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Abstract: The transient response of a flat circular plate to a sudden impact has been studied experimentally and theoretically. High-speed electronic speckle pattern interferometry reveals the presence of pulses that travel around the edge of the plate ahead of the bending motion initiated by the strike. It is found that the transient motion of the plate is well described by Kirchhoff thin-plate theory over a time approximately equal to the time required for the initial impulse to circumvent the plate; however, a more sophisticated model is required to describe the motion after this time has elapsed.

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1. Introduction

Studies of the motion of flat plates have been ongoing for many years. Typically, the interest has been in determining the frequencies and deflection shapes of the normal modes of plates, and there currently exists a firm theoretical and experimental basis for predicting the resonances of plates of various common shapes.[1] However, the transient response of impacted plates, even those with common geometries such as rectangular and circular, is still not well understood.

The struck flat plate is a common occurrence in a large number of situations, including those found in industry, the military, and music. There currently exists a small body of theoretical work on the transient response of impacted plates, and a few researchers have reported simulations of infinite, rectangular, and circular plates.[2–4] Reports of experimental work in this area are even more limited and are generally confined to acoustic measurements.[5,6] To our knowledge there are only two reports on the agreement between the predicted and actual deflection shapes of an impacted plate as a function of time,[7,8] and no reports of such a comparison where the effects of the shape of the plate or boundary conditions at the edges have been considered.

Here we report on an investigation of the transient response of a struck flat circular plate using high-speed electronic speckle pattern interferometry. We compare the results with the predictions of the Kirchhoff model of a thin plate, which was numerically solved using a finite-difference scheme. We show that there is initially excellent agreement between the model and the experiment; however, once the impulse has had sufficient time to traverse the plate the predictions of the model significantly diverge from the observed deflection of the plate.

2. Experiments

To investigate the deflection of a plate as it is being struck, a flat circular plate was held firmly at the center and left free at the edges. The plate was placed in the object plane of an electronic speckle pattern interferometer with the capability of high-speed acquisition. The light source for the interferometer was a 5 W solid-state laser with a wavelength of 532 nm. Configured in this manner, the interferometer produced an image containing fringes of equal displacement, where each fringe on the interferogram represents an increase in the out of plane deflection of 266 nm. The camera used for imaging had a resolution of 320×240 pixels and an acquisition rate of 33 057 frames per second.

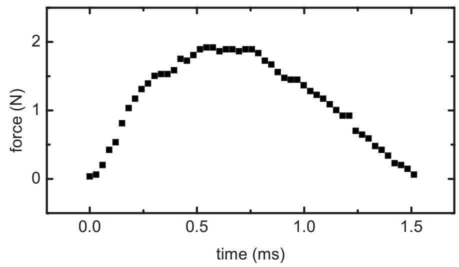

Fig. 1. Plot of the force exerted on the plate as a function of time.

The plate under consideration was a circular brass plate of diameter 134.0 ± 0.1 mm and thickness 4.7 ± 0.1 mm. It was held rigidly in the center by a screw that was secured to a 1.5 in. diameter aluminum post, which was in turn secured to an optical table. The plate was secured to the post to provide a rigid support, but it was offset from the post by a washer with a radius of 0.25 in. The edges of the plate were free to vibrate.

The plate was struck from behind at a point approximately 90% of the radius from the center by a 2.1 mm diameter screw attached to a load sensor. The strike lasted approximately 1.5 ms and had a maximum force of approximately 1.9 N. The output from the force sensor is shown in Fig. 1.

Figure 2 contains the series of interferograms produced during the strike, each representing the deflection of the plate approximately 30 μs after the previous one. The plate was struck from behind in the upper right corner. It is important to note that the fringe traversing the center of the plate represents the line of zero deflection; the point of maximum deflection is at the edge nearest point of impact and is displaced approximately 1.73 μm toward the observer in the final image.

Interferograms such as those shown in Fig. 2 are somewhat difficult to interpret since motion out of the plane of the interferogram creates the same image as motion into the plane. This ambiguity can usually be resolved by following the evolution of the lines of zero deflection. Figure 3 shows one image taken from Fig. 2 with several of the fringes annotated with the magnitude of the deflection at that point. Deflection toward the observer is labeled as being positive while deflection away from the observer is designated as being negative. To assist in interpreting the interferograms, fringes representing zero deflection are annotated by white arrows in Fig. 2.

An inspection of Fig. 2 shows that while the strike is occurring the deflection of the plate increases at the point of the strike as one would expect; however, the pulse due to the

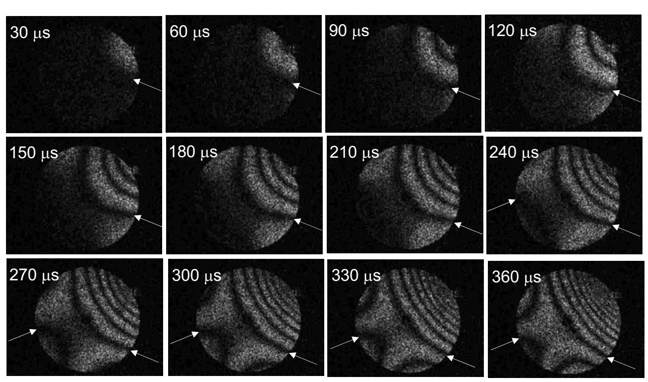

Fig. 2. Interferograms of the plate during the strike. The numbers in the upper left indicate the time since the beginning of the impact. The white arrows designate fringes of zero displacement.

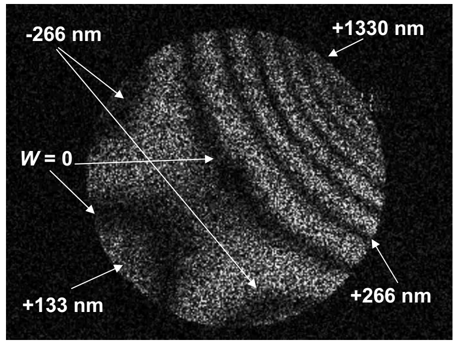

Fig. 3. One of the interferograms shown in Fig. 2 with several of the fringes labeled with the magnitude of the deflection. Motion toward the observer is noted as being positive while motion away from the observer is negative.

impact is transferred around the circumference of the plate without traversing the fringe of zero displacement. That is, the portion of the plate being driven directly by the impulse is being displaced toward the observer, but this displacement gives rise to symmetric pulses that are preceding the outward bending motion. These precursor pulses are characterized by displacement into the plane of the image and are being transferred around the circumference of the plate ahead of the outward displacement. The result is that the diametrical fringe representing zero displacement never traverses past the center of the plate.

By 90 μs after the initiation of the strike, the two symmetric pulses propagating around the circumference of the plate that have displacements into the plane of the interferogram are clearly visible. Surprisingly, these pulses lead to a third pulse, displaced out of the plane of the interferogram, which appears at a position diametrically opposite to the striking point. As will be shown later, eventually the two pulses that are displaced away from the observer coalesce, forcing the entire lower left half of the plate to be displaced into the plane of the interferogram. When this happens the two semicircular portions of the plate are displaced out of phase with one another, with the nodal line being through the center of the plate.

3. Modeling

It is reasonable to ask if the complex dynamics of the struck plate described above can be accurately modeled by a simple theory. Given the dimensions of the plate, one would expect that it can be considered to be thin and therefore Kirchhoff thin-plate theory should be adequate for the task. To test this assertion we modeled the plate using Kirchhoff thin-plate theory and solved the resulting differential equation explicitly using the method of finite differences.

Kirchhoff thin-plate theory describes the displacement w of a plate by the fourth-order differential equation[1]

$$-D\nabla^4 w(r,\phi,t) + p(r_0,\phi_0,t) - R\frac{\partial w(r,\phi,t)}{\partial t} = \sigma\frac{\partial^2 w(r,\phi,t)}{\partial t^2}, \tag{1}$$

where σ is the mass per unit area of the plate, R is a damping coefficient that is proportional to the velocity (and is negligible in this calculation), and $p(r_0,\phi_0,t)$ is the applied force per unit area at the point of impact. The factor D is the flexural rigidity and is given by

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{2}$$

where E is the elastic modulus of the plate, h is the thickness and ν is Poisson's ratio.

While it is logical to use the method of finite differences to solve Eq. (1), the circular symmetry and the complex boundary conditions on the free edge of the plate make this a difficult problem to solve accurately. A Cartesian mesh does not fit the symmetry of the problem;

therefore, to properly apply the boundary conditions one must employ cylindrical coordinates. However, if one employs a uniform radial mesh, the area enclosed between adjacent radial and angular mesh points increases with radius; this tends to decrease the accuracy of the solution, especially on the outer edge. To overcome this difficulty, we employed a nonlinear radial mesh that varies inversely with the radial coordinate:

$$dr = \frac{d\rho}{r}. \tag{3}$$

With this change of coordinates, the term involving ∇^4 can be written as

$$\nabla^4 w = 8\frac{\partial^2 w}{\partial \rho^2} + 16\rho\frac{\partial^3 w}{\partial \rho^3} + 4\rho^2\frac{\partial^4 w}{\partial \rho^4} + \frac{1}{4\rho^2}\frac{\partial^4 w}{\partial \phi^4} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + 2\frac{\partial^4 w}{\partial \rho^2 \partial \phi^2}. \tag{4}$$

The boundary conditions at the center of the plate are merely that the displacement and its first derivative must be identically zero. The boundary conditions at the free edge, however, are significantly more complicated since there the bending, shear, and twisting moments must all be zero. Additionally, to implement the finite difference scheme for any point on the mesh it is necessary to know the displacements of each of the nearest neighbors as well as the displacements of the points twice removed from that point. If the mesh ends at the edge of the plate, it is impossible to implement the finite difference scheme. Therefore, two virtual rings were added to the mesh and the displacements of the points on those rings were determined so that the boundary conditions at the actual edge of the plate were satisfied. The two virtual rings have no physical significance and are not included in the results, but they do allow one to determine finite difference solutions to Eq. (1) and accurately enforce the free-edge boundary conditions. The mesh used in all of the work reported here had 200 radial and 180 angular points. To achieve a stable solution with this spatial mesh it was necessary to use a time step of 2×10^{-9} s. The driving force $p(r_0, \phi_0, t)$ was modeled with a portion of a sine function scaled to approximate the time-dependent force shown in Fig. 1.

To test the validity of the model the predicted normal-mode frequencies for an annular plate with an inner radius equal to 30% of the outer radius were compared to the analytically calculated values found using Ref. 1. The plate was assumed to be clamped on the inner radius and free on the outer edge. The variation between the analytical and numerical calculations ranged from 0.2% to 5.3%, with the average variation being approximately 2%. These differences are similar to those found when comparing the results using the different analytic solutions presented in Ref. 1.

Once the model had been validated, it was used to determine the time-resolved displacement predicted by the Kirchhoff theory for the plate used in the experiment described above. After the computation was complete, the calculated displacement at each point of the plate was converted into equivalent quarter-wavelengths of the illuminating light and plotted as a gray-scale from black (even integer multiples of quarter-wavelengths) to white (odd integer multiples). Using this method of display, the results of the simulation appeared similar to an interferogram of the deformed plate. The results of the simulation are found in Mm. 1, where the actual interferogram appears on the left and the results of the simulation appear on the right. The time elapsed since the strike is shown in the lower right corner.

[Mm. 1 Video of the interferograms beside the results of the simulation of the struck plate. The time elapsed since the strike appears in the lower right corner.] This is a file of type "avi" (19.8 Mb).

4. Discussion

The evolution of the transient response of the plate shown in Fig. 2 and Mm. 1 is surprising in many ways, as is the fact that it can be accurately modeled with such a simple theory. One important result of the modeling effort is that any ambiguity in the interferograms as to the direction of motion of the plate can be unequivocally determined due to the excellent agreement between the theory and experiment. The results confirm the assertion above that the first precursor pulses are indeed out of phase with the driving motion, and the nodal lines do not represent a simple minimum in the displacement of the plate. That is, the impact is pushing the top of the plate toward the observer but the displacement changes direction across nodal lines. Therefore, the first precursor pulses are actually displacements away from the observer while the one that appears diametrically opposite to the point of impact is displaced toward the observer.

There are a number of striking features in the results presented above that deserve comment, but clearly the most surprising feature is the evolution and propagation of the precursor pulses. The eventual rocking motion of the plate about a nodal diameter that occurs after approximately one millisecond is expected, but the manner in which this motion evolves is not. The presence of a precursor to the displacement directly attributable to an impact has been both predicted and observed in plates before the wave reaches the boundaries;[2,4,7] however, the nature of the dynamics of this pulse as it propagates around the edge of a circular plate is fundamentally different. In this case it is the edge of the plate that primarily determines how the pulses propagate. Furthermore, the fact that the precursor pulses lead to an initial displacement of the plate in the same direction as the impact, but at a position diametrically opposite to the point of impact, is counterintuitive. This motion is quickly counteracted by the arrival of the first two precursor pulses that are directed into the plane of the interferogram (in the opposite direction of the impulse), which cause the plate to then bend into the plane. The result is that eventually the plate flexes about the center line in a rocking motion.

The presence of a precursor with a displacement opposite to the motion directly created by the impact has previously been posited to be attributable to the dispersion of the flexural waves in a solid; however, the time evolution of the pulse shown in Fig. 2 and Mm. 1 demonstrates that this is not the case. The higher frequency waves do indeed have a higher speed in the plate, but these precursor pulses cannot be attributed to this phenomenon. The fact that the pulses are seen to precede the deflection that can be directly attributed to the motion caused by the strike, without traversing the line of zero-displacement, indicates that they are not attributable simply to the dispersive nature of the bending waves. Furthermore, a comparison of these results with Fig. 1 shows that these pulses propagate completely around the plate before the driving force has reached its maximum. Thus, the top of the plate is still moving toward the observer as these pulses propagate around the plate.

Instead of being attributable to dispersion, these results indicate that the precursor pulses are attributable to the stiffness of the plate and are a consequence of the fact that a positive displacement at one point in a stiff medium induces a displacement in the opposite direction at points close to it. The bending eventually produces a second-order effect that creates a third pulse; however, this also occurs while the initial impact is still in progress and the displacement does not traverse the nodal line. Therefore, this motion cannot be attributed to the dispersive nature of the material either.

The evolution of the motion of the plate also shows clearly that the impulse is not traveling around the plate, nor is the whole-body motion immediately induced about the central nodal line. Instead, a displacement is being induced in the plate at regions far from the point of the strike, which eventually evolves into the rocking whole-body motion characteristic of the (1,0) mode. One may safely assume that a larger plate would result in additional nodal lines and hence the number of induced pulses is not as important as the mere fact that they exist and propagate around the edge of the plate.

The excellent agreement between the model and the experiment is also of considerable interest. It has been recently reported by others that explicitly solving the Kirchhoff equations numerically for the case of a circular thin plate can result in unrealistic solutions.[9] Specifically, they cite low-frequency noise and a high-frequency cut-off in the solution. A comparison of the actual and predicted results of the struck plate presented here reveals that indeed there is poor agreement between this model and the experimentally observed motion in the long-time limit ($\geqslant 1$ ms). When the simulation is allowed to progress beyond approximately 1 ms the deflection shapes predicted by the model are strikingly different than those observed. (The beginning of this lack of agreement can be observed in the last few images of Mm. 1.) Furthermore, the predicted average power spectrum is quite different from the measured power spectrum; the predicted frequencies of the lower resonances do not agree with the actual result and the relative power in each mode is poorly predicted.

It is possible to approximately calculate the eigenfrequencies of the plate analytically by assuming that the ratio of the outer to inner radii is 0.100 rather than the actual ratio of 0.103. The necessary parameters are given in Ref. 1. When the analytically calculated frequencies of the first five modes are compared with those predicted by the numerical simulation it is found that they agree well, indicating that the differences between the model and the experiment are not due to numerical error. However, when these values are compared with the actual resonance frequencies, the lowest frequencies are found to be significantly different, with the lowest predicted frquency larger than the measured value by more than a factor of 3. Despite the differences in the predicted resonance frequencies, the excellent agreement between the model and the experimental results reported here leaves little doubt that Eq. (1) describes the initial dynamics of the plate quite well.

To resolve the discrepancies between the predictions of the model and the physical response of the plate in the long-time limit it will probably be necessary to include a frequency-dependent damping term in Eq. (1). Chainge and Lambourg have implemented such a method for a suspended square plate,[6] but the assumptions made in their work do not appear to apply to the case of a circular plate clamped in the center. Measurements of free rectangular plates reported in Ref. 6 showed that the damping constants are small and of a similar order of magnitude below some cutoff frequency and are much larger with little variation above this frequency. To determine if there is a similar relationship between frequency and damping in a clamped circular plate we experimentally determined the exponential damping constant for five of the modes of the circular plate used in these experiments. It was found that this constant can vary from less than unity to over 30, with no obvious systematic relationship between the frequency and the damping.

5. Conclusion

The work reported here demonstrates that the transient response of a circular flat plate to sudden impact has some interesting and surprising characteristics. The impact creates precursor pulses that propagate around the outer edge of the plate, eventually meeting at the point diametrically opposite to the point of impact. These pulses are evidently not due to the dispersive nature of the bending waves because they occur before the wave motion actually begins and they do not traverse the nodal line. Rather, they appear spontaneously ahead of the motion directly attributable to the impact and well before the impact point has time to complete a half-cycle of the motion.

This unusual transient motion can be accurately predicted by explicitly solving the Kirchhoff plate equations using the method of finite differences. A nonlinear mesh assists in the numerical solution and overcomes some of the difficulties cited in the past. However, the Kirchhoff thin-plate model does not accurately predict the long-term motion of the plate. Indeed, after the initial precursor pulses have reached the opposite side of the plate and coalesced, the motion is not well described by the model. It is probable that producing an accurate time-resolved model of a struck plate will require the inclusion of frequency dependent damping terms; however, the simple model of Eq. (1) appears to be sufficient to predict the initial transient response of the system to an off-center impact.

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