* (case A). In this case of the tent map, the slope $\Lambda = 2$.

$$|\eta_N - \eta^*| \leqslant A. \tag{31}$$

In this case, we obtain

$$\Delta \leqslant \Delta_c \equiv A \Lambda^{-(L-\eta^*)/(\eta^*-h/L)}, \tag{32}$$

which is the same as the previous expression [Eq. (29)] without a prefactor A.

C. Barrier crossing probability and Δ_c

In this subsection we argue the relationship between the maximum distance Δ_c and the barrier crossing probability. First, we restrict ourselves to the chaotic noise generated by the tent map function for demonstration. This function has two unstable fixed points, $\eta_-^* = -1/2$ and $\eta_+^* = 1/6$, the effect of which corresponds to the case of Figs. 3 and 4, respectively. The former case is discussed here (Fig. 5). Suppose that the barrier crossing event of the particle starts when the chaotic noise is injected in the Δ neighbor of the unstable fixed point η^*. Then the event occurs only when the chaotic noise was in an interval, Δ/Λ, just before the start of the barrier crossing as shown in Fig. 5. Thus, the sum of the invariant density over the interval, Δ/Λ, gives the barrier crossing probability [30]. Therefore, we get the barrier crossing probability of the particle in a negative direction caused by the effect of the unstable fixed point, η_-^*, of the tent map function:

$$P_- = \Delta_c/\Lambda, \tag{33}$$

because the invariant density of the tent map function is uniform: $\rho(\eta) = 1$.
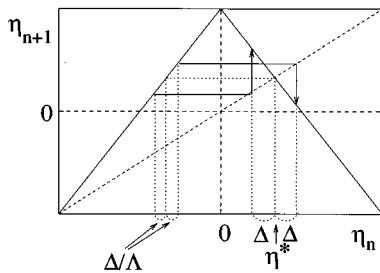

FIG. 6. Injection mechanism of chaotic sequence near to the Δ neighbor of the unstable fixed point η^* (case B). In this case of the tent map, the slope $\Lambda = 2$.
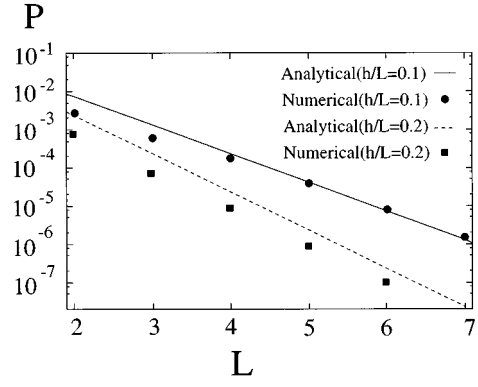

FIG. 7. Barrier crossing probabilities of the dissipative particle subject to a tent map chaos, where the slopes of a piecewise linear potential are $h/L = 0.1$ and 0.2. Both numerical and analytical results are shown.

The barrier crossing probability in a positive direction caused by η_+^* can be obtained in the same way (Fig. 6). However, the barrier crossing probability in a positive direction is much smaller than that in a negative direction because the amplitude of the unstable fixed point, η_+^*, in a positive direction is much smaller than that in a negative direction: $|\eta_-^*| \gg |\eta_+^*|$. This is immediately confirmed by the expression $P \propto \Delta_c \propto \Lambda^{-L/\eta^*}$ [see Eqs. (18) and (30)]. Therefore, the effect of one unstable fixed point, η_-^*, dominates the overall barrier crossing probability of the system with a sufficient barrier width L.

The comparison between the analytical result and the numerical one is shown for the noise generated by the tent map chaos (Fig. 7). The present theory sufficiently predicts the exponential decreased rate of the barrier crossing probability as to the potential width L for $L \gg |\eta|_{\max}$. The disagreement of the constant factor, especially for the case $h/L = 0.2$, may be attributed to the ambiguity of the assumption of the initial condition of the particle: $x_1 = 0$. When the chaotic noise is generated by the tent map, the noise drives the particle by $\eta_0 \sim +0.5$ in the positive direction just before the coherent drives caused by $\eta_-^* = -0.5$ in a negative direction for the barrier crossing. The greater the slope of the potential, the stronger this effect works, because this antidrive effect due to $\eta_0 \sim +0.5$ needs additional kicks in a negative direction for the noise to drive the particle, roughly equal to $L/(\eta^* - h/L)$, similar to in the next subsection.

In general, the invariant density of a chaotic map is not uniform [26,27]. Thus, a simple analytical expression of the barrier crossing probability cannot generally be obtained. Therefore, we derive a formal solution of the barrier crossing probability. Let there be a set I: $I = \{\eta | f(\eta) \in U(\eta^*;\Delta_c)$ and $\eta \notin U(\eta^*;\Delta_c)\}$, where $U(\eta^*;\Delta)$ is the Δ neighborhood of an unstable fixed point η^* and f is a chaotic map. Then, the sum of the invariant density over a set I gives the desired expression of the barrier crossing probability:

$$P = \int_{\eta \in I} d\eta \, \rho(\eta), \tag{34}$$

where $\rho(\eta)$ is the invariant density of the chaotic map.

If the invariant density of a chaotic map in the region I is so smooth that it can be approximated by a constant in the small region, the scaling form of the barrier crossing probability P is the same as that of Δ_c without a prefactor (such as Λ^{-1}). This explains the coincidence of the scaling forms between Δ_c [Eq. (18)] and the barrier crossing probability P [Eq. (19)].

D. Intuitive interpretation of barrier crossing probability

The analytical result of the barrier crossing probability is easily understandable by a physical insight. As mentioned in the last subsection, the barrier crossing probability and the critical distance Δ_c can have the same scaling form for several kinds of chaotic noise where an invariant density is so smooth that it can be approximated by a constant in a small region I for the injection to the unstable fixed point. We discuss this case for simplicity.

Then, the scaling form of the barrier crossing probability as in Eqs. (17) and (30) is

$$P \propto \Delta_c \propto \Lambda^{-L/(\eta^* - h/L)}. \tag{35}$$

The factor $\eta^* - h/L$ is a displacement of the particle when the particle is kicked by one chaotic noise at an unstable fixed point. Therefore the value $L/(\eta^* - h/L)$ gives the number of the kicks necessary for the particle at a basin of the potential to cross the potential barrier; we used an approximation where a chaotic noise having almost the same value as the unstable fixed point η^* during the condition, Eq. (6) or Eq. (20) is satisfied. The approximation can be justified at least for the piecewise linear maps, as found in the analytical derivation of the barrier crossing probability.

Because the value $L/(\eta^* - h/L)$ gives the number of necessary kicks by chaotic noise, we can understand the exponential dependence of the barrier crossing probability on the value $L/(\eta^* - h/L)$. Suppose that

$$|\eta_1 - \eta^*| \leqslant \Delta_s, \tag{36}$$

which satisfies Eq. (6) or Eq. (20), and the particle crosses the barrier. If the factor $L/(\eta^* - h/L)$ increases by one, the following inequality for the noise, η', must be satisfied for the barrier crossing:

$$|\eta_1' - \eta^*| \leqslant \Delta_s \Lambda^{-1}, \tag{37}$$

because Eq. (37) is equivalent to $|\eta_2' - \eta^*| \leqslant \Delta_s$, where the noise continues to drive the particle N times after $n=2$. Thus the unit increase of $L/(\eta^* - h/L)$ decreases the measure of set I by Λ^{-1}. Therefore, the barrier crossing probability depends exponentially on the value $L/(\eta^* - h/L)$, with the base Λ when the distribution of the invariant density is sufficiently smooth.

IV. AN APPLICATION TO THE LOGISTIC MAP

In the previous discussions including Ref. [20], we have argued only the effect of chaotic noise generated by a piecewise linear map with uniform invariant density. Therefore we consider in this section the validity of the present analytical results by applying them to the map, which is a nonpiece-
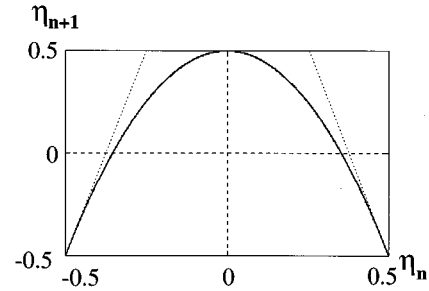

FIG. 8. A form of a logistic map, $\eta_{n+1} = 0.5 - 4\eta_n^2$. Dotted lines show the linearized slope for approximation of the analytical approach. The linearized slope $\Lambda = 4$.

wise linear map and has a nonuniform invariant density. For this purpose, we use a logistic map [26,27] as a chaotic noise, because an analytical expression of the invariant density is available. The chaotic sequence of the logistic map appears as follows:

$$\eta_{n+1} = f(\eta_n) = 1/2 - 4\eta_n^2. \tag{38}$$

The invariant density of the logistic map is $\rho(\eta) = 1/\pi\sqrt{(\eta + 1/2)(\eta - 1/2)}$, $[-1/2 \leqslant \eta < 1/2$, otherwise $\rho(\eta) = 0]$.

Because analytical expressions of Δ_c for the barrier crossing probability were derived only for a piecewise linear map, we have to make an approximation to obtain the barrier crossing probability induced by the logistic map. As noted previously, the clustering event of chaotic noise near an unstable fixed point occurs when η_n is injected near the unstable fixed point. Therefore, it seems valid to linearize the logistic map both around an unstable fixed point ($\eta \sim -0.5$) and around the corresponding region ($\eta \sim 0.5$) to be injected near the unstable fixed point. With this procedure, we get the linearized slope of the map: $\Lambda = 4$ (see Fig. 8).

By use of this approximation, we can evaluate the barrier crossing probability for a non-uniform invariant density by using Eq. (34). The set I of Eq. (34) is $I = \{\eta | 1/2 - \Delta_c/4 \leqslant \eta < 1/2\}$. Therefore, we get an analytical expression of the barrier crossing probability:

$$P = \int_{1/2 - \Delta_c/4}^{1/2} d\eta \frac{1}{\pi\sqrt{(\eta + 1/2)(\eta - 1/2)}} = \frac{2}{\pi}\left(\sin^{-1}\sqrt{\frac{\Delta_c}{4}}\right), \tag{39}$$

where Δ_c is given by Eq. (14).

The theoretical barrier crossing probabilities of the logistic map are shown with the numerically obtained barrier crossing probabilities (Fig. 9). The data of the larger slope of the potential ($h/L = 0.2$) agree better with the theoretical predictions [Eq. (39)] than that of $h/L = 0.1$. The disagreement for the smaller slope of the potential may be attributed to the assumption of the present theory that the particle should start climbing from $x_1 = 0$ when the chaotic sequence is injected near an unstable fixed point. If the slope of the potential is small, the particle fluctuates much around one of the bottoms of the periodic potential. This fluctuation effect caused by the logistic map may be strong because the invariant density
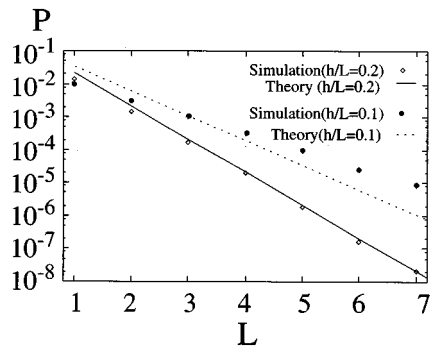
FIG. 9. Barrier crossing probabilities under the logistic map chaos, where the slopes of the piecewise linear potentials are $h/L=0.2$ and 0.1. Both numerical and theoretical results are shown.

goes to infinity at both edges of the map, $\eta = \pm 1/2$: this decreases the validity of the assumption of the present theory.

V. SUMMARY AND DISCUSSION

We showed in this paper a macroscopic feature of chaotic dynamics emerging in a multistable system: the effect of chaotic noise on the multistable system is attributed to its unstable fixed points, which reminds us of the deterministic nature of chaos. This feature appears effectively in a multistable system when the slope of the unstable fixed point of the noise, Λ, is near the ''edge of chaos,'' because the local Lyapunov index λ at the unstable fixed point is $\lambda = \ln\Lambda$ [31]. This is consistent at least with recent literature of neural networks [16,32].

The unidirectional motion of the dissipative particle in a periodic potential has been discussed in relation to the dynamics of motor proteins [24,25]. We showed that the unidirectional motion can be induced by an apparently symmetric chaotic noise even if the particle is in a symmetric multistable potential. Similar results have recently appeared with several variations [25] of our first report [29]. The authors of the papers claim that the unidirectional motion can occur in a symmetric multistable potential if an additive noise is ''temporally asymmetric.'' However, the asymmetric effect of the noises they used can be attributed to asymmetric distribution of the probability density of the noise. In this sense, the effect of the *temporally asymmetric* noise is rather static, and thus the effect of ''temporal asymmetry,'' should be discriminated from that of ''dynamical asymmetry.''

Finally, we mention that the present analytical method to estimate barrier crossing probabilities may be applied to the escape rate problem induced by other types of time correlation of the fluctuation including an intermittent chaos and nonchaotic time series: the escape rate is found to be strongly dependent on the *transient* time correlation of the additive noise [33].

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[31] If $\Lambda < 1$, η^* becomes a stable fixed point, where the map cannot generate a noisy time series. Therefore, the state of the edge of chaos, $\Lambda = 1 + \epsilon$ ($\epsilon \ll 1$), maximizes the barrier crossing probability induced by the chaotic noise.

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