# Energetics of a forced thermal ratchet

| | |
|---|---|
| 著者 | 本堂　毅 |
| journal or publication title | Physical review letters |
| volume | 80 |
| number | 24 |
| page range | 5251-5254 |
| year | 1998 |
| URL | http://hdl.handle.net/10097/35479 |

doi: 10.1103/PhysRevLett.80.5251

# Energetics of a Forced Thermal Ratchet

Hideki Kamegawa,* Tsuyoshi Hondou,† and Fumiko Takagi‡
*Department of Physics, Tohoku University, Sendai 980-8578, Japan*
(Received 17 February 1998)

We formulate energetics of the forced thermal ratchet [M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993)] and evaluate its efficiency of energy transformation. We show that the presence of thermal fluctuation cannot increase the efficiency of the energy transformation in the original system of Magnasco, which is contrary to his claim that "There is a region of the operating regime where the efficiency is optimized at finite temperatures." We also discuss the maximum efficiency of the forced thermal ratchet. [S0031-9007(98)06408-4]

PACS numbers: 05.40.+j, 87.10.+e

Molecular motors are known to have the high efficiency of energy transformation even in the presence of thermal fluctuation [1]. Motivated by the interesting fact, recent studies of thermal ratchet models [2] are showing how work should be extracted from nonequilibrium fluctuations [3–11]. Fluctuation-induced work has been a subject not only for biological interest but also for the foundation of statistical physics: Thermal fluctuation-induced motion in ratchet systems were also investigated earlier [2–4].

One of the important findings among ratchet models was by Magnasco [7] where he showed that the Brownian particle in periodic potential with broken symmetry, the so-called *ratchet*, can exhibit a nonzero net drift if the particle is subject to an external fluctuation having sufficient time correlation. He also studied the temperature dependence on the fluctuation-induced current in the system and showed that the current can be maximized at a finite temperature. This interesting finding has been interpreted that the existence of thermal fluctuation does *not* disturb the fluctuation-induced motion and even facilitates the efficiency of energy transformation.

The latter claim is quite surprising, because thermal fluctuation is naively considered to disturb effective operation of a *machine*. In mesoscopic systems as in molecular motors, one cannot escape from the effect of thermal fluctuation. Therefore, Magnasco's finding has been followed and analyzed further by much literature (see references in Ref. [11]). We show, however, this interpretation is incorrect, by energetic analysis [12,13] of Magnasco's original system [7]. The efficiency of energy transformation is not maximized at finite temperature: The maximum efficiency is realized in the absence of thermal fluctuation. It turns out that the following problem has not yet been solved: Can thermal fluctuation facilitate the efficiency of energetic transformation from force fluctuation into work in general ratchet systems?

Let us consider a forced ratchet system subject to an external load against global motion:

$$\frac{dx}{dt} = -\frac{\partial V_0(x)}{\partial x} + \xi(t) + F(t) - \frac{\partial V_L(x)}{\partial x}, \quad (1)$$

where $x$ represents the state of the ratchet, $\xi(t)$ is a thermal noise satisfying $\langle \xi(t)\xi(t')\rangle = 2kT\delta(t - t')$, "$\langle \cdots \rangle$" is an operator of ensemble average, $F(t)$ is an external fluctuation with temporal period $\tau$, $F(t + \tau) = F(t)$, $\int_0^\tau dt\, F(t) = 0$, and $V_L$ is a potential due to the load, $\frac{\partial V_L}{\partial x} = l > 0$. The geometry of the potential, $V(x) = V_0(x) + V_L(x)$, is displayed in Fig. 1. $V_0(x)$ is a piecewise linear and periodic potential with period $\lambda$. The spatial period of the potential is $\lambda = \lambda_1 + \lambda_2$, and $\Delta = \lambda_1 - \lambda_2 (>0)$ is a symmetry breaking amplitude. For a finite work against load, we assume the condition $l \leq A$ throughout this paper. The ratchet system can transform the external fluctuation into work (see, for review, Ref. [11]). The model that Magnasco discussed [7] is a system where the external load is omitted.

The Fokker-Planck equation [14] of the system is written

$$\frac{\partial P(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0,$$
$$J(x,t) = -kT\frac{\partial P(x,t)}{\partial x} + \left(-\frac{\partial V_0(x)}{\partial x} + F(t) - l\right)P(x,t), \quad (2)$$

where $P(x,t)$ is a probability density and $J(x,t)$ is a probability current. We apply the periodic boundary condition on the probability density, $P(x,t) = P(x + \lambda, t)$, and normalize it in the spatial period $\lambda$. If $F(t)$ changes slowly enough, $P(x,t)$ can be treated as quasistatic. For such fluctuation $F(t)$ of square wave of amplitude $A$ [7], an average current over the period of the fluctuation is obtained,

$$J_{\mathrm{sqr}} = \tfrac{1}{2}[J(A) + J(-A)], \quad (3)$$

where $J(A)$ is a current induced by a constant force $F(t) = A$ in Eq. (2) [15].

  

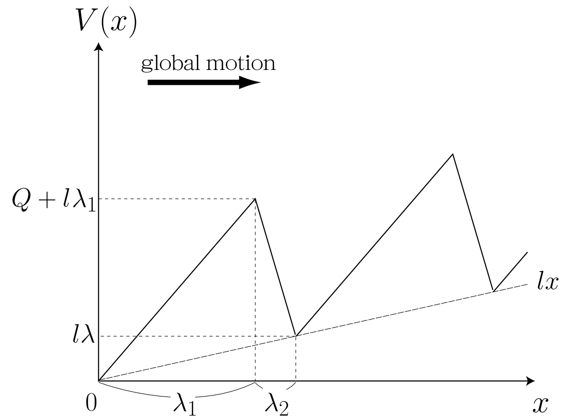


FIG. 1. Schematic illustration of the potential, $V(x) = V_0(x) + V_L(x)$.

It is reported as to this system by Magnasco [7] that "There is a region of the operating regime where the efficiency is optimized at finite temperatures." The result has been interpreted that the operation of the forced thermal ratchet is helped by thermal fluctuation. This discovery has been followed and confirmed by many literatures (see the references in Ref. [11]) with various situations. We first confirm the previous report and then analyze it energetically. We can distinguish the behavior of the current $J_{\rm sqr}$ on the temperature into three regimes, (a) low amplitude regime, $A < \frac{Q}{\lambda_1} + l < \frac{Q}{\lambda_2} - l$; (b) moderate amplitude regime, $\frac{Q}{\lambda_1} + l < A < \frac{Q}{\lambda_2} - l$; and (c) high amplitude regime, $\frac{Q}{\lambda_1} + l < \frac{Q}{\lambda_2} - l < A$ (the distinction is not explicitly described in the paper [7]). We confirmed that $J_{\rm sqr}$ is certainly maximized at finite temperature in regimes (a) and (b) [Figs. 2(a) and 2(b)]. In regime (c), $J_{\rm sqr}$ is a monotonically decreasing function of the temperature [Fig. 2(c)]. One finds that $J_{\rm sqr}$ becomes negative in the extremely high temperature regions. This is understandable because the effect of the ratchet (sawtooth potential) disappears in the high temperature limit and therefore the system suffers only the effect of the load, $-\partial V_l/\partial x = -l$.

We have to notice at this stage that the fluctuation-induced current $J$ is *not* an energetic quantity, and therefore $J$ is only the mimic of the *energetic* efficiency. The lack of discussion of the forced ratchet system by the *real* efficiency is attributed to the lack of construction of energetics of the systems described by Langevin or equivalently by Fokker-Planck equations. Recently, an energetics of these systems was systematically constructed by Sekimoto [12]. Therefore, we will go into the realm of the energetics of the forced thermal ratchet and analyze the *real* efficiency.

According to the energetics [12], the input energy $R$ (per unit time) from external fluctuation to the ratchet and the work $W$ (per unit time) that the ratchet system extracts from the fluctuation are written, respectively,

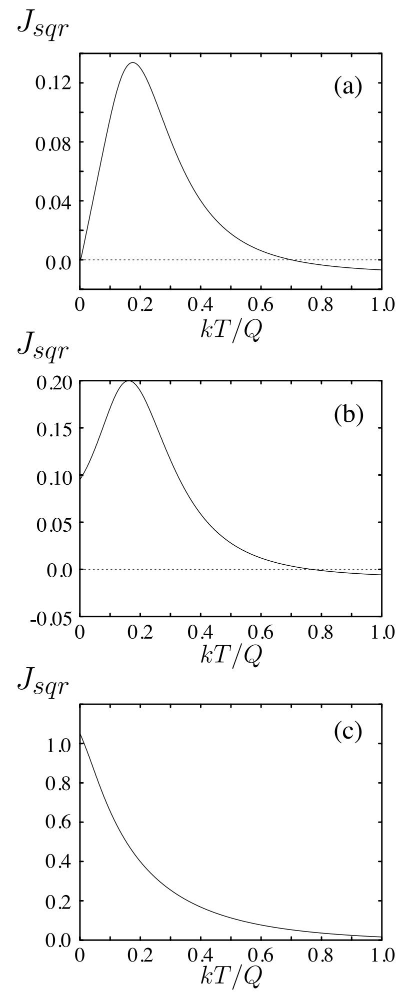


FIG. 2. Plot of the current $J_{\rm sqr}$ as a function of $kT/Q$. The first regime (a), $A < \frac{Q}{\lambda_1} + l < \frac{Q}{\lambda_2} - l$ ($\lambda = 1.0$, $\Delta = 1.0$, $l = 0.01$, $A = 1.0$); the second regime (b), $\frac{Q}{\lambda_1} + l < A < \frac{Q}{\lambda_2} - l$ ($\lambda = 1.0$, $\Delta = 1.0$, $l = 0.01$, $A = 1.2$); and the third regime (c), $\frac{Q}{\lambda_1} + l < \frac{Q}{\lambda_2} - l < A$ ($\lambda = 1.0$, $\Delta = 0.6$, $l = 0.01$, $A = 6.0$).

$$R[F(t)] = \frac{1}{t_f - t_i} \int_{x=x(t_i)}^{x=x(t_f)} F(t)\, dx(t)\,, \tag{4}$$

$$W = \frac{1}{t_f - t_i} \int_{x=x(t_i)}^{x=x(t_f)} dV[x(t)]\,. \tag{5}$$

For the square wave with amplitude $A$, they yield

$$\langle R_{\rm sqr}\rangle = \tfrac{1}{2}[\langle R(A)\rangle + \langle R(-A)\rangle]$$
$$= \tfrac{1}{2}A[J(A) - J(-A)]\,, \tag{6}$$

$$\langle W_{\rm sqr}\rangle = \tfrac{1}{2}l[J(A) + J(-A)]\,. \tag{7}$$

Therefore, we obtain the efficiency of the energy transformation $\eta$ [16],

$$\eta = \frac{\langle W_{\rm sqr}\rangle}{\langle R_{\rm sqr}\rangle} = \frac{l[J(A) + J(-A)]}{A[J(A) - J(-A)]}\,. \tag{8}$$

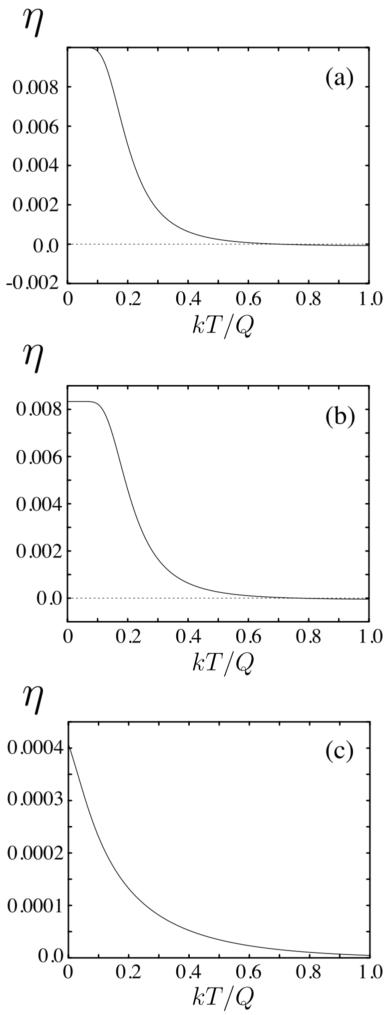


FIG. 3. Plot of the efficiency $\eta$ as a function of $kT/Q$. In each regime (a), (b), and (c), the condition is the same as in Fig. 2.

In Eq. (8), we can discuss the effect of thermal fluctuation on the *energetic* efficiency of the forced thermal ratchet. As demonstrated in Fig. 3, it is proved that the efficiency is a monotonically decreasing function of temperature in all three regimes. Let us remember the behavior of the probability current $J_{\mathrm{sqr}}$ that can be maximized at finite temperature (Fig. 2).

Why does the efficiency $\eta$ show different behavior in contrast to the current $J_{\mathrm{sqr}}$? Because $\frac{J(-A)}{J(A)} < 0$ [17], Eq. (8) is rewritten

$$\eta = \frac{l}{A}\left(1 - \frac{2|\frac{J(-A)}{J(A)}|}{1 + |\frac{J(-A)}{J(A)}|}\right). \tag{9}$$

Equation (9) shows that the efficiency $\eta$ depends on $|\frac{J(-A)}{J(A)}|$. On the other hand, the current $J_{\mathrm{sqr}}$ depends on the sum of $J(A)$ and $J(-A)$ [Eq. (3)]. In regimes (a) and (b), $|J(-A)|$ increases slower than $|J(A)|$ when the temperature increases. Therefore the current $J_{\mathrm{sqr}}$ can be maximized at finite temperature. This difference between $J(A)$ and $J(-A)$ is attributed to the symmetry breaking of the potential as illustrated in Fig. 1. However, as found in Fig. 4, $|\frac{J(-A)}{J(A)}|$ is a monotonically increasing function of the temperature. Therefore the efficiency $\eta$ is a decreasing function of the temperature. This result certainly shows that the presence of thermal fluctuation *does not* help efficient energy transformation by the ratchet, which is in contrast to the previous interpretation that thermal fluctuation could increase the efficiency.

In the limit $|\frac{J(-A)}{J(A)}| \to 0$ in regimes (a) and (b), the maximum efficiency of the energy transformation for given load $l$ and force amplitude $A$ is realized: $\eta_{\mathrm{max}} = \frac{l}{A}$. We note that the energy transformation from fluctuation to coherent work is impossible in the region where $J_{\mathrm{sqr}}$ is negative. Therefore, we have neglected the regime where $l > A$, because $J_{\mathrm{sqr}}$ is always negative.

We can learn here that the efficiency should be discussed energetically: The condition of maximum current does not correspond to that of the maximum efficiency. The difference is attributed to the observation that the efficiency is a ratio of the extracted work $W$ to the consumed energy $R$. The extracted work $W$ is surely proportional to the current $J_{\mathrm{sqr}} = \frac{1}{2}[J(A) + J(-A)]$ [Eq. (3)]. However, the consumed energy is not a constant but varies sensitively according to the temperature. Therefore the efficiency $\eta$ is not simply proportional to the induced current $J$. It turned out that the important problem is left for future studies whether the existence of thermal fluctuation can facilitate the efficiency of energy transformation in the general frame of forced ratchets. An experiment is also expected toward this interesting problem whether the molecular machine can use ambient thermal fluctuation energetically.

Finally, we mention the complementarity relation [18] of the forced ratchet system. We found that the maximum efficiency, $\eta_{\mathrm{max}} = 1$, can be realized if all of the following conditions are satisfied: $A \to Q/\lambda_1 + l + 0$,

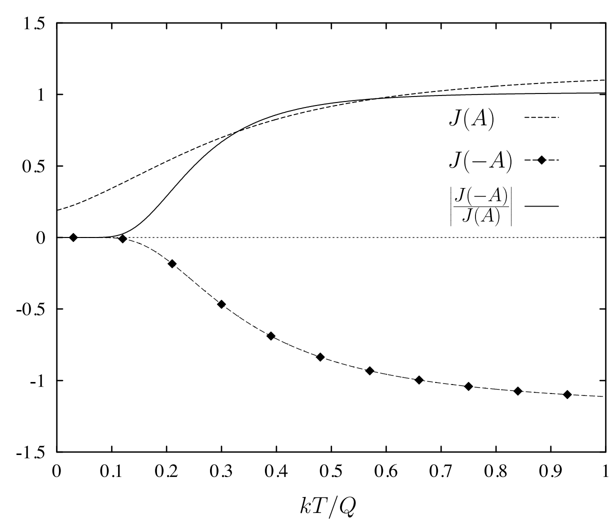


FIG. 4. Plot of currents $J(A)$, $J(-A)$, and $|\frac{J(-A)}{J(A)}|$. The condition is the same as in the second regime (b) in Fig. 2.

$Q \to +0$, and $T \to 0$. In this limit, the speed of the energy transformation goes to zero. That is to say, this maximum efficiency of the forced ratchet is realized in thermodynamically quasistatic process. As we increase the velocity of this *engine*, the efficiency is decreased. The result also emphasizes the importance of time scales of the operation of the ratchet as Jülicher *et al.* pointed out [11]. Detailed analysis of the loss of the efficiency may be analyzed by the formal theory of the complementarity relation [18] between the time lapse of thermodynamic process and the irreversible heat.

We thank K. Sekimoto, S. Sasa, T. Shibata, Y. Sawada, T. Fujieda, and T. Tsuzuki for helpful comments and discussions. This work is supported by the Japanese Grant-in-Aid for Science Research Fund from the Ministry of Education, Science and Culture (No. 09740301).

*Present address: Research Institute of Electrical Communication, Tohoku University, Sendai 980-8577, Japan.
Electronic address: kame@cmpt01.phys.tohoku.ac.jp
†Electronic address: hondou@cmpt01.phys.tohoku.ac.jp
‡Electronic address: fumiko@cmpt01.phys.tohoku.ac.jp

[1] For example, R. C. Woledge, N. A. Curtin, and E. Homsher, *Energetic Aspect of Muscle Contraction* (Academic Press, New York, 1985).
[2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures in Physics* (Addison-Wesley, Reading, MA, 1966).
[3] M. Büttiker, Z. Phys. B **68**, 161 (1987).
[4] R. Landauer, J. Stat. Phys. **53**, 233 (1988).
[5] R. D. Vale and F. Oosawa, Adv. Biophys. **26**, 97 (1990).
[6] A. Ajdari and J. Prost, C. R. Acad. Sci. Paris II **315**, 1635 (1992).
[7] M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993).
[8] A. Ajdari, D. Mukamel, D. Peliti, and J. Prost, J. Phys. I (France) **4**, 1551 (1994).
[9] A. R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994).
[10] T. Hondou and Y. Sawada, Phys. Rev. Lett. **75**, 3269 (1995).
[11] F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).
[12] K. Sekimoto, J. Phys. Soc. Jpn. **66**, 1234 (1997).
[13] An energetics of forced ratchet under chaotic noise was investigated in T. Hondou, J. Phys. Soc. Jpn. **67**, 1818 (1998).
[14] C. W. Gardiner, *Handbook of Stochastic Methods 2nd ed.* (Springer-Verlag, Berlin, 1990).
[15] $J(A)$ is obtained by replacing $F$ of Eq. (3) of Ref. [7] by $A - l$. $J(A)$ does not depend on the spatial coordinate $x$, nor time $t$. $J(-A)$ is a current induced by a constant force $F(t) = -A$.
[16] $\frac{\langle W_{\rm sqr}\rangle}{\langle R_{\rm sqr}\rangle}$ is identical with $\langle \frac{W_{\rm sqr}}{R_{\rm sqr}} \rangle$ when the period of the square wave and the operation time of the ratchet are sufficiently large, respectively.
[17] $J(-A) < 0$ is obvious for the symmetry of the system. For the system which extracts finite work against load, $J(A) > 0$ is also requested. Therefore, $\frac{J(-A)}{J(A)} < 0$ is satisfied here.
[18] K. Sekimoto and S. Sasa, J. Phys. Soc. Jpn. **66**, 3326 (1997).