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Modeling Public Opinion

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1 Introduction

Population models are used to study the dynamics of a population. In particular, dynamic population models are applied to populations that gain and lose members, unlike a fixed population. A dynamic population is said to be stable if the sizes of all subgroups remain constant. Epidemiological models typically focus on describing the transmission of communicable diseases through individuals. One well-known epidemiological model is the SIR model, which computes the theoretical number of people infected with a contagious disease in a closed population by modeling the flow of people between three states: susceptible (S), infected (I), and resistant (R).

In this paper, we adapt the epidemiological models presented by [1] to model the dynamics of public opinion. By definition, a public opinion is any view prevalent among the general public. Our model considers any topic or issue in which the public has two decisive and opposing viewpoints. In order to understand a certain opinion it is important to ascertain the following: if an individual has adopted the opinion, their particular level of adoption, and if they are capable of and/or active in spreading their ideals to other individuals in the opinion population.

Public opinion research developed from market research. In 1935 George Gallup, the American public opinion statistician, began conducting nationwide surveys in the United States on social and political opinions. Opinion polls in the United States continued to spread from the 1930s, conducted by both commercial and academic practitioners. At the same time, polling organizations developed in countries of Europe, Asia, and Latin America. In 1947 the World Association for Public Opinion Research was founded and regional studies continued to develop around the world with the support of NGOs, national governmental agencies, and university research programs [2]. In today's society, public opinion research is important in weighing public perception on a particular issue. Public opinion reflects public concerns, beliefs, and values, and plays a key role in making both local and national policy decisions [3].

The modeling of social processes began as early as 1952, when Rapoport and Rebhun analyzed the mathematical theory of rumor spread [4]. In 1965, Daley and Kendall presented a stochastic model for the spread of rumors. In their model, the population is divided into three social states defined as "ignorant", "spreaders", and "stiflers" and transitions are determined from contacts between the classes[5]. More recently, Dan and Cook presented a differential equation model capable

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of predicting public opinions and behaviors from persuasive information [6]. In [1], Bettencourt et al convert the typical SIR model to address the spread of an idea, namely the growth of the adoption of Feynman diagrams. In their models, the population is divided into five classes: susceptible (S), incubator (E), adopters (I), skeptics (Z), and immune (R).

In this paper, we modify the epidemiological models presented by Bettencourt et al [1] to create a system of ordinary differential equations that models the population dynamics of a public opinion with two opposing sides. In Section 2 we introduce our system and provide information on the equation dynamics. Section 3 presents a study of the total population. Lastly, in Sections 4-6 we discuss the mathematical properties of our model and various results.

2 Presentation of the Model

Below is the preliminary system to model public opinion on an issue with two opposing sides. The developed system is composed of four ordinary differential equations representing each of the following opinion classes: the adopters (A), the moderately adoptive (B), the moderately skeptic (Y), and the skeptics (Z). It is assumed that individuals are entering the system as either moderately adoptive (B) or moderately skeptic (Y). Once an individual enters the adoptive (A) or skeptic (Z) populations, their opinion on the issue is assumed to be solidified until the issue is no longer a topic of conversation. Parameters are defined in Table 2. The model is presented below:

$$\begin{aligned}
\frac{dA}{dt} &= \alpha_a d\frac{AB}{N} + \alpha_z\left(1-e\right)\frac{BZ}{N} + \epsilon B - \mu A \\
\frac{dB}{dt} &= \lambda + \beta_a e\frac{AY}{N} + \beta_z\left(1-d\right)\frac{YZ}{N} - \alpha_a\frac{AB}{N} - \alpha_z\frac{BZ}{N} - \epsilon B - \mu B \\
\frac{dY}{dt} &= \lambda + \alpha_z e\frac{BZ}{N} + \alpha_a\left(1-d\right)\frac{AB}{N} - \beta_a\frac{AY}{N} - \beta_z\frac{YZ}{N} - \epsilon Y - \mu Y \\
\frac{dZ}{dt} &= \beta_a\left(1-e\right)\frac{AY}{N} + \beta_z d\frac{YZ}{N} + \epsilon Y - \mu Z
\end{aligned} \tag{1}$$

We shall further examine each term in more detail, beginning with the equation for the adopters (A):

$$\frac{dA}{dt} = \alpha_a d\frac{AB}{N} + \alpha_z\left(1-e\right)\frac{BZ}{N} + \epsilon B - \mu A$$

The first term represents the increase in adopters based on contact with the moderately adoptive (B) population. The adoptive and moderately adoptive populations interact at a rate of α_a and the probability that a moderately adoptive individual will transition to adoptive given contact with (A) is represented by d. The following term shows the increase of adopters from contact between moderately adoptive (B) and skeptics (Z). Moderately adoptive and skeptic populations interact at a rate of α_z with a $(1-e)$ probability that the interaction causes the moderately adoptive to transition to adoptive. The third term is the rate of increase in adopters due to the moderately adoptive population manifesting the opinion on their own. Here, the average adoptive time is represented by ϵ. Finally, the adoptive population is decreasing at a rate of μ, which represents the average lifetime of the topic. We now move to the equation for the moderately adoptive (B):

$$\frac{dB}{dt} = \lambda + \beta_a e\frac{AY}{N} + \beta_z\left(1-d\right)\frac{YZ}{N} - \alpha_a\frac{AB}{N} - \alpha_z\frac{BZ}{N} - \mu B - \epsilon B$$

The recruitment rate into thinking about the issue is given by λ. The next term shows the increase in the moderately adoptive population related to interaction between the adoptive and moderately skeptic populations. The (A) and (Y) populations interact at a rate of β_a with probability e that the interaction will cause the moderately skeptic to transition to moderately adoptive. The third term shows the increase of moderately adoptive from contact between moderately skeptic and skeptic populations. The (Y) and (Z) populations interact at a rate of β_z with a $(1-d)$ probability that the contact will cause the moderately skeptic to transition to moderately adoptive. The next two terms represent the decrease in the moderately adoptive population based on contact with either the adopters or the skeptics at their respective contact rates, α_a and α_z. Finally, the moderately adoptive population is decreasing at a rate of μ for the lifetime of the topic and a rate of ϵ for the population that has naturally adopted the opinion and transitioned to (A). The equations for the moderately skeptic (Y) and skeptic (Z) populations are similar to those of the moderately adoptive (B) and adoptive (A), respectively. Below, Figure 1 demonstrates the population dynamics of the system.

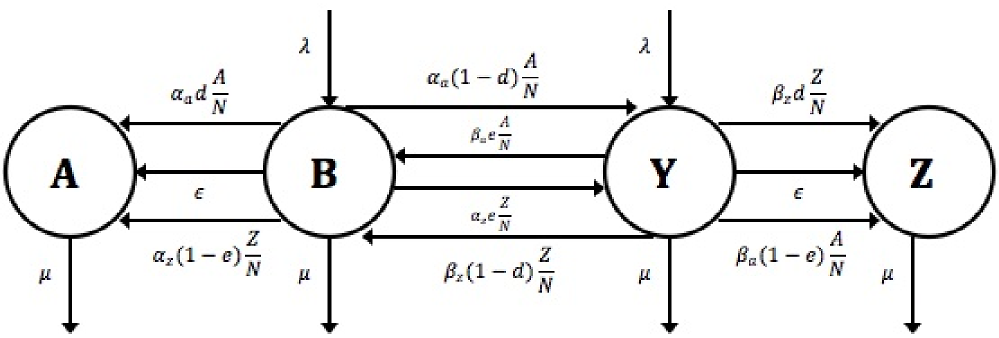


Figure 1: Model Diagram

Table 1: Parameter List

Parameter	**Symbol**
Total population	$N = A + B + Y + Z$
Recruitment rate	λ
Average lifetime of a topic	μ
Average adoption time	ϵ
AB contact rate	α_a
BZ contact rate	α_z
AY contact rate	β_a
YZ contact rate	β_z
$B \to A$ transition probability given contact with A	d
$Y \to Z$ transition probability given contact with Z	d
$Y \to B$ transition probability given contact with Z	$(1 - d)$
$B \to Y$ transition probability given contact with A	$(1 - d)$
$Y \to B$ transition probability given contact with A	e
$B \to Y$ transition probability given contact with Z	e
$B \to A$ transition probability given contact with Z	$(1 - e)$
$Y \to Z$ transition probability given contact with A	$(1 - e)$

3 Total Population

Summing the four differential equations for (A, B, Y, Z) results in the following differential equation representing the change in total population (N):

$$\frac{dN}{dt} = 2\lambda - \mu N$$

Solving the above differential equation, we find that the total population (N) can be modeled by the following equation, where c represents some constant:

$$N(t) = \left(\frac{2\lambda}{\mu}\right) + ce^{-\mu t}$$

Setting $\frac{dN}{dt} = 0$, we find that the total population reaches equilibrium when $\mu N = 2\lambda$. Thus, for any equilibrium solutions for the entire system, the population will be at $\frac{2\lambda}{\mu}$. Since the derivative of $\frac{dN}{dt}$ is negative, we know that the population equilibrium solution $N = \frac{2\lambda}{\mu}$ is stable. Given this information, we can now investigate equilibrium solutions under certain cases.

4 Proportion Model

The total population (N) will be a constant if $\lambda = \mu = 0$ or $N = \frac{2\lambda}{\mu}$. In order to examine the cases where the total population (N) is a constant, we modify the original system of differential equations to represent the proportion of the engaged population that is in each of the four opinion classes. Proportions for the four opinion classes are represented by:

$$\hat{A} = \frac{A}{N}$$
$$\hat{B} = \frac{B}{N}$$
$$\hat{Y} = \frac{Y}{N}$$
$$\hat{Z} = \frac{Z}{N}$$

Each differential equation illustrates the rate of change of the respective proportion. The modified model is presented below:

$$\begin{aligned}
\frac{d\hat{A}}{dt} &= \alpha_a d\hat{A}\hat{B} + \alpha_z (1-e)\hat{B}\hat{Z} + \epsilon\hat{B} - \mu\hat{A} \\
\frac{d\hat{B}}{dt} &= \frac{\lambda}{N} + \beta_a e\hat{A}\hat{Y} + \beta_z (1-d)\hat{Y}\hat{Z} - \alpha_a\hat{A}\hat{B} - \alpha_z\hat{B}\hat{Z} - \epsilon\hat{B} - \mu\hat{B} \\
\frac{d\hat{Y}}{dt} &= \frac{\lambda}{N} + \alpha_z e\hat{B}\hat{Z} + \alpha_a (1-d)\hat{A}\hat{B} - \beta_a\hat{A}\hat{Y} - \beta_z\hat{Y}\hat{Z} - \epsilon\hat{Y} - \mu\hat{Y} \\
\frac{d\hat{Z}}{dt} &= \beta_a (1-e)\hat{A}\hat{Y} + \beta_z d\hat{Y}\hat{Z} + \epsilon\hat{Y} - \mu\hat{Z}
\end{aligned} \tag{2}$$

5 Equilibrium Solutions

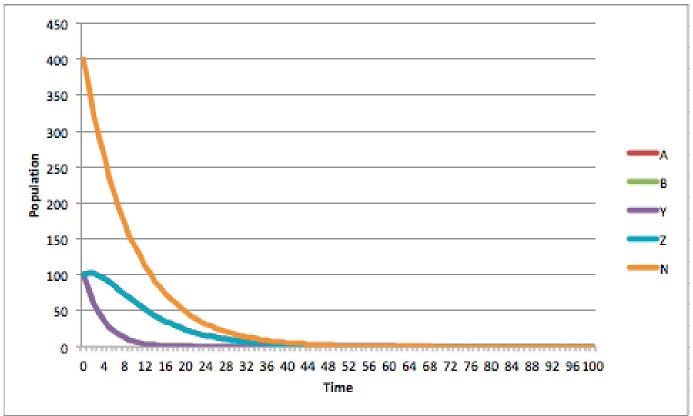
If an equilibrium solution exists, then $\mu N = 2\lambda$. Under this constraint, we consider the following two cases: $\lambda = 0$, and $\lambda > 0$.

6 Single Outbreak ($\lambda = 0$)

If $\lambda = 0$, then there are no new people being recruited into the system. Therefore, we can consider this case a single outbreak of an opinion. Given a recruitment rate of zero, we now analyze $\mu > 0$ and $\mu = 0$.

6.1 Case 1: $\mu > 0$

If $\mu > 0$ and $\lambda = 0$, then a steady proportion of people is leaving the system but no new people are being recruited into the system. Thus, the equation for the total population (N) is an exponential decay model and the only equilibrium solution that exists is when $N = 0$. Therefore, a stability point will be reached when $A = B = Y = Z = 0$. The result from this case with the given parameter values is presented in Figure 2.

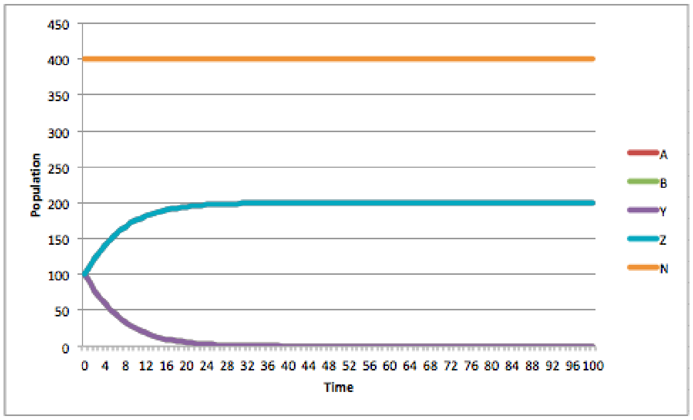


Parameter	Value
λ	0
μ	0.1
A_0, B_0, Y_0, Z_0	100
N_0	400
ϵ	0.1
α_a, β_z	0.1
α_z, β_a	0.05
d,e	0.7

Figure 2: Graphical Representation of Case 1

6.2 Case 2: $\mu = 0$

If $\lambda = \mu = 0$, then $\frac{dN}{dt} = 0$ and the total population (N) is a constant. Figures 3 and 4 illustrate results from Case 2 given different parameter values.



Parameter	Value
λ	0
μ	0
A_0, B_0, Y_0, Z_0	100
N_0	400
ϵ	0.1
α_a, β_z	0.1
α_z, β_a	0.05
d,e	0.7

Figure 3: Graphical Representation of Case 2

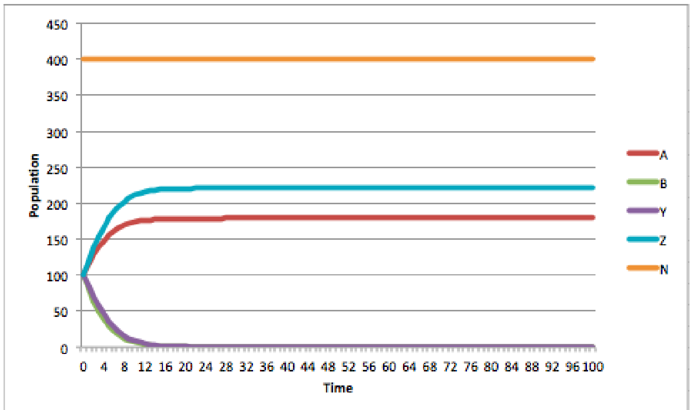


Figure 4: Graphical Representation of Case 2

Parameter	Value
λ	0
μ	0
A_0, B_0, Y_0, Z_0	100
N_0	400
ϵ	0.1
α_a, β_a	0.1
α_z, β_z	0.5
d,e	0.7

Question: What are the sizes of A, B, Y, Z?
Answer: We think that $B = Y = 0$ and A and Z vary depending on parameters.

6.2.1 Symmetric Case General Equations

Let $\hat{A}_0 = \hat{Z}_0$ and $\hat{B}_0 = \hat{Y}_0$. Let $\alpha_a = \beta_z$ and $\beta_a = \alpha_z$. Due to symmetry, this causes $\hat{A} = \hat{Z}$ and $\hat{B} = \hat{Y}$. Define $m = \frac{\alpha_a d + \alpha_z(1-e)}{2}$. From (2) we have

$$\begin{aligned} \frac{d\hat{A}}{dt} &= \alpha_a d \hat{A}\hat{B} + \alpha_z (1-e) \hat{B}\hat{Z} + \epsilon \hat{B} \\ \frac{d\hat{B}}{dt} &= \beta_a e \hat{A}\hat{Y} + \beta_z (1-d) \hat{Y}\hat{Z} - \alpha_a \hat{A}\hat{B} - \alpha_z \hat{B}\hat{Z} - \epsilon \hat{B} \end{aligned} \tag{3}$$

Substituting in the above parameters to (3),

$$\begin{aligned} \frac{d\hat{A}}{dt} &= 2m\hat{A}\hat{B} + \epsilon \hat{B} \\ \frac{d\hat{B}}{dt} &= -2m\hat{A}\hat{B} - \epsilon \hat{B} \end{aligned} \tag{4}$$

Consider the following:

$$\begin{aligned} \hat{A} &= \frac{1}{2} - \frac{k}{2m + Ce^{kt}} \\ \hat{B} &= \frac{k}{2m + Ce^{kt}} \end{aligned} \tag{5}$$

where $k = \epsilon + m$ and $C = \frac{k\hat{A}_0}{\hat{B}_0}$. Substituting these values into (4), we have

$$\begin{aligned}
2m\hat{A}\hat{B} + \epsilon B &= 2m\left(\frac{1}{2} - \frac{k}{2m + Ce^{kt}}\right)\left(\frac{k}{2m + Ce^{kt}}\right) + \epsilon\left(\frac{k}{2m + Ce^{kt}}\right) \\
&= \frac{mk}{2m + Ce^{kt}} - \frac{2mk^2}{\left(2m + Ce^{kt}\right)^2} + \frac{ek}{2m + Ce^{kt}} \\
&= \frac{2m^2k + mkCe^{kt} + 2mk^2 + 2\epsilon km + ekCe^{kt}}{\left(2m + Ce^{kt}\right)^2} \\
&= \frac{k^2Ce^{kt} + 2m\left(mk - k^2 + ek\right)}{\left(2m + Ce^{kt}\right)^2} \\
&= \frac{k^2Ce^{kt}}{\left(2m + Ce^{kt}\right)^2} \\
&= \frac{d\hat{A}}{dt}
\end{aligned}$$

Notice that in both (4) and (5), $\frac{d\hat{B}}{dt} = -\frac{d\hat{A}}{dt}$. Therefore, we have shown that (5) represents the general equations for $\hat{A}$ and $\hat{B}$ in the symmetric case.

7 Steady Recruitment ($\lambda > 0$)

If $\lambda > 0$, then there is a steady recruitment of people into the system. Therefore, we can consider this case a topic of conversation in which opinions are formed over a long period of time. Given a positive recruitment rate, we now analyze $\mu > 0$ and $\mu = 0$.

7.1 Case 1: $\mu > 0$

In the case that $\mu > 0$ and $\lambda > 0$, we have a constant population leaving and being recruited into the system. For long times, and regardless of the distribution of (A, B, Y, Z), recruitment into the system and exits from the system from the average lifetime of the topic will balance each other out so that the $\lim_{x\to\infty} N(t) = N^* = \frac{2\lambda}{\mu}$. To examine the equilibrium solutions in this situation, we consider a symmetric case.

7.1.1 Symmetric Case Equilibrium Solution

Let $\hat{A}_0 = \hat{Z}_0$ and $\hat{B}_0 = \hat{Y}_0$. Let $\alpha_a = \beta_z$ and $\beta_a = \alpha_z$. Due to symmetry, this causes $\hat{A} = \hat{Z}$ and $\hat{B} = \hat{Y}$. Define $c = \alpha_a d + \alpha_z(1-e)$. From (2) we have

$$\begin{aligned}
\frac{d\hat{A}}{dt} &= \alpha_a d\hat{A}\hat{B} + \alpha_z(1-e)\hat{B}\hat{Z} + \epsilon\hat{B} - \mu\hat{A} \\
\frac{d\hat{B}}{dt} &= \frac{\lambda}{N} + \beta_a e\hat{A}\hat{Y} + \beta_z(1-d)\hat{Y}\hat{Z} - \alpha_a\hat{A}\hat{B} - \alpha_z\hat{B}\hat{Z} - \epsilon\hat{B} - \mu\hat{B}
\end{aligned} \tag{6}$$

Substituting in the above parameters to (6),

$$\begin{aligned}\frac{d\hat{A}}{dt} &= c\hat{A}\hat{B} + \epsilon\hat{B} - \mu\hat{A} \\ \frac{d\hat{B}}{dt} &= \frac{\lambda}{N} - c\hat{A}\hat{B} - \epsilon\hat{B} - \mu\hat{B}\end{aligned} \tag{7}$$

Setting $\frac{dA}{dt} = 0$,

$$0 = c\hat{A}\hat{B} + \epsilon\hat{B} - \mu\hat{A} \tag{8}$$

Given the symmetry of this case, we know that $A + B = \frac{N}{2}$. Thus

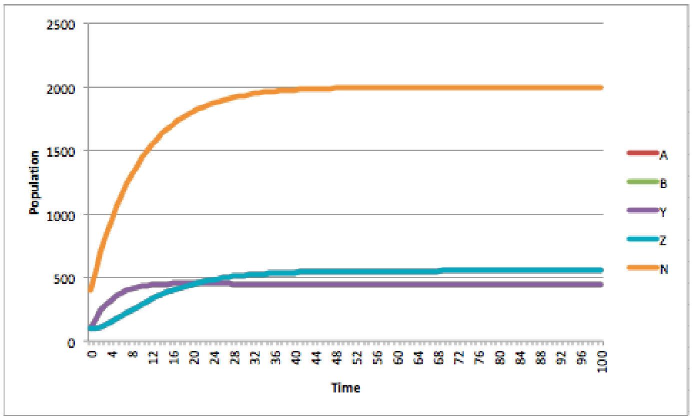
$$\hat{A} + \hat{B} = \frac{1}{2} \tag{9}$$

Solving equations (8) and (9) for $\hat{A}$ and $\hat{B}$ gives us the following solutions:

$$\hat{A} = \frac{-\left(\epsilon + \mu + \frac{1}{2}c\right) \pm \sqrt{\left(\epsilon + \mu + \frac{1}{2}c\right)^2 + 2c\epsilon}}{2c}$$

$$\hat{B} = \frac{1}{2} - \hat{A}$$

Since $\hat{A} = \hat{Z}$ and $\hat{B} = \hat{Y}$, an equilibrium solution exists when the above solutions are positive and satisfied by the given parameters. Figure 5 illustrates a result of the symmetric case with the given parameter values.



Parameter	Value
λ	100
μ	0.1
A_0, B_0, Y_0, Z_0	100
N_0	400
ϵ	0.1
α_a, β_z	0.1
α_z, β_a	0.05
d,e	0.7

Figure 5: Graphical Representation of the Symmetric Case

Question: When do you get $\hat{A} = \hat{Z}$ and $\hat{B} = \hat{Y}$ as an equilibrium solution?
Answer: We think whenever λ and μ are not zero and $\alpha_a = \beta_z$ and $\beta_a = \alpha_z$.

7.1.2 Extreme-free Equilibrium Solution

Consider a constant population of $N = \frac{2\lambda}{\mu}$. Assume $\alpha_a = \beta_z$, which means that $\hat{A}$ and $\hat{Z}$ behave symmetrically. Let $d = e = 1$. That is, contact with $\hat{A}$ moves $\hat{B}$ and $\hat{Y}$ towards $\hat{A}$ with probability of 1 and contact with $\hat{Z}$ moves $\hat{B}$ and $\hat{Y}$ towards $\hat{Z}$ with probability of 1. Let $\epsilon = 0$. Substituting these parameters into (2) we have

$$\begin{aligned}
\frac{d\hat{A}}{dt} &= \alpha_a \hat{A}\hat{B} - \mu\hat{A} \\
\frac{d\hat{B}}{dt} &= \frac{\mu}{2} + \beta_a \hat{A}\hat{Y} - \alpha_a \hat{A}\hat{B} - \alpha_z \hat{B}\hat{Z} - \mu\hat{B} \\
\frac{d\hat{Y}}{dt} &= \frac{\mu}{2} + \alpha_z \hat{B}\hat{Z} - \beta_a \hat{A}\hat{Y} - \alpha_a \hat{Y}\hat{Z} - \mu\hat{Y} \\
\frac{d\hat{Z}}{dt} &= \alpha_a \hat{Y}\hat{Z} - \mu\hat{Z}
\end{aligned} \tag{10}$$

Our reproductive number for (10) is $R_0 = \frac{\alpha_a}{2\mu}$. In this case, the reproductive number is interpreted as the average number of people that an extreme individual in the A and Z populations converts in the next time period. Having $R_0 = \frac{\alpha_a}{2\mu}$ represents the probability of an individual in either A or Z converting someone in one unit of time, $\left(\frac{\alpha_a}{2}\right)$, multiplied by the amount of time a person remains in an extreme opinion group, $\left(\frac{1}{\mu}\right)$.

Solving (10) for the extreme-free equilibrium solution, we find

$$\begin{aligned}
A^* &= 0 \\
B^* &= \frac{1}{2} \\
Y^* &= \frac{1}{2} \\
Z^* &= 0
\end{aligned}$$

To analyze the local stability of this equilibrium solution, we calculate the Jacobian for (10):

$$\begin{bmatrix}
\alpha_a \hat{B} - \mu & \alpha_a \hat{A} & 0 & 0 \\
\beta_a \hat{Y} - \alpha_a \hat{B} & -\alpha_a \hat{A} - \alpha_z \hat{Z} - \mu & \beta_a \hat{A} & -\alpha_z \hat{B} \\
-\beta_a \hat{Y} & \alpha_z \hat{Z} & -\beta_a \hat{A} - \alpha_a \hat{Z} - \mu & \alpha_z \hat{B} - \alpha_a \hat{Y} \\
0 & 0 & \alpha_a \hat{Z} & \alpha_a \hat{Y} - \mu
\end{bmatrix}$$

Plugging in our solution (A^*, B^*, Y^*, Z^*) to our Jacobian, we have

$$\begin{bmatrix}
\frac{\alpha_a}{2} - \mu & 0 & 0 & 0 \\
\frac{\beta_a}{2} - \frac{\alpha_a}{2} & -\mu & 0 & -\frac{\alpha_z}{2} \\
\frac{-\beta_a}{2} & 0 & -\mu & \frac{\alpha_z}{2} - \frac{\alpha_a}{2} \\
0 & 0 & 0 & \frac{\alpha_a}{2} - \mu
\end{bmatrix}$$

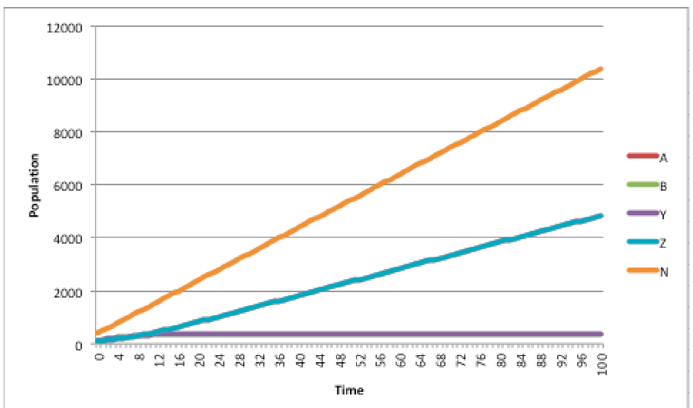
The eigenvalues for this solution are as follows:

$$\lambda_1 = \lambda_2 = \frac{1}{2}(\alpha_a - 2\mu)$$
$$\lambda_3 = \lambda_4 = -\mu$$

Whenever $R_0 < 1$, the eigenvalues are all negative and the extreme-free equilibrium (A^*, B^*, Y^*, Z^*) is locally stable. This implies that more individuals are leaving the extreme opinion groups than are being converted.

7.2 Case 2: $\mu = 0$

When $\lambda > 0$ and $\mu = 0$, then there is a steady recruitment of people into the system, but the population is never decreasing. Since the total population (N) is never in equilibrium, then no equilibrium solutions exist for this case. Figure 6 presents a result from this case with the given parameter values.



Parameter	Value
λ	50
μ	0
A_0, B_0, Y_0, Z_0	100
N_0	400
ϵ	0.1
α_a, β_z	0.1
α_z, β_a	0.05
d,e	0.7

Figure 6: Graphical Representation of Case 2

Question: Is there something interesting to examine when $\lambda > 0$ and $\mu = 0$?

References

[1] Luis M.A. Bettencourt, Ariel Cintron-Arias, David I. Kaiser, and Carlos Castillo-Chavez. The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models. *Elsevier*, 2005.

[2] W. Phillips Davison. Public opinion. *Encyclopædia Britannica, Inc.*, 2017.

[3] Amanda Barna. Why is public opinion research important? *Center for Marketing and Opinion Research, LLC.*

[4] Anatol Rapoport and Lionel I. Rebhun. On the mathematical theory of rumor spread. *The bulletin of mathematical biophysics*, 14(4):375–383, Dec 1952.

[5] D. J. Daley and D. G. Kendall. Stochastic rumours. *IMA Journal of Applied Mathematics*, 1(1):42–55, 1965.

[6] David P. Fan and R. Dennis Cook. A differential equation model for predicting public opinions and behaviors from persuasive information: Application to the index of consumer sentiment. *The Journal of Mathematical Sociology*, 27(1):29–51, 2003.