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# Nondegenerate normal-mode doublets in vibrating flat circular plates

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The vibrations of flat circular plates have been studied for hundreds of years and are well understood. However, when vibrating circular plates are discussed in textbooks, the relation between pairs of spatially orthogonal vibrational patterns that occur at each of the normal-mode frequencies often is ignored. Usually these orthogonal solutions are presented to the student as being degenerate in frequency. However, in practice the degeneracy of the doublet often is broken, and the two spatially orthogonal solutions are separated in frequency. We show theoretically and experimentally that the degeneracy can be broken by small density perturbations in the plate, and we derive a formula for predicting the magnitude of the frequency splitting. We have used electronic speckle pattern interferometry to investigate the phenomenon of doublet splitting and have confirmed the validity of the theory. 
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## I. INTRODUCTION

Investigations into the normal modes of vibration of flat plates began in earnest over 200 years ago with the work of Ernst Chladni. Since then, these investigations have occupied the minds of some of the greatest names in classical physics. Sarvart, Kirchoff, Rayleigh, and many more have applied their genius to the problem of describing the vibrations of a flat plate, and the system continues to be of interest today.[1]

Flat circular plates occur in many acoustic and mechanical devices, and the symmetry lends itself to relatively simple analysis. Therefore, the flat circular plate has been a popular geometry in which to study plate vibrations. Many hundreds of articles and book chapters have been written on the analysis of the normal modes of flat circular plates, and almost every textbook on acoustics or vibrations reproduces some version of the derivation of the normal modes of flat circular plates. Furthermore, articles appear regularly in scholarly journals on special cases of the vibrating circular plate.[2–4]

With all of the attention given the subject of vibrating circular plates, it is surprising that very little attention has been given to the fact that each normal mode is twofold degenerate in its angular solution. That is, the general solution for the amplitude of vibration for a flat circular plate consists of the radial solutions, which are Bessel functions, modulated by the angular solution, which is a linear superposition of a sine and cosine function. In most textbooks one of these orthogonal solutions is taken as the applicable solution, or at best the analysis leaves the superposition of the two terms explicit.[5,6]

For most practical purposes this type of analysis is adequate, because the position of an antinode on the plate is determined by the imposition of the driving element, and therefore the orientation of the angular solution is fixed. However, although it is seldom mentioned in print, often the two normal modes are not degenerate in frequency.

In common laboratory work or in a classroom demonstration of Chladni patterns, the normal modes of a circular plate can be visualized by placing sand on the plate while forcing it to vibrate. The classic driver is a violin bow, but magnetic or piezoelectric drivers work well in the modern laboratory. (A large collection of these patterns can be found in Ref. 7.) When observed in this manner, an antinode appears at the position of the driving mechanism and only one of the angular patterns appears. If the plate is driven from the center, it is common for both angular solutions to be present simultaneously, as predicted by theory, and circular rings appear on the plate. Likewise, sometimes the symmetry is broken for some reason (for example, the plate is touched), and a single angular solution will appear. In this case, the circular nodes will be bisected by one or more nodes that extend across the diameter of the plate.

As early as 1827 Sarvart observed that sometimes the radial nodes will rotate between the two orthogonal orientations after the driver has been removed from the plate. In 1887 Rayleigh proposed that the two orthogonal solutions would not be degenerate in frequency if the plate is not perfectly symmetric, and he suggested that the behavior observed by Savart was a result of this lack of symmetry. Rayleigh then proposed a perturbative technique to calculate the frequency difference between the two orthogonal modes, but did not explicitly derive the applicable equation.[8]

The splitting of the doublets in frequency has received little attention in the past century, probably because in most applications the driving mechanism is attached to the vibrating plate, and therefore only one set of modes is excited. Those involved in research on bells have been responsible for most of the reported work on this phenomenon, because after the clapper hits a bell, it is free to vibrate in all of its normal modes. For a bell, if the frequencies of the two orthogonal modes of the doublet are close together but not degenerate, the interference between the two modes causes a warble in the sound that is considered unpleasant. Although bells are more complicated than flat circular plates, the interest in the splitting of the doublets in bells has led to some research in the simpler system of the flat circular plate.[9] It is interesting to note, however, that probably the most complete work on the vibrations of plates does not mention this effect.[10]

Our purpose here is to present a detailed investigation of the doublet splitting in flat circular plates in such a way that it is accessible to students of physics. We begin with a derivation of the magnitude of the frequency splitting that we believe is more intuitive and easier to grasp than that outlined in Rayleigh's original work.

Following a review of the theory, we present an experimental verification of its applicability. Specifically, we report on observations of the doublet splitting of the normal modes of vibration of a flat circular plate supported in the center. The plate vibrations are driven acoustically so that no specific position for an antinode is imposed upon the plate, as occurs when the driving mechanism actually touches the plate. With this arrangement we can easily determine the orientation of the normal modes and their frequencies. Additionally, this type of driver has a very narrow bandwidth, enabling the precise determination of the normal mode frequencies. While the vibrations of the plate are being driven, we observe the modal patterns using electronic speckle pattern interferometry; therefore, the plate is completely unperturbed by the measurement process as well. After demonstrating the applicability of the theory, we show that we can determine some fundamental parameters of the plate from these measurements.

## II. THEORY

We begin by assuming that the plate is circular in shape, flat, thin, and has uniform density. We define a thin plate as one with a thickness that is significantly smaller than the radius of the plate and is much smaller than the wavelength of the sound propagating in the plate. In practice, we take the thickness to be at least an order of magnitude smaller than both of these parameters. We further assume that any variation from a circular shape or uniform density is not sufficiently large to alter the equations of motion that describe the vibration of a thin, flat, circular plate in cylindrical coordinates.

The derivation of the applicable equation for the deviation of the plate from equilibrium can be found in any one of numerous textbooks, and the solutions are the solutions to the Helmholtz equation, that is,[11]

$$(\nabla^2 \pm k^2) Z(r,\theta) = 0, \tag{1}$$

where $Z(r,\theta)$ represents the deviation of the plate from its equilibrium position, $r$ is the radial coordinate measured from the center of the plate, $\theta$ is the angular coordinate, and the Laplacian operator is assumed to be expressed in polar coordinates. The constant $k$ is given by

$$k = \left[\frac{\Omega^2 \rho (1-\nu^2)}{\kappa^2 E}\right]^{1/4}, \tag{2}$$

where $\Omega$ is the angular frequency of the vibration, $\rho$ is the density of the plate, $\nu$ is Poisson's ratio, $E$ is Young's modulus, and $\kappa$ is the radius of gyration, which for a circular plate of uniform thickness $h$ is given by

$$\kappa = \frac{h}{\sqrt{12}}. \tag{3}$$

The general solution to Eq. (1) is well known and is given by

$$Z(r,\theta) = Z_0[\alpha \cos(n\theta) + \beta \sin(n\theta)][J_n(kr) + I_n(kr)], \tag{4}$$

where $\alpha$ and $\beta$ are constants, and $J_n$ and $I_n$ are the ordinary and hyperbolic Bessel functions of order $n$, respectively. For our analysis we will assume that the plate is clamped at the center, and therefore all of the hyperbolic Bessel functions, as well as the $J_0$ ordinary Bessel function, are not solutions because they have a nonzero value at $r=0$. Therefore, the applicable equation for our study is

$$Z(r,\theta) = Z_0[\alpha \cos(n\theta) + \beta \sin(n\theta)] J_n(kr). \tag{5}$$

In the absence of some asymmetry of the plate, the frequencies at which the two radially orthogonal modes of each normal mode solution occur are identical; however, we assume that the presence of a small perturbation can break this degeneracy. Note that because $n$ cannot be zero, and the value of $n$ is equal to the number of nodal diameters of the mode, there can be no mode without a nodal diameter when a plate is clamped at the center unless the two radial orthogonal solutions are degenerate in frequency. If the two solutions are not degenerate, a mode with no nodal diameters may be observed only if the driving mechanism has a bandwidth that exceeds the difference in frequency between the two modes.

To derive a relation between the frequencies of the orthogonal modes in each doublet, we assume that there is a small perturbation in the symmetry of the plate. In any given plate there may actually be many small perturbations in the density, or the plate may be slightly asymmetrical in shape. We will treat the aggregate of these perturbations and asymmetries as a single mass at a point $(r_0, \theta_0)$. The effect of this mass will be modified by the presence of the nodal circles and diameters. Therefore, we refer to this hypothetical mass as the *modified perturbative mass*. Note that the angular coordinate of the modified perturbative mass is $n$-fold degenerate, and that the effect of any mass perturbation depends only upon its coordinate with respect to the closest antinode and not with respect to a single coordinate on the plate.

### A. Relation between resonance frequency and the position of the perturbing mass

The normal-mode frequencies of vibration are determined by the boundary conditions, and by the size and density of the plate. Regardless of the actual dimensions of the plate, however, the addition of a small mass to some point on the plate will always lower the resonance frequency unless the additional mass is placed on a circular node of the vibrational pattern. Although the addition of a mass can fundamentally change Eq. (1), if the added mass is significantly smaller than the mass of the plate, the problem can be treated without having to derive an alternative to Eq. (1).

If we assume that the additional mass is indeed significantly less than the total mass of the plate, the normal-mode solutions given by Eq. (5) are still valid, and the only necessary change is in the normal-mode frequencies.

To derive a relation between the perturbed resonance frequency $\omega$ and the unperturbed resonance frequency $\Omega$, we initially assume a thin circular plate of uniform density. We further assume that there is no significant loss due to the radiation of sound. The differential element of the kinetic energy of a point on the plate is given by

$$dT = \tfrac{1}{2}(dm) v^2, \tag{6}$$

where $v$ is the instantaneous speed of the point and $dm$ is the differential element of mass.

The speed of an arbitrary point on the plate is determined both by its position on the plate and the mode in which the plate is vibrating. If we assume that all points on the plate execute simple harmonic motion with angular frequency $\Omega$, the deviation from the equilibrium position is given by

$$Z(r,\theta,t) = Z_0[\alpha \cos(n\theta) + \beta \sin(n\theta)] J_n(kr) \cos(\Omega t). \tag{7}$$

We assume that the plate is vibrating in a normal mode and therefore, for the moment, the choice of the sine or cosine function as the angular solution is arbitrary. For convenience we choose the cosine function as the applicable angular solution, but in reality the angular part of the solution will be indeterminate.

The speed of any point on the plate is the time derivative of Eq. (7), that is,

$$\frac{dZ}{dt} = -Z_0 \Omega J_n(kr)\cos(n\theta)\sin(\Omega t). \tag{8}$$

We substitute Eq. (8) into Eq. (6) and take the time average over one period and obtain an expression for the differential average kinetic energy in a single cycle:

$$d\overline{T} = \tfrac{1}{4} Z_0^2 \Omega^2 J_n^2(kr)\cos^2(n\theta)\,dm. \tag{9}$$

In cylindrical coordinates the differential element of mass may be expressed as

$$dm = \rho r\, dr\, d\theta\, dz, \tag{10}$$

where $\rho$ is the mass density of the plate. We evaluate the integral of the average kinetic energy over a plate of radius $a$ and thickness $h$ and obtain

$$\overline{T} = \frac{\rho Z_0^2 \Omega^2}{4} \int_0^h dz \int_0^{2\pi} \cos^2(n\theta)\, d\theta \int_0^a J_n^2(kr) r\, dr. \tag{11}$$

If we use the fact that[12]

$$\int_0^a J_n^2(kr) r\, dr = \frac{a^2}{2}\left\{\left[\left.\frac{dJ_n(kr)}{d(kr)}\right|_{r=a}\right]^2 + \left[1-\left(\frac{n}{ka}\right)^2\right] J_n^2(ka)\right\}, \tag{12}$$

Eq. (11) evaluates to

$$\overline{T} = \frac{1}{8} Z_0^2 \Omega^2 M \left\{\left[\left.\frac{dJ_n(kr)}{d(kr)}\right|_{r=a}\right]^2 + \left[1-\left(\frac{n}{ka}\right)^2\right] J_n^2(ka)\right\}, \tag{13}$$

where $M$ is the total mass of the plate.

For ease of calculation we may substitute the recurrence formula for the Bessel functions into Eq. (13), that is,

$$2\frac{dJ_n(x)}{dx} = J_{n-1}(x) - J_{n+1}(x). \tag{14}$$

Equation (13) then becomes

$$\overline{T} = \frac{1}{8} Z_0^2 \Omega^2 M \left\{\frac{1}{4}[J_{n-1}(ka) - J_{n+1}(ka)]^2 + \left[1-\left(\frac{n^2}{ka}\right)\right] J_n^2(ka)\right\}. \tag{15}$$

We now assume that the plate is modified by the addition of a small mass at the position $(r_0, \theta_0)$. We assume that the mass is so small that it does not significantly affect the form of the vibrations. Therefore, the motion is assumed to still be simply harmonic, and the normal modes are adequately described by Eq. (7).

If an infinitesimal point mass of magnitude $\delta m$ is added to the plate and the frequency of vibration is left unchanged, the average kinetic energy is changed by an amount $\delta T$. The change in kinetic energy of the plate is given by

$$\delta T = \tfrac{1}{2}\delta m v^2, \tag{16}$$

where $v$ is still given by Eq. (8). The change in kinetic energy of the point averaged over one period is then given by

$$\delta\overline{T} = \tfrac{1}{4}\Omega^2 Z_0^2 \delta m \cos^2(n\theta_0) J_n^2(kr_0). \tag{17}$$

According to Rayleigh's principle, the vibrations of the plate will always reorient such that the point mass is situated on either a radial node or antinode.[13] When the mass is situated on a radial node, the addition of the mass causes no change in the physical situation, and the plate resonates at the natural frequency of the unperturbed plate. When the mass is not located on a radial node, a different frequency is required to induce resonance. In this case the maximum amount of energy will be transferred to the plate when the perturbing mass is vibrating with the maximum displacement. Therefore, if the mass is oscillating at all, the vibrational pattern of the plate will become oriented such that an antinode occurs at the position of the perturbing mass. Hence, $\cos(n\theta)$ is either unity or zero. If $\cos(n\theta)=0$, the mass is located at a node and $\delta\overline{T}$ is identically zero. In the more interesting case of $\cos(n\theta)=1$, there are an infinite number of combinations of $\delta m$ and $r_0$ that can result in the same value of $\delta\overline{T}$.

Physically, the case $\cos(n\theta)=1$ presents the possibility of different masses at different radial positions on the plate changing the kinetic energy the same amount. When located near a nodal circle, more mass is required to produce the same effect on the resonance frequency as would be produced by a smaller mass located near a circular antinode. (Note that the nodal circles will not significantly shift their position with the addition of a point mass because the position of the circular nodes is primarily determined by the boundary conditions of the plate.)

In an actual physical system there will be no change in the total kinetic energy of the plate with the addition of mass, and the required energy will be compensated by a lowering of the resonance frequency. The relation between the change in resonance frequency and the change in kinetic energy is straightforward to derive from Eq. (15), because $\overline{T}$ is quadratic in $\Omega$ except for the weak dependence of $\sqrt{\Omega}$ within $k$ in the argument of the Bessel functions. Therefore, to better than first order, the relationship between the change in resonance frequency and the change in kinetic energy is

$$\frac{\delta\Omega}{\Omega} = \frac{-\delta\overline{T}}{2\overline{T}}, \tag{18}$$

where $\delta\overline{T}$ is the change in the average kinetic energy that would occur with the addition of some mass, provided there was no change in frequency. We define the change in frequency as

$$\delta\Omega = \Omega - \omega, \tag{19}$$

where $\omega$ is the resonance frequency of plate that has been altered by the addition of the small perturbative mass. We then substitute Eqs. (15) and (17) into Eq. (18) and find

$$\frac{\omega}{\Omega}=1-\frac{4\,\delta m J_n^2(kr_0)}{M\left\{[J_{n-1}(ka)-J_{n+1}(ka)]^2+4\left[1-\left(\frac{n}{ka}\right)^2\right]J_n^2(ka)\right\}}. \tag{20}$$

For any specific plate the value of the second term on the right-hand side of Eq. (20) is directly proportional to $\delta m/M$. Therefore, we define the constant

$$\gamma_n=\frac{4J_n^2(kr_0)}{\left\{[J_{n-1}(ka)-J_{n+1}(ka)]^2+4\left[1-\left(\frac{n}{ka}\right)^2\right]J_n^2(ka)\right\}}, \tag{21}$$

and we can rewrite Eq. (20) as

$$\frac{\omega}{\Omega}=1-\gamma_n\frac{\delta m}{M}, \tag{22}$$

clearly showing that the ratio of the two frequencies varies linearly with $\delta m$ for any given mode number $n$.

### B. Calculating the position of the modified perturbative mass

The position and value of the modified perturbative mass depends critically on both the specifics of the plate and the mode in which the plate is vibrating. The dependence on the mode results from the fact that the circular nodes will occur at different parts of the plate as the mode changes, and it is possible that a large fraction of the perturbing masses fall on a circular node for one mode but not for others. Also, the perturbing mass must be located at an angle associated with a radial antinode for the mode in each doublet with the lower frequency, but it is impossible to know along which antinodal diameter it is located. In other words, even if we can find the radial position of the theoretical point mass, the angular position is $2n$-fold degenerate due to the symmetry of the modal patterns.

It is very difficult to determine the position and magnitude of the modified perturbative mass either theoretically or experimentally; however, it is quite easy to measure the frequencies $\Omega$ and $\omega$ in the laboratory. From this knowledge we can determine the possible values for the magnitude of $\delta m$ and the position $r_0$ by rewriting Eq. (22) as

$$\frac{\delta m}{M}=\left(1-\frac{\omega}{\Omega}\right)\frac{1}{\gamma_n}. \tag{23}$$

Using Eq. (23) we may plot the normalized perturbing mass versus the radial position, yielding the relationship between the necessary mass perturbations and the possible radial locations of that mass. Note that only the radial position is important because the angular position is determined by the position of the antinodes, and as noted above this coordinate is $2n$-fold degenerate.

Unless there are multiple perturbations in the density of the plate that are of similar magnitude, each set of normal modes will have the same orientation. We may then assume that the only significant difference among modes within these sets are the positions of the nodal circles. For a set of modes in which the number of nodal diameters $n$ is the same, but the number of nodal circles differ, the plots of $\delta m/M$ vs $r_0/a$ for each mode should intersect at the magnitude and radial position of the modified perturbative mass. The only uncertainty in the location of the intersection will be the difference caused by the addition of nodal circles, and the subsequent possibility that a large perturbation in the density falls near one of them.

## III. EXPERIMENT

### A. Imaging modal patterns

To experimentally investigate the frequency dependence of the normal modes of vibration of a flat circular plate, we have designed an experiment to view modal patterns in real time without the driving mechanism touching the plate or asymmetrically perturbing it in any way. Because the mechanism that drives the vibrations does not touch the plate, it does not impose an antinode on the modal structure. Therefore, the modal patterns are free to form based solely on the characteristics of the plate.

The plate used in our investigations was an austenitic stainless steel circular plate of radius $17.0\pm0.1$ cm and thickness $0.20\pm0.01$ cm. The plate was mounted on a horizontal 0.25-in.-diameter post attached to the center of the plate, which was in turn attached to a commercially available 1-in.-diameter vibration-damping post. The entire apparatus was mounted on an optical table which is isolated from ambient vibrations by pneumatic legs inside an anechoic chamber.

To image the modal patterns we designed and built an electronic speckle pattern interferometer.[14] An electronic speckle pattern interferometer detects out of plane vibrations by digitally recording the image of the speckle pattern on a structure that is illuminated by coherent light from a laser. The speckle pattern prior to the onset of movement is then digitally subtracted in real time from the speckle image after movement, producing a final image that is black where no movement has occurred (due to the subtraction of identical speckle patterns) and light where there has been a change in the speckle pattern.

The interferometer used in these experiments was built from discrete optical components and mounted on the same optical table as the plate. The laser used to illuminate the system resides on an optical table outside of the anechoic chamber, which is also isolated from ambient vibrations by pneumatic legs. The 532 nm beam from the laser enters the anechoic chamber through a small hole in the wall. The processing of the image is accomplished in real time using a computer outside of the anechoic chamber. The software for image subtraction and image recording was written by the authors using LABVIEW.

The interferometer can image out of plane vibrations on the order of 100 nm. Therefore, the driving mechanism does not need to transfer a significant amount of energy to the plate in order for the vibrations to become visible. This system allows us to drive the vibrations acoustically with a speaker mounted on the floor of the anechoic chamber. We found that the position of the speaker does not affect the structure of the normal modes of vibration of the plate, and therefore the actual position of the speaker relative to the plate is arbitrary.

The vibrations of the plate were driven acoustically using a sine wave generated by a high quality function generator. The bandwidth of the driver was less than 0.1 Hz. To ensure that all of the normal modes of vibration were found, the plate was struck and the resulting acoustic power spectrum was used to determine the normal modes of vibration. Addi-

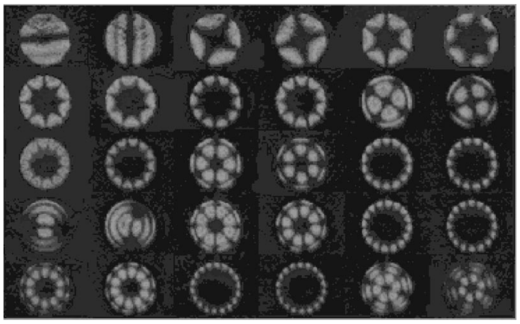

Fig. 1. Series of non-frequency-degenerate modes of a flat circular plate. The plate is driven acoustically and is viewed using electronic speckle pattern interferometry.

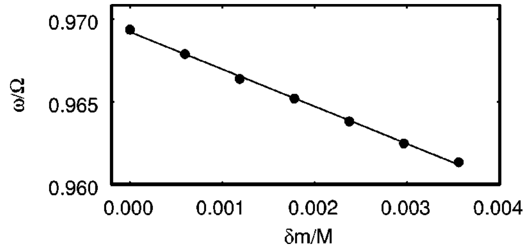


Fig. 2. Plot of the frequency splitting vs the perturbing mass for seven different masses. The line is a linear fit to the data.

tionally, the function generator was swept through the acoustic spectrum from 10 Hz to 10 kHz and all steady-state modes were noted. All of the normal modes of the plate that occur between 10 Hz and 10 kHz were imaged. Some typical modes are shown in Fig. 1.

In Fig. 1, the nodes are imaged as black while the antinodes are imaged as white. To distinguish one mode from another, it is common to refer to the modes by the number of nodal circles and the number of nodal diameters. For instance, the mode $(m,n)$ has $m$ nodal circles [resulting from the term $J_n(kr)$] and $n$ nodal diameters [resulting from the term $\cos(n\theta)$]. Using this notation the three modes shown horizontally across the top in Fig. 1 are (0,1), (0,2), and (0,3), respectively. The frequencies associated with the modes shown in Fig. 1 are given in Table I. Note that in the cases shown, as in the majority of cases we observed, the doublets are not degenerate in frequency. The breaking of the degeneracy of the doublets is due to the fact that the mounting hole for this particular plate was not drilled precisely in the center of the plate. This asymmetry caused the doublets to be nondegenerate in frequency.

### B. Effects of mass on doublet splitting

In order to demonstrate that Eq. (22) actually predicts the effect of perturbing the mass of the plate, we measured the frequencies of the resonances for the (1,2) mode as a small mass was added to the edge of the plate. Small amounts of putty were used to add mass to the plate without significantly altering the other parameters.

Table I. Frequencies of the modes shown in Fig. 1. The uncertainties are $\pm 0.05$ Hz.

| Mode | $\Omega_1/2\pi$ (Hz) | $\Omega_2/2\pi$ (Hz) |
|---|---|---|
| (0,1) | 127.20 | 134.40 |
| (0,2) | 336.30 | 345.20 |
| (0,3) | 775.30 | 775.70 |
| (0,4) | 1349.20 | 1350.20 |
| (0,5) | 2057.50 | 2058.50 |
| (1,2) | 2132.60 | 2145.20 |
| (0,6) | 2897.30 | 2898.00 |
| (1,3) | 3199.40 | 3204.00 |
| (0,7) | 3865.30 | 3865.80 |
| (2,1) | 3987.80 | 4055.40 |
| (1,4) | 4433.90 | 4438.80 |
| (0,8) | 4958.80 | 4959.90 |
| (1,5) | 5823.30 | 5828.90 |
| (0,9) | 6176.50 | 6177.80 |
| (2,3) | 6736.60 | 6743.60 |

Figure 2 shows a plot of the ratio of the frequencies $\omega/\Omega$ vs $\delta m/M$ as the mass $\delta m$ was added to the plate. It clearly shows that the ratio of the two frequencies varies linearly with the addition of mass as predicted by Eq. (22). The uncertainty in both axes is smaller than the data points, with the slight deviation from perfect linearity being attributable to the finite size of the perturbing mass.

When $\delta m/M$ exceeds approximately 0.004, the physical size of the perturbing mass exceeds the width of the node and the plot of $\omega/\Omega$ vs $\delta m/M$ becomes slightly nonlinear. However, even if the physical size of the mass were infinitesimal, a similar result would occur if the value of $\delta m/M$ exceeds approximately 0.1, because in this case the assumptions used in deriving Eq. (22) would be violated.

The line shown in Fig. 2 is a linear fit to the data. The slope, found by linear regression, is $-2.49\pm 0.05$. Note that $\omega/\Omega$ is not equal to unity when no perturbing mass is attached to the plate, due to the asymmetry of the plate caused by the slight off-center mounting hole as noted above. A perfectly symmetric plate would intercept the vertical axis at unity. However, there would be no evidence of the nondegenerate doublets of the unperturbed plate as shown in Fig. 1. The linear nature of the response to the addition of mass shown in Fig. 2 clearly indicates that the asymmetry of the plate is not significant enough to impede the applicability of Eq. (22).

## IV. ANALYSIS

Because the mass that was added to the edge of the plate was placed on a nodal diameter of the normal mode, the magnitude of the slope of the graph in Fig. 2 is the constant $\gamma_2$ defined by Eq. (21) with $r_0=a$. Therefore, applying a linear regression analysis to the data shown in Fig. 2 provides an experimental value for $\gamma_2$. Once $\gamma_2$ is known, it can be used to determine the value of the propagation constant $k$, which is a constant of the plate which is otherwise very difficult to precisely determine.

The theoretical value for $k$ is given by Eq. (2), but calculating $k$ requires a complete knowledge of the exact shape of the plate and the material of which it is made. By applying mass to the edge of the plate, as was the procedure in our experiments, Eq. (22) can be fit to the data using the value of $k$ as the only free parameter. However, because $k$ is proportional to the square root of the frequency, it is more convenient to define the constant

$$\xi = \frac{k}{\sqrt{\Omega}}. \tag{24}$$

The data shown in Fig. 2 indicate that for the plate used in our experiments, $\xi = 0.50 \pm 0.02\ \mathrm{s}^{1/2}/\mathrm{m}$, which is in good agreement with the estimated theoretical value of $0.56\ \mathrm{s}^{1/2}/\mathrm{m}$ for a perfectly circular stainless steel plate. The knowledge of this value enables further analysis of the plate, including a calculation for the position of the modified perturbative mass using Eq. (23).

## V. CONCLUSION

We have presented a derivation of the frequency splitting of the doublet modes of vibrating circular plates. We have also shown that this derivation is applicable to our experiment by observing the splitting of the doublets while driving the vibrations of the plate without actually touching the plate. We have further used this data to measure the value of the propagation constant $k$.

Although the experiments described here require extensive and expensive equipment, it is possible to demonstrate these effects in the classroom. We have used a thin aluminum plate clamped at the center and mounted such that the surface is parallel to a table top; normal mode vibrations can then be driven by a large public address speaker connected to a function generator. Although we have found it difficult to make measurements comparable to those that we have reported here, Chladni patterns can be observed by sprinkling sand on the plate just as when the plate is driven by more usual means. By mounting the plate at the center, patterns with diametrical nodes will form; slightly perturbing the plate with a small amount of putty at an antinode will rotate the pattern by $\pi/2$ as described above. An alternative method of observing such patterns without the driving mechanism touching the plate is to use dry ice as described by Waller.[7]

We hope that this work will encourage educators to discuss the phenomenon of frequency splitting of the doublet modes of vibrating symmetrical structures in the classroom. It is a fascinating phenomenon that appears to be seldom addressed outside of specialized applied physics journals, yet this phenomenon is easily understood, and a discussion can be included whenever Chladni patterns are shown to students.

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