Role of low-l component in deformed wave functions near the continuum threshold

doi: 10.1103/PhysRevC.72.064311

Role of low-l component in deformed wave functions near the continuum threshold

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(Received 26 September 2005; published 20 December 2005)

The structure of deformed single-particle wave functions in the vicinity of zero energy limit is studied using a schematic model with a quadrupole deformed finite square-well potential. For this purpose, we expand the single-particle wave functions in multipoles and seek for the bound state and the Gamow resonance solutions. We find that, for the $K^\pi = 0^+$ states, where K is the z-component of the orbital angular momentum, the probability of each multipole components in the deformed wave function is connected between the negative energy and the positive energy regions asymptotically, although it has a discontinuity around the threshold. This implies that the $K^\pi = 0^+$ resonant level exists physically unless the $l = 0$ component is inherently large when extrapolated to the well bound region. The dependence of the multipole components on deformation is also discussed.

DOI: 10.1103/PhysRevC.72.064311 PACS number(s): 21.10.Pc, 21.60.Jz, 24.30.Gd

I. INTRODUCTION

The physics of nuclei located far from the β stability line has been one of the main current subjects of nuclear physics. One of the unique properties of drip-line nuclei is that the Fermi level lies close to zero. An understanding of single-particle levels in the continuum is essential in describing the nuclear structure close to, and beyond, the drip line, since the shell structure of both bound and continuum levels plays an important role in many-body correlations such as deformation and pairing.

It has been argued recently that, as the binding energy approaches zero, the s-wave component of a bound single-particle wave function behaves uniquely in a deformed potential, and plays a dominant role in Nilsson levels with $\Omega^\pi = 1/2^+$ [1,2]. Naively, resonant levels can be considered as an extension of bound states into the positive energy regime. Therefore, if the s-wave component remains dominant in the continuum, the level with $\Omega^\pi = 1/2^+$ might not exist as a physical state. Notice that, for a Nilsson Hamiltonian [3], single-particle levels with $\Omega = 1/2$ belonging to a high-j orbit comes down in energy in a prolately deformed potential. These states play an important role in generating the deformed shell structure. It is therefore crucially important to investigate the role of the low-l component in a deformed wave function for $\Omega = 1/2$ states and its transition from bound to resonant levels.

The structure of deformed single-particle levels in the continuum has been investigated in a few publications. In Ref. [4], the resonance energy of negative parity states was studied by employing the Gamow wave function. The analytic continuation in the coupling constant (ACCC) method was applied to study single-particle resonance states in spherical and deformed nuclei [5]. Using the multichannel scattering approach, Ref. [6] has studied how the single-particle energies change from bound to resonant levels when the depth of the potential is varied. In order to fully understand the structure of deformed single-particle levels in the continuum, however, a detailed study of the wave function components is still necessary, in addition to the resonance energy itself.

In this paper, we investigate the structure of deformed wave functions around zero energy using the Gamow state representation for a resonant state. To this end, we use a schematic model: A Y_{20} deformed finite square-well potential without spin-orbit force. This enables us to determine the single-particle wave function analytically. To use the Gamow state for resonance has a certain advantage in analyzing the deformed wave function. That is, we are able to treat the bound and the resonant levels on the same footing, because the Gamow states are normalizable just like the bound states [7]. It is then straightforward to see how the fraction of each component in the deformed wave functions changes when the single-particle level changes its character from bound to resonant. A slight disadvantage of this approach is that the expectation value with the Gamow states, including the probability of wave function components, becomes complex numbers. However, this is not a big defect for our purpose, since the physical quantity of the expectation values can be obtained by taking their real part [8,9].

The paper is organized as follows. In the next section, we present our model for a deformed single-particle potential. Numerical results and discussion are given in Sec. III. Finally, we summarize the paper in Sec. IV.

II. MODEL

Our purpose is to study the structure of the wave function in a deformed single-particle potential. To this end, we employ a schematic model for the single-particle potential, that is, a deformed square-well potential without the spin-orbit force,

$$V(\boldsymbol{r}) = -V_0\,\theta(R(\hat{\boldsymbol{r}}) - r), \tag{1}$$

where $R(\hat{\boldsymbol{r}}) = R_0(1 + \beta_2 Y_{20}(\hat{\boldsymbol{r}}))$. For simplicity, we expand this potential up to the first order of deformation parameter β_2

and obtain

$$V(\boldsymbol{r}) \simeq -V_0 \left[\theta(R_0 - r) + R_0\beta_2 Y_{20}(\hat{\boldsymbol{r}})\delta(r - R_0)\right]. \quad (2)$$

In order to solve the Schrödinger equation with this potential, we expand the wave function in the multipoles as

$$\Psi_K(\boldsymbol{r}) = \sum_l \frac{u_{lK}(r)}{r} Y_{lK}(\hat{\boldsymbol{r}}), \quad (3)$$

where the quantum number $K(=\Lambda)$ is the z-component of the orbital angular momentum l. By projecting out each multipole component, we obtain the coupled equations for the radial wave functions given by

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - V_0\theta(R_0 - r) + \frac{\hbar^2 l(l+1)}{2mr^2} - E\right] u_{lK}(r) = V_0 R_0 \beta_2 \delta(r - R_0) \sum_{l'} \langle lK|Y_{20}|l'K\rangle u_{l'K}(r). \quad (4)$$

For the positive energy solution, $E > 0$, we impose the boundary condition corresponding to the Gamow state for resonance. That is, the wave function is regular at the origin and satisfies the outgoing boundary condition $u(r) \sim e^{ikr}$ asymptotically. This boundary condition is satisfied only if the energy is complex, $E = \hbar^2 k^2/2m = E_R - i\Gamma/2$, where E_R and Γ are the resonance energy and the width, respectively. In the case of $\Gamma = 0$ and $E_R < 0$, the Gamow state wave function is equivalent to the bound state wave function, which satisfies the decaying asymptotics $u(r) \sim e^{-\alpha r}$, where $\alpha = \sqrt{-2mE_R/\hbar^2}$.

The solutions of the coupled-channels equations (4) therefore read (we omit the subscript K for simplicity of notation),

$$u_l(r) = \begin{cases} A_l\, r j_l(k_1 r) & (r < R_0), \\ B_l\, r h_l^{(+)}(kr) & (r \geqslant R_0), \end{cases} \quad (5)$$

where $k_1 = \sqrt{2m(E + V_0)/\hbar^2}$, $k = \sqrt{2mE/\hbar^2}$, and $j_l(x)$, $h_l^{(+)}(x)$ are the spherical Bessel and Hankel functions, respectively. The amplitudes A_l and B_l are determined by the matching condition at $r = R_0$ given by

$$u_l(R_-) = u_l(R_+), \quad (6)$$

$$-\frac{\hbar^2}{2m}[u_l'(R_+) - u_l'(R_-)] = V_0 R_0 \beta_2 \sum_{l'} \langle lK|Y_{20}|l'K\rangle u_{l'}(R_0), \quad (7)$$

where $R_\pm$ represents $\lim_{\varepsilon\to 0} R_0 \pm \varepsilon$.

The bound state wave function is normalized as

$$1 = \int d\boldsymbol{r}\, |\Psi_K(\boldsymbol{r})|^2 = \sum_l N_l, \quad (8)$$

where

$$N_l = \int_0^\infty dr |u_l(r)|^2. \quad (9)$$

The Gamow state wave function can be also normalized by introducing the regularization factor as Zel'dovich proposed [10]

$$N_l = \lim_{\epsilon\to 0} \int_0^\infty dr e^{-\epsilon r^2} \{u_l(r)\}^2 \quad (10)$$

$$= \int_0^{R_0} dr \{A_l r j_l(k_1 r)\}^2 + \lim_{\epsilon\to 0} \int_{R_0}^\infty dr e^{-\epsilon r^2} \{B_l r h_l^{(+)}(kr)\}^2. \quad (11)$$

Using a property of the spherical Bessel function [11], one can evaluate the first term as

$$\int_0^{R_0} dr \{A_l\, r j_l(k_1 r)\}^2 = \frac{A_l^2 R_0^3}{2}(\{j_l(k_1 R_0)\}^2 - j_{l-1}(k_1 R_0) j_{l+1}(k_1 R_0)). \quad (12)$$

The second term can be also evaluated using the contour integral method or equivalently the complex scaling method (CSM). The result is given by [12]

$$\lim_{\epsilon\to 0} \int_{R_0}^\infty dr e^{-\epsilon r^2} \{B_l\, r h_l^{(+)}(kr)\}^2 = -\frac{B_l^2 R_0^3}{2}(\{h_l^{(+)}(kR_0)\}^2 - h_{l-1}^{(+)}(kR_0) h_{l+1}^{(+)}(kR_0)). \quad (13)$$

Note that the fraction of multipole components N_l is in general a complex number for the Gamow state wave function.

III. RESULTS AND DISCUSSION

Let us now discuss the behavior of the low-l components in deformed wave functions. In Sec. III A, we vary the potential depth for a fixed deformation parameter, while we vary the deformation parameter for a fixed potential depth in Sec. III B.

A. Dependence on potential depth

We first study the wave functions at a fixed deformation, $\beta_2 = 0.5$. Figure 1 shows the real and imaginary parts of the energy for a $K^\pi = 0^+$ state in varying the potential depth V_0.

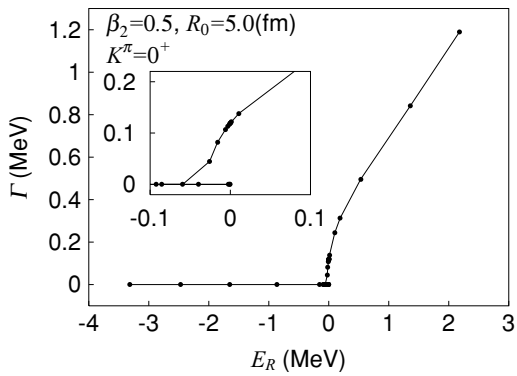

FIG. 1. The real part of the energy and the resonance width for a $K^\pi = 0^+$ state with various potential depths. In the inset, the behavior around zero energy is enlarged. The corresponding potential depths are shown in Fig.2.

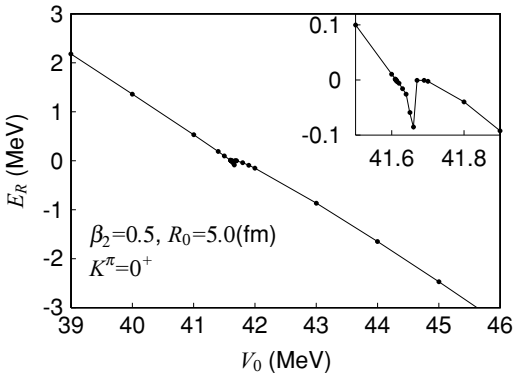

FIG. 2. The real part of the energy for $K^\pi = 0^+$ state as a function of the potential depth. In the inset, the behavior around zero energy is enlarged.

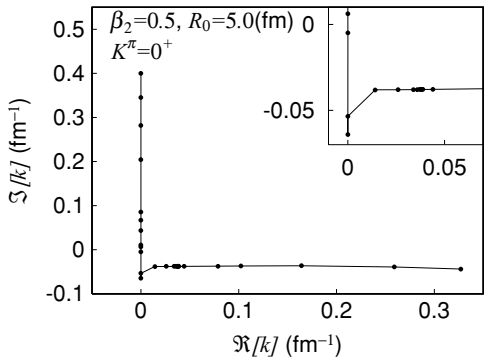

FIG. 4. The trajectory for the pole of S-matrix in the complex momentum plane. In the inset, the behavior around zero momentum is enlarged.

The correspondence between the potential depth and the real part of the energy is shown in Fig. 2. We observe that the width is quite large even for small values of positive energy. This large width is caused by the admixture of the $l = 0$ component in the wave function. Indeed, as shown in Fig. 3, in the small positive energy region (0.1 MeV $< \Re(E) <$ 1.0 MeV), the behavior of the width is consistent with the relation expected for the s-wave resonance state [4,6,13],

$$\Gamma \propto \Re(E)^{l+1/2} \times \Re(N_l)\Big|_{l=0}, \tag{14}$$

where $\Re(E)$ denotes the real part of E.

Below E_R =0.1 MeV, the width is larger than the solid line, which predicts $\Gamma = 0$ at $E_R = 0$. Also, we found a nonmonotonic behavior in the eigenenergy between $V_0 =$ 41.62 and 41.68 MeV, where the solution has $E_R < 0$ and $\Gamma > 0$, as is shown in the inset of Fig. 2. These facts might be related to the possible presence of the antibound ($\kappa = 0, \gamma > 0$) and/or "crazy" resonant ($0 < \kappa < \gamma$) states, where $k = \kappa - i\gamma$, as presented in Ref. [14] for a spherical square-well potential (see Fig. 1 of Ref. [14]). In order to study the presence of these states in the present deformed potential, we plot the trajectory for the pole of S-matrix in the complex momentum plane in Fig. 4. As the potential depth is made shallower, the pole comes down along the imaginary axis. In the present example, the pole goes through $k = 0$ and comes into the antibound (virtual) state region. With a shallower potential, the pole begins to have a finite real part, which corresponds to the "crazy" resonance, and eventually comes into the normal resonance region ($\kappa > \gamma > 0$). It is thus apparent that this state has finite width even in the limit of $E_R \to 0+$.

Above 1.0 MeV also, the width is larger than that expected by Eq. (14). This is due to the fact that the relation Eq. (14) is valid only for small values of k [13].

In Fig. 2, we see that the slope of the single-particle energy as a function of the potential depth, dE/dV_0, or equivalently dE/dA, where A is the mass number, becomes smaller in approaching the zero binding energy. For a spherical square-well potential, it has been shown that $dE_l/dA \to 0$ for $l = 0$ in the limit of zero binding [15]. This is due to the fact that the s-wave function can be easily extended outside the nuclear potential and also the kinetic energy is reduced due to the absence of the centrifugal barrier [15]. This property implies that the $l = 0$ component becomes dominant in a deformed wave function around the zero-binding region. On the other hand, the slope has a finite value in the positive energy region even in the limit of zero energy, thus the slope has a discontinuity around zero energy. Therefore, care must be taken, as discussed in Ref. [14], when one estimates the energy of a deformed resonant level with $K^\pi = 0^+$ by using the ACCC method [5].

We now discuss the energy dependence of the fraction of the multipole components in the deformed wave function. Figure 5 shows the real part of the fraction for each multipole component in the Gamow state wave function with $K^\pi = 0^+$. When the binding energy approaches zero, the s-wave component in the deformed wave function becomes dominant. In contrast, in the positive energy region, all the multipole components have a finite value even in the zero energy limit and show a similarity with the well bound cases. As we will discuss in the next section (see Fig. 10 below), the state shown in Fig. 5 originates from the $2d$ orbit in the spherical limit. This state couples with the lower-lying $2s$, $1g$ and the higher-lying $3s$ states. The dominant component is $l = 4$ both in well bound and in resonant levels, as one sees in Fig. 5. This suggests that both the well bound and the resonant levels have a similar property to each other and the intuitive picture that the resonant

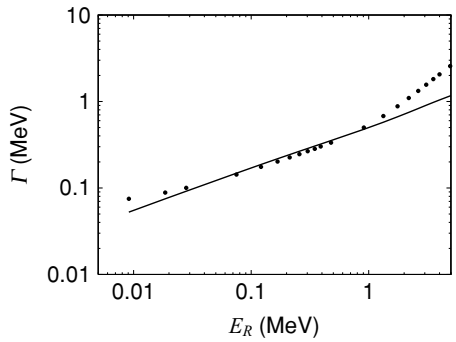

FIG. 3. Same as Fig. 1, but in the logarithmic scale. The solid line is an expectation for the pure s-wave configuration given by Eq. (14).

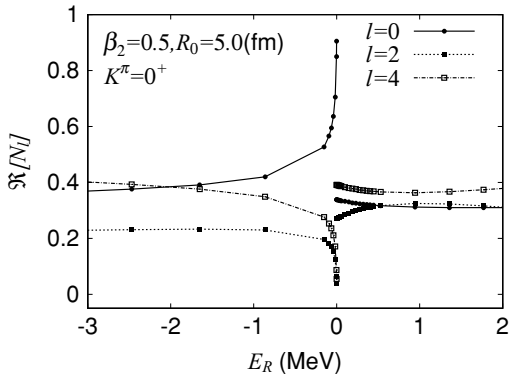

FIG. 5. The real part of the fraction for each multipole component N_l for the $K^\pi = 0^+$ state. The solid, dotted, and dot-dashed lines indicate the $l = 0$, 2 and $l = 4$ components, respectively.

level is an extension of a bound level into the continuum is valid.

Only at the limit of zero binding, the singular behavior of the $l = 0$ component appears. This is entirely due to the property of the normalization integral, Eq. (13). Since the Gamow state wave function is equivalent to the bound state wave function for $E_R < 0$, $\Gamma = 0$, Eq. (13) holds both for the resonance and the bound states. For small values of k, Eq. (13) is proportional to k^{2l-1} as discussed in Refs. [1,16], that diverges only for $l = 0$ as $k \to 0$. When the total wave function Ψ_K is normalized according to Eq. (8), then only the s-wave component is allowed in the wave function [1]. This condition is always met for the bound state when the binding energy approaches the threshold. In principle, the same consideration can apply also to the resonance state when the resonance energy approaches zero from the positive energy side. However, as we show in Fig. 1, the resonance state acquires a relatively large width even when the real part of the energy is infinitesimally small. Since k is defined as $k = \sqrt{2m(E_R - i\Gamma/2)/\hbar^2}$, it remains a constant even if E_R itself approaches zero. This leads to the disappearance of the "s-wave dominance" in the positive energy side.

We next study the case for $K^\pi = 0^-$. In Figs. 6 and 7, we show the dependence of the single-particle energy on the potential depth. In contrast to the case for $K^\pi = 0^+$,

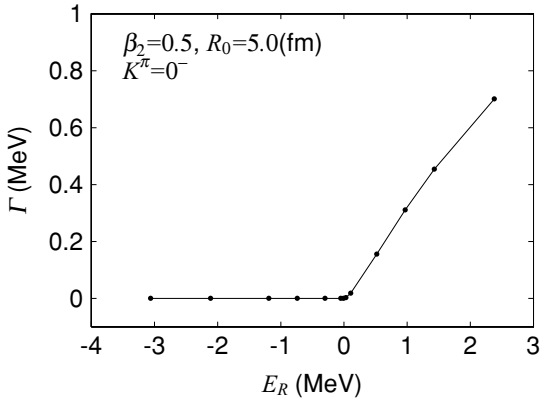

FIG. 6. Same as Fig. 1, but for a $K^\pi = 0^-$ state.

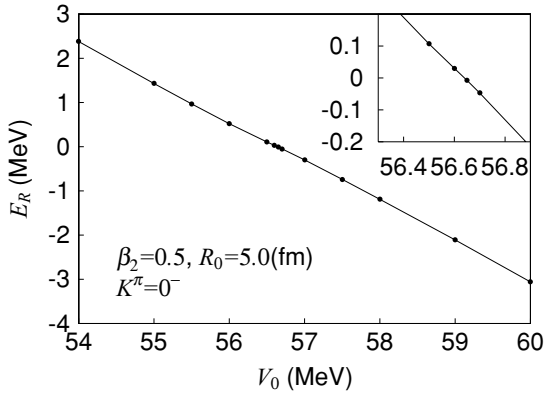

FIG. 7. Same as Fig. 2, but for the $K^\pi = 0^-$ state.

due to the presence of the centrifugal barrier, we do not see any singular behavior around zero energy. Single-particle energies are connected smoothly when changing the potential depth, and the width increases gradually in the small positive energy region. Figure 8 shows the fraction of each multipole component in the Gamow state wave function. As the binding energy approaches zero, the p-wave component becomes relatively large, that is consistent with the dominance of low-l component in the limit of zero binding energy discussed in Ref. [2]. The fractions are connected smoothly and asymptotically in the bound and resonant regions.

B. Deformation dependence

In this subsection, we study the deformation dependence of the low-l component in deformed wave functions for a fixed potential depth. In the realistic situation, the location of single-particle levels changes as a function of nuclear deformation. Especially, the levels of $\Omega = 1/2$ ($K = 0$) with (without) spin-orbit force belonging to high-j (high-l) orbit in the spherical limit play an important role in nuclear deformation.

Figure 9 shows the resonance energy and width when the deformation parameter is varied from $\beta_2 = 0.0$ to 0.5. The potential depth V_0 and the radius R_0 are set to be 45.0 MeV

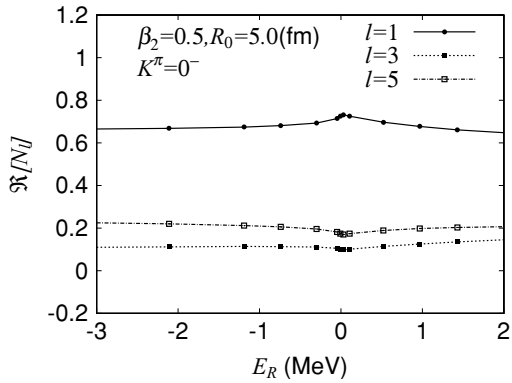

FIG. 8. The real part of the fraction for each multipole components N_l for the $K^\pi = 0^-$ state. The solid, dotted, and dot-dashed lines indicate the $l = 1$, 3 and $l = 5$ components, respectively.

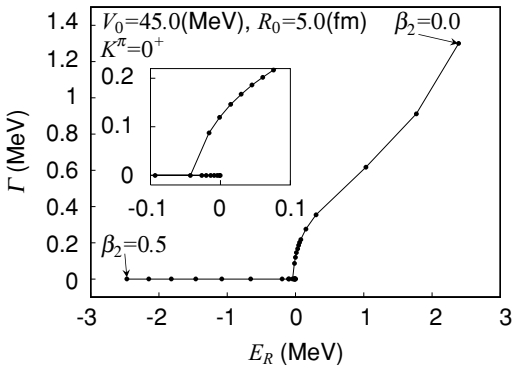

FIG. 9. Same as Fig. 1 except for varying the deformation. This level corresponds to the one originating from the 2d orbit in the spherical limit. The deformation dependence of the single-particle energies is shown in Fig. 10.

and 5.0 fm, respectively. This state belongs to the 2d orbit at $\beta_2 = 0.0$ as shown in Fig. 10. At around zero energy, we see the similar behavior as in Fig. 1: The width is quite large even for the small values of positive energy, which implies that the $l = 0$ component is responsible for the width of the resonant level.

The corresponding wave function components for this state are shown in Fig. 11. As in Fig. 5, we see the singular behavior for the s-wave component at around zero-binding energy, corresponding to the "s-wave dominance" in the limit of zero binding. Except for the zero-energy region, however, we see that the fraction of each multipole components is linked asymptotically and is smoothly connected to the d-state resonant level in the spherical limit. From this calculation, it is evident that the singular behavior of the $l = 0$ component for the $K^\pi = 0^+$ state occurs only just below the continuum threshold and this state is connected to the physical resonant level in the continuum. Furthermore, the fraction of each-l components in the deformed wave function is connected smoothly from the bound to the resonant levels except for the region near the threshold.

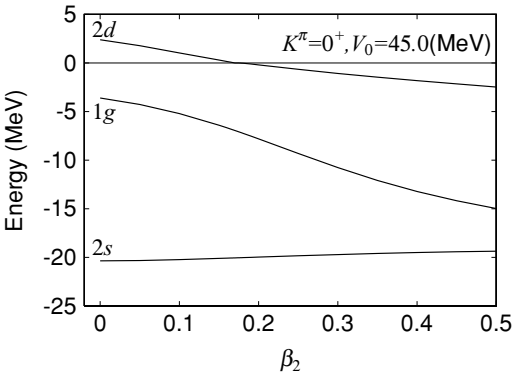

FIG. 10. Single-particle energies for the $K^\pi = 0^+$ state as a function of deformation parameter β_2. The potential depth is $V_0 =$ 45.0 (MeV), and the potential radius $R_0 = 5.0$ (fm).

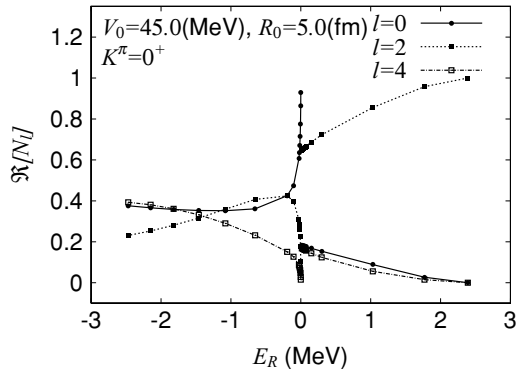

FIG. 11. Same as Fig. 5 except for varying the deformation.

IV. SUMMARY

We have analyzed the structure of the deformed wave functions around zero energy using the Gamow state wave function for resonance, with which one can treat the resonant and bound levels on the same footing and thus analyze the wave function continuously from the negative to the positive energy regions. For this purpose, we developed a schematic model with a deformed square-well potential. Since the wave functions can be obtained analytically with this model, detailed analyses of the deformed wave functions were possible. For a $K^\pi = 0^+$ state, we have found a singularity in the resonance width as well as in the s-wave component in the deformed wave function at around zero energy. That is, the width becomes considerably large even in the small positive energy region and the $l = 0$ component approaches unity in the limit of zero binding. We have shown that the "s-wave dominance" occurs only at the threshold of continuum. Far from the zero energy region, the probability of each-l component is connected asymptotically. This implies that the $K^\pi = 0^+$ resonant level exists unless the $l = 0$ component is large inherently when extrapolated to the well bound region. In contrast, for the $K^\pi = 0^-$ state, we did not find any singular behavior even in the zero-energy limit. The single-particle energies are connected smoothly when changing the potential depth, and the width increases gradually in the small positive energy region. The probability of each-l component in the wave function is also connected smoothly and asymptotically between the bound and the resonant regions.

ACKNOWLEDGMENTS

We thank Prof. K. Matsuyanagi for his continuing encouragement and for many useful discussions. We also thank Dr. T. Myo for useful discussions on the imaginary part of expectation values for a Gamow state. We acknowledge discussions with the member of the Japan-U.S. Cooperative Science Program "Mean-Field Approach to Collective Excitations in Unstable Medium-Mass and Heavy Nuclei." This work was supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology by Grant-in-Aid for Scientific Research under Contract No. 16740129.

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